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# Singular Reaction-Diffusion System Arising from Quenching

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**Abstract:** In this paper, we study a singular parabolic reaction-diffusion system with positive Dirichlet boundary conditions. It is shown that certain conditions are sufficient to guarantee finite-time quenching and global existence of solutions. This system appears in the modeling of the quenching phenomena.

**Keywords:** *reaction-diffusion system; quenching; singular parabolic equations.*

**Mathematics Subject Classification (2010):** 35K57, 35K67, 93A30.

## 1 Introduction

Quenching refers to the process of rapidly cooling a material from a high temperature to a lower temperature. This is done to alter the material's physical or mechanical properties such as hardness or strength. The rapid cooling prevents the material from undergoing a gradual cooling process, which would allow the material to form larger crystals that could weaken the material's structure. Quenching can be accomplished using different methods, including immersion in water, oil, or air, depending on the desired outcome. The study of this important phenomenon began in 1975 with a paper by Kawarada [5], where he studied a model in one space dimension. That paper was an introduction to the large-scale studies of the quenching problem by many researchers in several scientific fields. For a detailed survey, we refer to Chan [3], Levine [7], Rouabah et *al.* [13], Zouaoui et *al.* [20].

By using reaction-diffusion models, researchers can simulate the behavior of quenching processes and predict the resulting microstructure and mechanical properties of the metal. This can help in the design of new quenching techniques and in the optimization of

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existing ones. For more research on the phenomenon of quenching via reaction-diffusion systems, we refer the readers to Bonis [2], Ji et *al.* [4], Mesbahi [8], Mu et *al.* [11], Pei and Li [12], Salin [14–16], Wang [17], Zheng and Song [18], Zheng and Wang [19] and the references therein, where we will also find, in addition to the results by Mesbahi [9] and [10], many theoretical and numerical methods frequently used to study such problems.

In biology, quenching is a process that involves the rapid cooling of a sample in order to interrupt or halt certain biological processes. This procedure has several uses, including stopping metabolic processes and preserving metabolite profile of a sample in metabolomics. Protein synthesis and degradation can also be stopped for protein level and modification analysis in cells or tissues, while RNA in cells or tissues can be preserved for the analysis of gene expression. Moreover, microbial cultures can be preserved for long-term storage or transport by rapidly cooling them to halt growth and metabolic activity.

Quenching has many applications in medicine. One common medical application of quenching is cryotherapy, where extreme cold to treat disease or injury is used. This can include using liquid nitrogen to freeze and destroy cancerous tissue, or the use of ice packs to reduce swelling and inflammation. Another application is controlling the release of drugs from drug delivery systems. Rapid cooling of the system can halt or slow down drug release, enabling sustained release over time. Furthermore, quenching can aid in the preservation of biological samples such as blood or tissue samples for analysis or storage. Rapid cooling can prevent degradation of the sample and preserve its integrity for later use.

Quenching is also an important process in the manufacture of contact lenses. Typically, after the lenses are shaped, they undergo thermal quenching by being immersed in cold water. This process helps in solidifying their structure and preventing any deformation or distortion during handling and further processing. Furthermore, it enhances the mechanical and optical features of the lenses making them stronger, more resistant to damage, and long-lasting. Chemical quenching is also used by manufacturers to adjust the properties of the lenses. For instance, to crosslink the polymer chains in the lenses or to enhance their strength and flexibility. For better understanding, we refer to Barka et *al.* [1], Khurshid et *al.* [6].

In this work, we are interested in the study of the following reaction-diffusion system with general singular terms and positive Dirichlet boundary conditions that can be applied to the quenching phenomenon:

$$\begin{cases} (u_1)_t - \Delta u_1 = -f_1(u_2) & \text{in } (0,T)\times\Omega, \\ \vdots & \vdots \\ (u_{m-1})_t - \Delta u_{m-1} = -f_{m-1}(u_m) & \text{in } (0,T)\times\Omega, \\ (u_m)_t - \Delta u_m = -f_m(u_1) & \text{in } (0,T)\times\Omega, \\ u_1 = u_2 = \cdots = u_m = 1 & \text{on } (0,T)\times\partial\Omega, \\ u_1(0,x) = u_{10}(x), \dots,\ u_m(0,x) = u_{m0}(x) & \text{in } \Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded domain with smooth boundary. The functions $f_j$ $(1 \leq j \leq m)$ are positive on $(0,1]$. The initial data satisfy

$$\begin{cases} u_{10},\ u_{20}, \dots,\ u_{m0} \in C^2(\Omega) \cap C^1\left(\overline{\Omega}\right), \\ u_{j0} = 1, \text{ for all } 1 \leq j \leq m, \text{ on } \partial\Omega, \\ 0 < u_{j0} \leq 1, \text{ for all } 1 \leq j \leq m, \text{ in } \overline{\Omega}. \end{cases} \tag{2}$$

The rest of this paper is organized as follows. In the next section, we state our main results. In the third section, we prove some important preliminary results. The fourth section is devoted to the proof of the main results. The paper ends with a concluding remarks and perspectives.

## 2 Statement of Main Results

### 2.1 Assumptions

For this model, the finite-time quenching phenomena are caused by singular nonlinearities in the absorption terms of (1).

**Definition 2.1** We say the solution $(u_1, \ldots, u_m)$ of problem (1) quenches if $(u_1, \ldots, u_m)$ exists in the classical sense and is positive for all $0 \leq t < T$, and also satisfies $\inf_{t \to T} \min_{x \in [0,1]} \{(u_1(t,x), \ldots, u_m(t,x)\} = 0$. In this case, $T$ is called quenching time.

To study problem (1), we also assume that the positive functions $f_j : (0,1] \to (0, +\infty)$, $1 \leq j \leq m$, satisfy the following simple assumptions which allow them to be chosen from a wide range:

$(H_1)$ The functions $f_j$, $1 \leq j \leq m$, are locally Lipschitz on $(0,1]$,

$(H_2)$ $f_j'(s) < 0$ on $(0,1]$ for all $1 \leq j \leq m$,

$(H_3)$ $\lim_{s \to 0^+} f_j(s) = +\infty$ for all $1 \leq j \leq m$.

In order to state our results more conveniently, we denote by $\varphi$ the first eigenfunction associated with the first eigenvalue $\lambda_1$ of the problem

$$\begin{cases} \Delta\varphi + \lambda\varphi = 0 & \text{in } \Omega, \\ \varphi = 0 & \text{on } \partial\Omega, \end{cases}$$

normalized by $\int_\Omega \varphi(x)dx = 1$, with $\varphi(x) > 0$ in $\Omega$.

### 2.2 The main results

The following theorem gives us a sufficient condition for finite-time quenching.

**Theorem 2.1** *Under hypotheses $(H_1) - (H_3)$, the solution of problem (1) quenches in finite time for any initial data provided that $\lambda_1$ is small enough.*

Many quenching studies confirm that time-derivatives blow-up while the solution itself remains bounded. We refer, for example, to Chan [3] and Kawarada [5]. Throughout this paper, without any special explanation, we assume that the initial data $u_{10}, \ldots, u_{m0}$ satisfy

$$\Delta u_{10} - f_1(u_{20}) < 0, \ldots, \ \Delta u_{m0} - f_m(u_{10}) < 0 \text{ in } \Omega. \tag{3}$$

Thus, the global existence of solutions can be described by the following theorem.

**Theorem 2.2** *If the diameter of $\Omega$ is small enough and the initial data satisfies $0 < \varepsilon \leq u_{10}, \ldots, u_{m0} \leq 1$ in $\overline{\Omega}$, then under hypotheses $(H_1)-(H_3)$, the solution of problem (1) does not quench in finite time. In this case, we say that the solution $(u_1, \ldots, u_m)$ exists globally.*

## 3 Preliminary Results

We will prove two important lemmas which we will use to prove our main results.

**Lemma 3.1** *Assume that the initial data satisfy* (3)*, then* $(u_1)_t, \ldots, (u_m)_t < 0$ *in* $(0,T) \times \Omega$.

***Proof.*** Let $I_j(t,x) = (u_j)_t(t,x)$ for all $1 \le j \le m$ and $(t,x) \in (0,T) \times \Omega$. Differentiating system (1) with respect to $t$, we have

$$\begin{cases} \frac{\partial}{\partial t} I_1 = \Delta (u_1)_t - (u_2)_t \ f_1'(u_2) & \text{in } (0,T) \times \Omega, \\ \vdots & \vdots \\ \frac{\partial}{\partial t}(I_m)(x,t) = \Delta (u_m)_t - (u_1)_t f_m'(u_1) & \text{in } (0,T) \times \Omega, \\ I_1 = I_2 = \cdots = I_m = 0 & \text{on } (0,T) \times \partial\Omega, \\ I_j(0,x) < 0, \text{ for all } 1 \le j \le m & \text{in } \Omega, \end{cases}$$

which, after simplification, gives

$$\begin{cases} \frac{\partial}{\partial t} I_1 - \Delta I_1 = -I_2 f_1'(u_2) & \text{in } (0,T) \times \Omega, \\ \vdots & \vdots \\ \frac{\partial}{\partial t} I_m - \Delta I_m = -I_1 f_m'(u_1) & \text{in } (0,T) \times \Omega, \\ I_1 = I_2 = \cdots = I_m = 0 & \text{on } (0,T) \times \partial\Omega, \\ I_j(0,x) < 0, \text{ for all } 1 \le j \le m & \text{in } \Omega. \end{cases} \tag{4}$$

By the comparison principle, we have, for all $(t,x) \in (0,T) \times \Omega$,

$$I_j(t,x) = (u_j)_t(t,x) < 0 \text{ for all } 1 \le j \le m.$$

This shows that $u_1, \ldots, u_m$ are strictly decreasing in time.

Now, we consider the radial solutions of problem (1) on $\Omega = Br = \{x \in \mathbb{R}^N : |x| < R\}$.

**Lemma 3.2** *Let* $(u_1, \ldots, u_m)$ *be the global solution of problem* (1) *with* $(u_{10}, \ldots, u_{m0}) \equiv (1, \ldots, 1)$*,* $u_1, \ldots, u_m \ge b$ *in* $(0,\infty) \times \overline{B}_R$ *for some* $b \in (0,1)$*. Then* $(u_1, \ldots, u_m)$ *approaches uniformly from above to a solution* $(U_1, \ldots, U_m)$ *of the steady-state problem*

$$\begin{cases} \Delta U_1 = f(U_2) & \text{in } B_R, \\ \vdots & \vdots \\ \Delta U_{m-1} = f(U_m) & \text{in } B_R, \\ \Delta U_m = f(U_1) & \text{in } B_R, \\ U_1 = U_2 = \cdots = U_m = 1 & \text{on } \partial B_R. \end{cases} \tag{5}$$

***Proof.*** Since $(1, \ldots, 1)$ is a strict super-solution of problem (1), by Lemma 3.1, we have $(u_1)_t, \ldots, (u_m)_t < 0$ in $(0,\infty) \times B_R$. Define the functions

$$Q_j(t,x) = \int_{B_R} G(x,y)\, u_j(t,y)\, dy, \text{ in } (0,\infty) \times B_R, \text{ for all } 1 \le j \le m,$$

where $G(x,y)$ is Green's function associated with the operator $-\Delta$ on $B_R$ under Dirichlet boundary conditions. Hence

$$\begin{aligned}
\frac{\partial}{\partial t}(Q_1) &= \int_{B_R} G(x,y)(u_1)_t(t,y)\,dy \\
&= \int_{B_R} G(x,y)\,\Delta u_1(t,y)\,dy - \int_{B_R} G(x,y)\,f_1(u_2(t,y))\,dy, \\
&\vdots \\
\frac{\partial}{\partial t}(Q_m) &= \int_{B_R} G(x,y)(u_m)_t(t,y)\,dy \\
&= \int_{B_R} G(x,y)\,\Delta u_m(t,y)\,dy - \int_{B_R} G(x,y)\,f_m(u_1(t,y))\,dy,
\end{aligned}$$

this gives us

$$\begin{aligned}
\frac{\partial}{\partial t}(Q_1) &= 1 - u_1(x,y) - \int_{B_R} G(x,y)\,f_1(u_2(t,y))\,dy, \\
&\vdots \\
\frac{\partial}{\partial t}(Q_m) &= 1 - u_m(x,y) - \int_{B_R} G(x,y)\,f_m(u_1(t,y))\,dy.
\end{aligned}$$

It follows from $(u_j)_t < 0$ for all $1 \le j \le m$, that

$$G(x,y)\,f_1(u_2(t,y)),\ldots,\; G(x,y)\,f_{m-1}(u_m(t,y)) \text{ and } G(x,y)\,f_m(u_1(t,y))$$

are nondecreasing with respect to $t$. According to the monotone convergence theorem with

$$b \le U_j(x) = \lim_{t\to 0} u_j(t,x) \text{ for all } 1 \le j \le m,$$

we have

$$\begin{aligned}
\lim_{t\to 0}\frac{\partial}{\partial t}(Q_1) &= 1 - U_1(x) - \int_{B_R} G(x,y)\,f_1(U_2(y))\,dy, \\
&\vdots \\
\lim_{t\to 0}\frac{\partial}{\partial t}(Q_m) &= 1 - U_m(x) - \int_{B_R} G(x,y)\,f_m(U_1(y))\,dy.
\end{aligned}$$

Furthermore, since $Q_1,\ldots,Q_m$ are bounded, $(Q_1)_t,\ldots,(Q_m)_t \le 0$, and by $(u_1)_t,\ldots,(u_m)_t < 0$, we have

$$\lim_{t\to 0}\frac{\partial}{\partial t}(Q_j) = 0 \text{ for all } 1 \le j \le m,$$

which yields

$$\begin{aligned}
U_1(x) &= 1 - \int_{B_R} G(x,y)\,f_1(U_2(y))\,dy, \\
&\vdots \\
U_m(x) &= 1 - \int_{B_R} G(x,y)\,f_m(U_1(y))\,dy,
\end{aligned}$$

and therefore $(U_1,\ldots,U_m)$ is a solution of problem (5), and the uniform convergence is ensured by Dini's theorem.

## 4 Proofs of the Main Results

***Proof.*** [of Theorem 2.1] Let $(u_1,\ldots,u_m)$ be the solution of problem (1) with the maximal existence time $T$. By the maximum principle, we have $0 \le u_j \le 1$ for all $1 \le j \le m$, in $(0,T)\times\Omega$. Let

$$\psi_j(t) = \int_\Omega (1-u_j)\,\varphi dx \ \text{ for all } 1 \le j \le m,\ t \in [0,T) \tag{6}$$

and

$$\Psi(t) = \psi_1(t) + \cdots + \psi_m(t), \quad t \in [0,T). \tag{7}$$

By hypotheses $(H_1)-(H_3)$ and the corresponding Taylor expansions, we can easily get

$$f_1(u_2) \ge \delta(1-u_2) + c_1\ ,\ldots,\ f_m(u_1) \ge \delta(1-u_1) + c_m, \tag{8}$$

where $\delta$, $c_1,\ldots,c_m$ are positive constants determined by $f_1(u_2),\ldots,f_m(u_1)$.

By a straight-forward computation and (8), we have

$$\begin{aligned}
\psi_1'(t) &= -\int_\Omega \Delta u_1\varphi dx + \int_\Omega f_1(u_2)\,\varphi dx \\
&= \int_\Omega \Delta(1-u_1)\,\varphi dx + \int_\Omega f_1(u_2)\,\varphi dx \\
&\ge -\lambda_1\int_\Omega (1-u_1)\,\varphi dx + \delta\int_\Omega (1-u_2)\,\varphi dx + c_1\int_\Omega \varphi dx \\
&= -\lambda_1\psi_1(t) + \delta\psi_2(t) + c_1.
\end{aligned}$$

In the same way, with $\psi_2(t),\ldots,\psi_m(t)$, we finally get the following inequalities:

$$\begin{aligned}
\psi_1'(t) &\ge -\lambda_1\psi_1(t) + \delta\psi_2(t) + c_1, \\
&\ \ \vdots \\
\psi_m'(t) &\ge -\lambda_1\psi_m(t) + \delta\psi_1(t) + c_m.
\end{aligned}$$

Using (7), we get

$$\Psi'(t) \ge (\delta-\lambda_1)\,\Psi(t) + C\ ,\ \text{with } C = c_1 + \cdots + c_m. \tag{9}$$

Since $0 \le u_j \le 1$ in $(0,T)\times\Omega$, then $0 \le 1-u_j \le 1$ in $(0,T)\times\Omega$, which clearly implies by (6) that $0 \le \psi_j(t) \le 1$ for all $1 \le j \le m$, consequently, $1 \le \Psi(t) \le m$. Since $\lambda_1$ is small enough, it is obvious that $(\delta-\lambda_1)\,\Psi(t) + C > 0$. Then, by (9), we have

$$\frac{d\Psi}{(\delta-\lambda_1)\,\Psi(t)+C} \ge dt\ ,\ t \in [0,T),$$

which gives, by integration from 0 to $T$,

$$t \le \begin{cases} \dfrac{1}{\delta-\lambda_1}\log\left(\dfrac{(\delta-\lambda_1)\,\Psi(t)+C}{(\delta-\lambda_1)\,\Psi(0)+C}\right) & \text{if } \delta \ne \lambda_1, \\ \dfrac{1}{C}\left(\Psi(t)-\Psi(0)\right) & \text{if } \delta = \lambda_1. \end{cases} \tag{10}$$

Now, letting $t \to T^-$ in (10) and combining $\lim\limits_{t\to T^-} \Psi(t) \le m$, we get

$$T \le \begin{cases} \dfrac{1}{\delta - \lambda_1} \log\left(\dfrac{m(\delta - \lambda_1) + C}{(\delta - \lambda_1)\Psi(0) + C}\right) & \text{if } \delta \ne \lambda_1, \\ \dfrac{1}{C}(m - \Psi(0)) & \text{if } \delta = \lambda_1. \end{cases} \tag{11}$$

Since $1 \le \Psi(t) \le m$, we can easily arrive at the positivity of the right-hand side of (11), which shows finite time quenching of the solutions in system (1). This ends the proof of Theorem 2.1.

***Proof.*** Consider the auxiliary system

$$\begin{cases} \frac{\partial}{\partial t}\bar{u}_1 = \Delta\bar{u}_1 - f(\bar{u}_2) & \text{in } (0,T)\times\Omega, \\ \vdots & \vdots \\ \frac{\partial}{\partial t}\bar{u}_m = \Delta\bar{u}_m - f(\bar{u}_1) & \text{in } (0,T)\times\Omega, \\ \bar{u}_1 = \cdots = \bar{u}_m = 1 & \text{on } (0,T)\times\partial\Omega, \\ \bar{u}_1(0,x) = \cdots = \bar{u}_m(0,x) = 1 & \text{in } \overline{\Omega}. \end{cases}$$

By the comparison principle, we have $u_j \le \bar{u}_j$ for all $1 \le j \le m$.

We first consider the following system:

$$\begin{cases} \Delta\bar{u}_1^* = f_1(1) & \text{in } B_R, \\ \vdots & \vdots \\ \Delta\bar{u}_m^* = f_m(1) & \text{in } B_R, \\ \bar{u}_1^* = \cdots = \bar{u}_m^* = 1 & \text{on } \partial B_R. \end{cases}$$

By Green's function, the solution is $(\bar{u}_1^*, \ldots, \bar{u}_m^*)$ denoted as follows:

$$\bar{u}_j^* = \frac{f_j(1)\left(|x|^2 - R^2\right)}{2N} + 1\ ,\ 1 \le j \le m$$

and

$$\min \bar{u}_j^* = \frac{-f_j(1) R^2}{2N} + 1\ ,\ 1 \le j \le m.$$

Clearly, $(\bar{u}_1^*, \ldots, \bar{u}_m^*)$ is a super solution of (1). By Lemma 3.2, the solution $(u_1, \ldots, u_m)$ of (1) is global only if $\bar{u}_1^*, \ldots, \bar{u}_m^* > 0$.

## 5 Concluding Remarks and Perspectives

This contribution advances mathematical research on quenching phenomena. The results of this study can be used to study other singular reaction-diffusion phenomena. We managed to overcome some difficulties and achieved very important results. This leads us to think more about the problem and do further theoretical and numerical research under other conditions. These efforts will advance quenching technology and modeling in many scientific fields.

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