Nonlinear Dynamics and Systems Theory, **23** (5) (2023) 507–518

Maximum and Anti-Maximum Principles for Boundary Value Problems for Ordinary Differential Equations in Neighborhoods of Simple Eigenvalues

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Received: October 7, 2022;   Revised: October 20, 2023

Abstract: It has been shown that, under suitable hypotheses, for boundary value problems of the form $Ly + \lambda y = f$, $BCy = 0$, where L is a linear differentiable operator and BC denotes the linear boundary operator, there exists $\Lambda > 0$ such that $f \geq 0$ implies $\lambda y \geq 0$ for $\lambda \in [-\Lambda, \Lambda] \setminus \{0\}$, where y is the unique solution of $Ly+\lambda y = f$, $BCy = 0$. So, the boundary value problem satisfies a maximum principle for $\lambda \in [-\Lambda, 0)$ and the boundary value problem satisfies an anti-maximum principle if $\lambda \in (0, \Lambda]$. Moreover, this information is provided in the one inequality, $\lambda y \geq 0$. In this study, we shall provide suitable hypotheses such that for boundary value problems of the form $Ly+\beta y' = f$, $BCy = 0$, where L is an ordinary differentiable operator and BC denotes the boundary operator, there exists $\mathcal{B} > 0$ such that $f \geq 0$ implies $\beta y' \geq 0$ for $\beta \in [-\mathcal{B}, \mathcal{B}] \setminus \{0\}$, where y is the unique solution of $Ly + \beta y' = f$, $BCy = 0$. Under suitable boundary conditions, one obtains sign properties on solutions and derivatives of solutions. Two examples satisfying the suitable hypotheses are provided and one application of monotone methods is provided to illustrate an application of the main result.

Keywords: *maximum principle; anti-maximum principle; ordinary differential equation; boundary value problem.*

Mathematics Subject Classification (2010): 34B08; 34B18; 34B27; 34L15.

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1 Introduction

The maximum principle is an important tool in the study of differential equations and we refer the reader to the well-known book [14] for many applications. For example, for the specific boundary value problem for a second order ordinary differential equation, $y'' + \lambda y = f$, $y'(0) = 0$, $y'(1) = 0$, if $\lambda < 0$, then this boundary value problem satisfies a maximum principle. In particular, for $f \in C[0,1]$, the boundary value problem is uniquely solvable and f nonnegative implies y is nonpositive, where y is the unique solution associated with f. In the study of boundary value problems for ordinary differential equations, the maximum principle implies that the associated Green's function is of constant sign, and in this case, the Green's function is nonpositive on $(0,1) \times (0,1)$.

Clément and Peletier [8] were the first to discover an anti-maximum principle. They were primarily interested in partial differential equations, but they illustrated the anti-maximum principle with the boundary value problem, $y'' + \lambda y = f$, $y'(0) = 0$, $y'(1) = 0$, $0 < \lambda < \frac{\pi^2}{4}$. For this particular boundary value problem, if $0 < \lambda < \frac{\pi^2}{4}$, if $f \in C[0,1]$, the boundary value problem is uniquely solvable and f nonnegative implies y is nonnegative, where y is the unique solution associated with f.

Since the publication of [8], there have been many studies of boundary value problems with parameter and the change of behavior from maximum to anti-maximum principles as a function of the parameter. In the case of partial differential equations, we refer to [1, 2, 7, 9, 10, 12, 13, 15]. In the case of ordinary differential equations, we refer to [3–6, 16]. In this paper, we shall continue to study the change in behavior of boundary value problems for ordinary differential equations, with respect to maximum and anti-maximum principles, through simple eigenvalues.

In an interesting study produced in [7], those authors began with a differential equation

$$y''(t) + \lambda y(t) = f(t), \quad t \in [0,1], \tag{1}$$

and considered either periodic boundary conditions or Neumann boundary conditions. Key to their argument is that for $f = 0$, at $\lambda = 0$, the boundary value problem, (1) with periodic or Neumann boundary conditions, is at resonance since constant functions are nontrivial solutions. Further, $\lambda = 0$ is a simple eigenvalue and the eigenspace is $< 1 >$, where $< 1 >$ denotes the linear span of the 1 function. Employing the resolvent, the inverse of $(D^2 + \lambda\mathcal{I})$ for $\lambda \neq 0$, under the imposed boundary conditions, if it exists, and the partial resolvent for $\lambda = 0$, and under the assumption that $f \geq 0$ (with $f \in \mathcal{L}[0,1]$), the authors in [7] obtained sufficient conditions to construct an interval $[-\Lambda, \Lambda]$, $\Lambda > 0$, a constant $K > 0$, independent of f such that

$$\lambda y(t) \geq K|f|_1, \quad \lambda \in [-\Lambda, \Lambda] \setminus \{0\}, \quad 0 \leq t \leq 1,$$

where $|f|_1 = \int_0^1 |f(s)|ds$. With this one inequality, the authors showed that for $\Lambda \leq \lambda < 0$, the boundary value problem, (1) with periodic or Neumann boundary conditions, satisfies a maximum principle and for $0 < \lambda \leq \Lambda$, the boundary value problem (1) with periodic or Neumann boundary conditions, satisfies an anti-maximum principle. They proceeded to produce many nice examples in that paper.

Consider the boundary value problem

$$y''(t) + \beta y'(t) = f(t), \quad t \in [0,1], \tag{2}$$

$$y(0) = 0, \quad y'(0) = y'(1). \tag{3}$$

For $f = 0$, $\beta = 0$ is a simple eigenvalue and generates the eigenspace $< t >$, the linear span of t. For $\beta \neq 0$,

$$G(\beta; t, s) = \begin{cases} \frac{e^{-\beta(1-s)} - e^{-\beta} e^{-\beta(t-s)}}{\beta(1-e^{-\beta})}, & 0 \leq t \leq s \leq 1, \\ \frac{e^{-\beta(1-s)} - e^{-\beta} e^{-\beta(t-s)}}{\beta(1-e^{-\beta})} + \frac{1-e^{-\beta(t-s)}}{\beta}, & 0 \leq s \leq t \leq 1, \end{cases} \tag{4}$$

is the Green's function for the boundary value problem (2), (3). Note that

$$\beta G(\beta; t, s) > 0, \quad (t, s) \in (0, 1] \times [0, 1], \text{ and } \beta \frac{\partial}{\partial t} G(\beta; t, s) > 0, \quad [t, s) \in [0, 1] \times [0, 1].$$

So, if y denotes the solution of (2), (3), then $f \geq 0$ implies $\beta y' \geq 0$ and $\beta y \geq 0$. This observation indicates that the principle obtained in [7] can be extended to other order derivatives.

Our goal in this paper is to study boundary value problems for ordinary differential equations containing a parameter β such that $\beta = 0$ is a simple eigenvalue generating an eigenspace $< t - t_0 >$ for some constant t_0 and modify the methods produced in [7]; in particular, we shall assume $f \geq 0$ and obtain sufficient conditions to construct an interval $[-\mathcal{B}, \mathcal{B}]$, $\mathcal{B} > 0$, a constant $K > 0$, independent of f, and an inequality

$$\beta y'(t) \geq K|f|_1, \quad \beta \in [-\mathcal{B}, \mathcal{B}] \setminus \{0\}, \quad 0 \leq t \leq 1, \tag{5}$$

where y is a unique solution of the boundary value problem associated with f. It will follow that if $0 < |\beta| \leq \mathcal{B}$, and if $f \geq 0$, then $\beta y' \geq 0$.

In Section 2, following the lead of [7], we shall define the concept of a strong signed maximum principle in y'. In Section 3, we shall obtain sufficient conditions for (5) and hence obtain sufficient conditions for adherence to a strong signed maximum principle in y'. In Section 4, we shall illustrate the main result, Theorem 3.1, with two examples. In each example, the boundary conditions are such that (5) generates a natural partial order in $C^1[0, 1]$.

We close in Section 5 with an application of a monotone method applied to a nonlinear problem related to one of the examples produced in Section 4. At $\beta = 0$, the problem is at resonance. The problem is shifted [11] by $\beta y'$ and $\beta > 0$ or $\beta < 0$ is chosen as a function of the monotonicity properties of the nonlinearity.

2 Strong Signed Maximum Principle

Assume $\mathcal{A}$ is a linear operator with Dom $(\mathcal{A}) \subset C^1[0, 1]$ and Im $(\mathcal{A}) \subset C[0, 1]$. Let $Dy = y'$ for $y \in C^1[0, 1]$. The following definition is motivated by Definition 1 found in [7].

Definition 2.1 For $\beta \in \mathbb{R} \setminus \{0\}$, the operator $\mathcal{A} + \beta D$ satisfies a **signed maximum principle in** Dy if for each $f \in C[0, 1]$, the equation

$$(\mathcal{A} + \beta D)y = f, \quad y \in \text{ Dom } (\mathcal{A}),$$

has a unique solution, y, and $f(t) \geq 0$, $0 \leq t \leq 1$ implies $\beta Dy(t) \geq 0$, $0 \leq t \leq 1$. The operator $\mathcal{A} + \beta D$ satisfies a **strong signed maximum principle in** Dy if $f(t) \geq 0$, $0 \leq t \leq 1$ and $f(t) > 0$ on some interval of positive length, implies $\beta Dy(t) > 0$, $0 < t < 1$.

Remark 2.1 Throughout this study, the phrases "maximum principle" or "anti-maximum principle" may be used loosely. If so, we mean the following. If $f \geq 0$ implies $y \leq 0$ (or $Dy \leq 0$), the phrase, maximum principle, may be used. This is precisely the case for the classical second order differential equation with Dirichlet boundary conditions. If $f \geq 0$ implies $y \geq 0$ (or $Dy \geq 0$), the phrase, anti-maximum principle, may be used. This is the case observed in [8] where the phrase, anti-maximum principle, was coined.

Remark 2.2 As pointed out in the Introduction, for the boundary value problem $y''(t) + \beta y'(t) = f(t)$, with boundary conditions (3), $f(t) \geq 0$, $0 \leq t \leq 1$ implies $\beta Dy(t) \geq 0$, $0 \leq t \leq 1$, and $\beta y(t) \geq 0$, $0 \leq t \leq 1$. In the application of the main theorem, Theorem 3.1, one only concludes (5). In the examples produced in Section 4, the boundary conditions are such that (5) implies further that for some $t_0 \in [0,1]$, $\beta(t-t_0)y(t) \geq 0$, $0 \leq t \leq 1$. In particular, in each example, a signed maximum principle in Dy will generate a natural partial order on $C^1[a,b]$ in which monotone methods can be applied.

3 The Main Theorem

Let $C[0,1]$ denote the Banach space of continuous real-valued functions defined on $[0,1]$ with norm $|y|_0 = \max_{0\leq t\leq 1} |y(t)|$ and let $C^1[0,1]$ denote the Banach space of continuously differentiable real-valued functions defined on $[0,1]$ with

$$||y|| = \max\{|y|_0, |y'|_0\}.$$

Also, $C[0,1] \subset \mathcal{L} = L^1[0,1]$, and so, we shall also have use for $|f|_1 = \int_0^1 |f(s)|ds$. For $f \in \mathcal{L}$, set

$$\bar{f} = \int_0^1 f(t)dt,$$

and define

$$\tilde{\mathcal{C}} \subset C[0,1] = \{f \in C[0,1] : \bar{f} = 0\}, \quad \tilde{\mathcal{L}} \subset \mathcal{L} = \{f \in L^1[0,1] : \bar{f} = 0\}.$$

Let $t_0 \in \mathbb{R}$. Assume $\mathcal{A}:$ Dom $(\mathcal{A}) \to \mathcal{L}$ denotes a linear operator satisfying

$$\text{Dom }(\mathcal{A}) \subset C^1[a,b] \quad \text{Ker }(\mathcal{A}) =< t-t_0 >, \quad \text{Im }(\mathcal{A}) = \tilde{\mathcal{L}}, \tag{6}$$

where $< t-t_0 >$ denotes the linear span of $t-t_0$. Assume further that for $\tilde{f} \in \tilde{\mathcal{L}}$, the problem $\mathcal{A}y = \tilde{f}$ is uniquely solvable with solution $y \in \text{Dom}(\mathcal{A})$ and such that $\bar{(y')} = 0$. In particular, define

$$\text{Dom }(\tilde{\mathcal{A}}) = \{y \in \text{ Dom }(\mathcal{A}) : \bar{(y')} = 0\},$$

and then

$$\mathcal{A}|_{\text{ Dom }(\tilde{\mathcal{A}})} : \text{ Dom }(\tilde{\mathcal{A}}) \to \tilde{\mathcal{L}}$$

is one to one and onto. Moreover, if $\mathcal{A}\tilde{y} = \tilde{f}$ for $\tilde{f} \in \tilde{\mathcal{L}}$, $\tilde{y} \in \text{Dom}(\tilde{\mathcal{A}})$, assume there exists a constant $K_1 > 0$ depending only on $\mathcal{A}$ such that

$$|\tilde{y}'|_0 \leq K_1 |\tilde{f}|_1. \tag{7}$$

For $f \in \mathcal{L}$, define

$$\tilde{f} = f - \bar{f},$$

and for $y \in$ Dom $(\mathcal{A})$, define

$$\tilde{y} = y - \bar{y'}(t - t_0),$$

which implies

$$\tilde{y}' = y' - \bar{y'}.$$

Finally, assume there exists $\mathcal{A}' :$ Dom $(\mathcal{A}') \to \mathcal{L}$ such that $\mathcal{A} = \mathcal{A}'D$. In this context, we rewrite

$$\mathcal{A}y + \beta y' = f, \quad y \in \text{ Dom } (\mathcal{A}), \tag{8}$$

as

$$(\mathcal{A}' + \beta\mathcal{I})Dy = f, \quad Dy \in \text{ Dom } (\mathcal{A}'). \tag{9}$$

Define Dom $(\tilde{\mathcal{A}}') = \{v \in$ Dom $(\mathcal{A}') : \bar{v} = 0\} \subset C[0, 1]$ and it follows that

$$\mathcal{A}'|_{\text{ Dom } (\tilde{\mathcal{A}}')} : \text{ Dom } (\tilde{\mathcal{A}}') \to \tilde{\mathcal{L}}$$

is one to one and onto.

With the decompositions $\tilde{f} = f - \bar{f}$ and $\tilde{y} = y - \bar{y'}(t - t_0)$, it follows that $\tilde{f} \in \tilde{\mathcal{L}}$ and $\tilde{y} \in$ Dom $(\tilde{\mathcal{A}})$, or more appropriately, $D\tilde{y} \in$ Dom $(\tilde{\mathcal{A}}')$. So, equation (8) or equation (9) decouples as follows:

$$\mathcal{A}'D\tilde{y} + \beta D\tilde{y} = (\mathcal{A}' + \beta\mathcal{I})D\tilde{y} = \tilde{f}, \tag{10}$$

$$\beta D\bar{y'}(t - t_0) = \beta\bar{y'} = \bar{f}. \tag{11}$$

Denote the inverse of $(\mathcal{A}' + \beta\mathcal{I})$, if it exists, by $\mathcal{R}_\beta$ and denote the inverse of $\mathcal{A}'|_{\text{ Dom } (\tilde{\mathcal{A}}')}$ by $\mathcal{R}_0$. So, $\mathcal{R}_0 : \tilde{\mathcal{L}} \to C[0, 1]$ and

$$D\tilde{y} = \mathcal{R}_0\tilde{f} \text{ if, and only if, } \mathcal{A}'(D\tilde{y}) = \tilde{f}. \tag{12}$$

Note that (12) implies

$$D\tilde{y} = \mathcal{R}_0\mathcal{A}'D\tilde{y} \tag{13}$$

since $D\tilde{y} \in \text{Dom}(\tilde{\mathcal{A}}')$,

Since $\tilde{\mathcal{C}} \subset \tilde{\mathcal{L}}$, we can also consider $\mathcal{R}_0 : \tilde{\mathcal{C}} \to C[0, 1]$. Let

$$||\mathcal{R}_0||_{\tilde{\mathcal{C}}\to\tilde{\mathcal{C}}} = \sup_{|v|_0=1} |\mathcal{R}_0 v|_0, \quad v, \mathcal{R}_0 v \in \tilde{\mathcal{C}},$$

and

$$||\mathcal{R}_0||_{\tilde{\mathcal{L}}\to\tilde{\mathcal{C}}} = \sup_{|v|_1=1} |\mathcal{R}_0 v|_0, \quad v \in \tilde{\mathcal{L}}, \quad \tilde{\mathcal{R}}_0 v \in \mathcal{C}.$$

Since $D\tilde{y} \in \tilde{\mathcal{C}}$, it follows that $|\mathcal{R}_0 D\tilde{y}|_0 \leq ||\mathcal{R}_0||_{\tilde{\mathcal{C}}\to\tilde{\mathcal{C}}}|D\tilde{y}|_0$. Similarly, $\tilde{f} \in \tilde{\mathcal{L}}$ implies $|\mathcal{R}_0\tilde{f}|_0 \leq ||\mathcal{R}_0||_{\tilde{\mathcal{L}}\to\tilde{\mathcal{C}}}|\tilde{f}|_1$.

Theorem 3.1 *Assume* $\mathcal{A} :$ *Dom* $(\mathcal{A}) \to C[0, 1]$ *denotes a linear operator satisfying* (6) *and* (7)*, and assume that for* $\tilde{f} \in \tilde{\mathcal{L}}$*, the problem* $\mathcal{A}y = \tilde{f}$ *is uniquely solvable with solution* $y \in$ *Dom*$(\mathcal{A})$ *such that* $(\bar{y'}) = 0$*. Further, assume there exists* $\mathcal{A}' :$ *Dom* $(\mathcal{A}') \to C[0, 1]$ *such that* $\mathcal{A} = \mathcal{A}'D$*. Assume* $\tilde{\mathcal{A}}'|_{\text{Dom } (\tilde{\mathcal{A}}')} :$ *Dom* $(\tilde{\mathcal{A}}') \to \tilde{\mathcal{L}}$ *is one to one and* *onto. Then there exists* $B_1 > 0$ *such that if* $0 < |\beta| \leq B_1$*, then* $\mathcal{R}_\beta$*, the inverse of* $(\mathcal{A}' + \beta\mathcal{I})$*, exists. Moreover, if* $\tilde{f} \in \tilde{L}$*,* $B_1||\mathcal{R}_0||_{\tilde{\mathcal{C}}\to\tilde{\mathcal{C}}} < 1$ *and* $0 < |\beta| \leq B_1$*, then*

$$|\mathcal{R}_\beta \tilde{f}|_0 \leq \frac{||\mathcal{R}_0||_{\tilde{\mathcal{L}}\to\tilde{\mathcal{C}}}}{1 - B_1||\mathcal{R}_0||_{\tilde{\mathcal{C}}\to\tilde{\mathcal{C}}}}|\tilde{f}|_1. \tag{14}$$

Further, there exists $\mathcal{B} \in (0, B_1)$ *such that if* $0 < |\beta| \leq \mathcal{B}$*, then the operator* $(\mathcal{A} + \beta D)$ *satisfies a strong signed maximum principle in* Dy*.*

Proof. Employ (13) and apply $\mathcal{R}_0$ to (10) to obtain

$$D\tilde{y} + \beta\mathcal{R}_0 D\tilde{y} = \mathcal{R}_0\tilde{f}.$$

Note that (7) implies that $\mathcal{R}_0 : \tilde{\mathcal{L}} \to \tilde{\mathcal{C}}$ is continuous. Assume $|\beta|||\mathcal{R}_0||_{\tilde{\mathcal{C}}\to\tilde{\mathcal{C}}} < 1$. Then $(\mathcal{I} + \beta\mathcal{R}_0) : \tilde{C} \to \tilde{C}$ is invertible and

$$D\tilde{y} = (\mathcal{I} + \beta\mathcal{R}_0)^{-1}\mathcal{R}_0\tilde{f}.$$

So, assume $0 < B_1 < \frac{1}{||\mathcal{R}_0||_{\tilde{\mathcal{C}}\to\tilde{\mathcal{C}}}}$ and assume $|\beta| \leq B_1$. Then $\mathcal{R}_\beta = (\mathcal{I} + \beta\mathcal{R}_0)^{-1}\mathcal{R}_0$ exists. Moreover,

$$\begin{aligned} |D\tilde{y}|_0 - B_1||\mathcal{R}_0||_{\tilde{\mathcal{C}}\to\tilde{\mathcal{C}}}|D\tilde{y}|_0 &\leq |D\tilde{y}|_0 - |\beta|||\mathcal{R}_0||_{\tilde{\mathcal{C}}\to\tilde{\mathcal{C}}}|D\tilde{y}|_0 \\ &\leq |(\mathcal{I} + \beta\mathcal{R}_0)D\tilde{y}|_0 = |\mathcal{R}_0\tilde{f}|_0 \leq ||\mathcal{R}_0||_{\tilde{\mathcal{L}}\to\tilde{\mathcal{C}}}|\tilde{f}|_1 \end{aligned}$$

and (14) is proved since $D\tilde{y} = \mathcal{R}_\beta\tilde{f}$.

Now assume $f \in \mathcal{L}$ and assume $f \geq 0$ a.e. Then $\bar{f} = |f|_1$. Let $0 < |\beta| \leq B_1 < \frac{1}{||\mathcal{R}_0||_{\tilde{\mathcal{C}}\to\tilde{\mathcal{C}}}}$, write $f = \bar{f} + \tilde{f}$ and consider

$$\beta Dy = \beta\mathcal{R}_\beta f = \beta\mathcal{R}_\beta(\bar{f} + \tilde{f}).$$

Note that $\beta\mathcal{R}_\beta\bar{f} = \bar{f}$ since $(\mathcal{A}' + \beta\mathcal{I})\bar{f} = \beta\bar{f}$. So,

$$\begin{aligned} \beta Dy = \beta\mathcal{R}_\beta f &= \beta\mathcal{R}_\beta(\bar{f} + \tilde{f}) \\ &= \bar{f} + \beta\mathcal{R}_\beta\tilde{f} \geq |f|_1 - |\beta||\mathcal{R}_\beta\tilde{f}|_0. \end{aligned}$$

Continue to assume that $0 < |\beta| \leq B_1$; it now follows from (14) that

$$\beta Dy \geq |f|_1 - |\beta|\Big(\frac{||\mathcal{R}_0||_{\tilde{\mathcal{L}}\to\tilde{\mathcal{C}}}}{1 - B_1||\mathcal{R}_0||_{\tilde{\mathcal{C}}\to\tilde{\mathcal{C}}}}\Big)|\tilde{f}|_1.$$

Since $\tilde{f} = f - \bar{f}$, and $|\tilde{f}|_1 \leq |f|_1 + \bar{f} = 2|f|_1$, assume

$$\mathcal{B} < \min\Big\{B_1, \Big(\frac{1 - B_1||\mathcal{R}_0||_{\tilde{\mathcal{C}}\to\tilde{\mathcal{C}}}}{2||\mathcal{R}_0||_{\tilde{\mathcal{L}}\to\tilde{\mathcal{C}}}}\Big)\Big\}.$$

Then

$$\beta Dy \geq \Big(1 - 2\mathcal{B}\Big(\frac{||\mathcal{R}_0||_{\tilde{\mathcal{L}}\to\tilde{\mathcal{C}}}}{1 - B_1||\mathcal{R}_0||_{\tilde{\mathcal{C}}\to\tilde{\mathcal{C}}}}\Big)\Big)|f|_1$$

and (5) is valid with

$$K = \Big(1 - 2\mathcal{B}\Big(\frac{||\mathcal{R}_0||_{\tilde{\mathcal{L}}\to\tilde{\mathcal{C}}}}{1 - B_1||\mathcal{R}_0||_{\tilde{\mathcal{C}}\to\tilde{\mathcal{C}}}}\Big)\Big).$$

□

4 Examples

Example 4.1

Our first example considers boundary conditions that contain (3). Let $t_0 \in [0,1]$ and consider the boundary value problem

$$y'' + \beta y' = f, \quad 0 \le t \le 1, \tag{15}$$

$$y(t_0) = 0, \quad y'(0) = y'(1). \tag{16}$$

So, for the boundary value problem (15), (16), $\mathcal{A} = D^2$, $\mathcal{A}' = D$, $\text{Ker}(\mathcal{A}) =< t - t_0 >$ or $\text{Ker}(\mathcal{A}') =< 1 >$.

We point out that if $t_0 = 0$ or $t_0 = 1$, the Fredholm alternative will imply that $\text{Im}(\mathcal{A}) = \tilde{\mathcal{L}}$. If $t_0 = 0$, then $f \in$ Im $(\mathcal{A})$, if, and only if, f is orthogonal to solutions of the adjoint problem

$$y'' = 0, \quad 0 \le t \le 1, \quad y(0) = y(1), \quad y'(1) = 0.$$

Thus, f is orthogonal to the constant functions. If $t_0 = 1$, then f is orthogonal to solutions of the adjoint problem

$$y'' = 0, \quad 0 \le t \le 1, \quad y(0) = y(1), \quad y'(0) = 0,$$

and again, f is orthogonal to the constant functions.

However, if $t_0 \in [0,1]$, one can show directly that $\text{Im}(\mathcal{A}) = \tilde{\mathcal{L}}$. If $f \in \text{Im}(\mathcal{A})$, then there exists a solution y of

$$y''(t) = f(t), \quad 0 \le t \le 1, \quad y(t_0) = 0, \quad y'(0) = y'(1),$$

which implies

$$0 = y'(1) - y'(0) = \int_0^1 y''(t)dt = \int_0^1 f(t)dt,$$

and $f \in \tilde{\mathcal{L}}$. Likewise, if $f \in \tilde{\mathcal{L}}$, then

$$y(t) = \int_0^t (t-s)f(s)ds - \int_0^{t_0} (t_0 - s)f(s)ds \tag{17}$$

is a solution of

$$y''(t) = f(t), \quad 0 \le t \le 1, \quad y(t_0) = 0, \quad y'(0) = y'(1),$$

which implies $f \in \text{Im}(\mathcal{A}')$. Thus, if $t_0 \in [0,1]$, Im $(\mathcal{A}) = \tilde{\mathcal{C}}$.

To argue that $\mathcal{A}y = \tilde{f}$ is uniquely solvable with solution $y \in$ Dom $(\tilde{\mathcal{A}})$, (17) implies the solvability. For uniqueness, if y_1 and y_2 are two such solutions, then $(y_1 - y_2)(t) = c(t - t_0)$ and $y_1 - y_2 \in$ Dom $(\tilde{\mathcal{A}})$ implies $c = 0$.

Finally, (17) implies (7) is satisfied with $K_1 = 1$.

Theorem 3.1 applies and there exists $\mathcal{B} > 0$ such that if $0 < |\beta| \le \mathcal{B}$, then $(\mathcal{A} + \beta\mathcal{I})$ has the strong signed maximum principle in Dy. Thus, $f \ge 0$ implies $\beta Dy \ge 0$. Hence, a natural partial order in which to apply the method of upper and lower solutions and monotone methods to a nonlinear boundary value problem is

$$y \in C^1[0,1] \succeq 0 \Longleftrightarrow \beta(t - t_0)y(t) \ge 0, 0 \le t \le 1, \text{ and } \beta y'(t) \ge 0, 0 \le t \le 1. \tag{18}$$

In Section 5, we shall employ monotone methods with respect to this partial order and obtain sufficient conditions for the existence of maximal and minimal solutions of a nonlinear boundary value problem associated with the boundary conditions (16).

Example 4.2

For the second example, let $h > 0$, and we consider a family of boundary conditions

$$y(0) = hy(1), \quad y'(0) = y'(1). \tag{19}$$

The boundary conditions (19) contain the periodic boundary conditions at $h = 1$. In this example, however, we exclude $h = 1$.

For the boundary value problem (15), (19), $\mathcal{A} = D^2$ and $\mathcal{A}' = D$, $\text{Ker}(\mathcal{A}) =< t + \frac{h}{1-h} >$ or $\text{Ker}(\mathcal{A}') =< 1 >$. Appealing directly to the Fredholm alternative, $f \in \text{Im}(\mathcal{A})$ if, and only if, f is orthogonal to solutions of the adjoint problem,

$$y'' = 0, \quad 0 \le t \le 1, \quad y(0) = y(1), \quad hy'(0) = y'(1).$$

Thus, Im $(\mathcal{A}) = \tilde{\mathcal{L}}$. Again, $f \in \tilde{\mathcal{L}}$ implies Dom $(\mathcal{A}) = \{y \in \mathcal{B} : \bar{D}y = 0\}$. Again, K in (7) can be computed since if $\tilde{f} \in \tilde{\mathcal{C}}$, then

$$\tilde{y}(t) = \int_0^t (t-s)\tilde{f}(s)ds + \frac{h}{1-h}\int_0^1 (1-s)\tilde{f}(s)ds.$$

Thus, Theorem 3.1 applies and there exists $\mathcal{B} > 0$ such that if $0 < |\beta| \le \mathcal{B}$, then $(\mathcal{A}' + \beta D)$ satisfies the strong signed maximum principle in Dy.

To determine sign conditions on βy, four cases arise. If $0 < \beta \le \mathcal{B}$, one considers the two cases, $1 < h$ or $0 < h < 1$. If $0 < \beta \le \mathcal{B}$, then βy is increasing, which in turn implies y is increasing. If $h > 1$, then $\frac{y(0)}{y(1)} > 1$, and it follows that $y(t) < 0$ for $0 \le t < 1$. If $0 < h < 1$, then $0 < \frac{y(0)}{y(1)} < 1$, and it follows that $y(t) > 0$ for $0 < t \le 1$. Two analogous cases can be analyzed if $0 > \beta \ge -\mathcal{B}$. So, for example, if $\beta > 0$ and $1 < h$, a natural partial order in which to apply the method of upper and lower solutions and monotone methods to a nonlinear problem is

$$y \in C^1[0,1] \succeq 0 \Longleftrightarrow y(t) \ge 0, 0 \le t \le 1, \text{ and } y'(t) \ge 0, 0 \le t \le 1.$$

5 A Monotone Method

Let $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ be continuous. Let $t_0 \in [0,1]$ and consider the boundary value problem

$$y''(t) = f(t, y(t), y'(t)), \quad \le t \le 1, \tag{20}$$

$$y(t_0) = 0, \quad y'(0) = y'(1). \tag{21}$$

Assume that f satisfies the following monotonicity properties:

$$\begin{aligned} &f(t, y, z_1) < f(t, y, z_2) \text{ for } (t, y) \in [0,1] \times \mathbb{R}, \quad z_1 > z_2, \\ &f(t, y_1, z) < f(t, y_2, z) \text{ for } (t, z) \in [t_0, 1] \times \mathbb{R}, \quad y_1 > y_2, \\ &f(t, y_1, z) > f(t, y_2, z) \text{ for } (t, z) \in [0, t_0] \times \mathbb{R}, \quad y_1 < z_2. \end{aligned} \tag{22}$$

So, f is monotone decreasing in the third component; for $t_0 < t \le 1$, f is monotone decreasing in the second component and for $0 \le t < t_0$, f is monotone increasing in the second component.

Apply a shift to (20) and consider the equivalent boundary value problem

$$y''(t) + \beta y'(t) = f(t, y(t), y'(t)) + \beta y'(t), \quad 0 \le t \le 1,$$

with boundary conditions (21), where $\beta < 0$. Assume $|\beta|$ is small such that $|\beta| \le \mathcal{B}$, where $\mathcal{B} > 0$ is shown to exist in Theorem 3.1. Note that if $g(t, y, z) = f(t, y, z) + \beta z$ and f satisfies (22), then g satisfies (22).

Assume the existence of solutions, w_1 and v_1, of the following boundary value problems for differential inequalities

$$\begin{gathered} w_1''(t) \ge f(t, w_1(t), w_1'(t)), \quad 0 \le t \le 1, \quad v_1''(t) \le f(t, v_1(t), v_1'(t)), \quad 0 \le t \le 1, \\ w_1(t_0) = 0, \qquad w_1'(0) = w_1'(1), \qquad v_1(t_0) = 0, \qquad v_1'(0) = v_1'(1). \end{gathered} \tag{23}$$

Assume further that

$$(t - t_0)(v_1(t) - w_1(t)) \ge 0, \quad 0 \le t \le 1, \quad (v_1'(t) - w_1'(t)) \ge 0, \quad 0 \le t \le 1. \tag{24}$$

Motivated by (18) and noting that $\beta < 0$, define a partial order $\succeq$ on $C^1[0, 1]$ by

$$u \in C^1[0, 1] \succeq 0 \Longleftrightarrow (t - t_0)u(t) \le 0, 0 \le t \le 1, \text{ and } u'(t) \le 0, 0 \le t \le 1.$$

Then the assumption (24) implies $w_1 \succeq v_1$.

Define iteratively, the sequences $\{v_k\}_{k=1}^{\infty}$, $\{w_k\}_{k=1}^{\infty}$, where

$$\begin{gathered} v_{k+1}''(t) + \beta v_{k+1}'(t) = f(t, v_k(t), v_k'(t)) + \beta v_k'(t), 0 \le t \le 1, \\ v_{k+1}(t_0) = 0, \quad v_{k+1}'(0) = v_{k+1}'(1), \end{gathered} \tag{25}$$

and

$$\begin{gathered} w_{k+1}''(t) + \beta w_{k+1}'(t) = f(t, w_k(t), w_k'(t)) + \beta w_k'(t), 0 \le t \le 1, \\ w_{k+1}(t_0) = 0, \quad w_{k+1}'(0) = w_{k+1}'(1). \end{gathered} \tag{26}$$

Theorem 3.1 implies the existence of each v_{k+1}, w_{k+1} since if $0 < |\beta| \le \mathcal{B}$, the inverse of $(\mathcal{A} + \beta D)$ exists.

Theorem 5.1 *Assume $f : [0, 1] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous and assume f satisfies the monotonicity properties* (22)*. Assume the existence of two times continuously differentiable functions, v_1 and w_1, satisfying* (23)*. Define the sequences of iterates $\{v_k\}_{k=1}^{\infty}$, $\{w_k\}_{k=1}^{\infty}$ by* (25) *and* (26)*, respectively. Then, for each $k \in \mathbb{N}_1$,*

$$w_k \succeq w_{k+1} \succeq v_{k+1} \succeq v_k. \tag{27}$$

Moreover, $\{v_k\}_{k=1}^{\infty}$ converges in $C^1[0, 1]$ to a solution v of (20) *and $\{w_k\}_{k=1}^{\infty}$ converges in $C^1[0, 1]$ to a solution w of* (20) *satisfying*

$$w_k \succeq w_{k+1} \succeq w \succeq v \succeq v_{k+1} \succeq v_k. \tag{28}$$

Proof. Since v_1 satisfies a differential inequality given in (24),

$$v_2''(t) + \beta v_2'(t) = f(t, v_1(t), v_1'(t)) + \beta v_1'(t) \geq v_1''(t) + \beta v_1'(t), \quad 0 \leq t \leq 1.$$

Set $u = v_2 - v_1$ and u satisfies a boundary value problem for a differential inequality

$$u''(t) + \beta u'(t) \geq 0, \quad 0 \leq t \leq 1, \quad u(t_0) = 0, \quad u'(0) = u'(1).$$

The signed maximum principle applies and $u \succeq 0$; in particular, $v_2 \succeq v_1$. Similarly, $w_1 \succeq w_2$. Now, set $u = w_2 - v_2$ and

$$u''(t) + \beta u'(t) = (f(t, w_1(t), w_1'(t)) - f(t, v_1(t), v_1'(t))) + \beta(w_1'(t) - v_1'(t)), \quad 0 \leq t \leq 1,$$
$$u(t_0) = 0, \quad u'(0) = u'(1).$$

Since f satisfies (22) and $\beta(w_1'(t) - v_1'(t)) \geq 0$, $0 \leq t \leq 1$, it follows that

$$u''(t) + \beta u'(t) \geq 0, \quad 0 \leq t \leq 1,$$

and again, the signed maximum principle applies and $u \succeq 0$. In particular, $w_2 \succeq v_2$. Thus, (27) is proved for $k = 1$. It follows by a straightforward induction that (27) is valid using the arguments presented in this paragraph.

To obtain the existence of limiting solutions v and w satisfying (28), note that the sequence $\{v_k'\}$ is monotone and appropriately bounded. Thus, the sequence $\{v_k'\}$ is converging pointwise on $[0, 1]$. Dini's theorem then implies the uniform convergence of the sequence $\{v_k\}$ on $[0, 1]$ since $\{v_k(t)\}$ is monotone for each t and is appropriately bounded. This argument can be repeated to obtain the uniform convergence of $\{v_k'\}$ on $[0, 1]$. Since $v_{k+1}''(t) = f(t, v_k(t), v_k'(t)) + \beta(v_k'(t) - v_{k+1}'(t))$, the sequence $\{v_k''\}$ is converging pointwise on $[0, 1]$. Now, Dini's theorem implies the uniform convergence of the sequence $\{v_k'\}$ on $[0, 1]$. Again, employ $v_{k+1}''(t) = f(t, v_k(t), v_k'(t)) + \beta(v_k'(t) - v_{k+1}'(t))$, and it follows that the sequence $\{v_k''\}$ converges uniformly on $[0, 1]$. This implies that if $v \in C^1[0, 1]$ is the limit of $\{v_k\}$ (meaning v_k is converging to v uniformly and v_k' is converging to v' uniformly), then $\{v_k''\}$ converges uniformly to v'' on $[0, 1]$ and v is a solution of (20), (21) satisfying (28). Similarly, the solution w of (20), (21) satisfying (28) exists, and the theorem is proved. $\square$

Suppose now f satisfies the "anti"-inequalities to (22); that is, suppose f satisfies

$$f(t, y, z_1) > f(t, y, z_2) \text{ for } (t, y) \in [0, 1] \times \mathbb{R}, \quad z_1 > z_2, \tag{29}$$
$$f(t, y_1, z) > f(t, y_2, z) \text{ for } (t, z) \in [t_0, 1] \times \mathbb{R}, \quad y_1 > y_2,$$
$$f(t, y_1, z) < f(t, y_2, z) \text{ for } (t, z) \in [0, t_0] \times \mathbb{R}, \quad y_1 < z_2.$$

One can appeal to the signed maximum principle and apply a shift to (20) and consider the equivalent boundary value problem, $y''(t) + \beta y'(t) = f(t, y(t), y'(t)) + \beta y'(t), 0 \leq t \leq 1$, where $\beta > 0$. Note, if f satisfies (29) and $\beta > 0$, then $g(t, y, z) = f(t, y, z) + \beta z$ satisfies (29).

Now, assume the existence of solutions, w_1 and v_1, of the following differential inequalities

$$w_1''(t) \leq f(t, w_1(t), w_1'(t)), \quad 0 \leq t \leq 1, \quad v_1''(t) \geq f(t, v_1(t), v_1'(t)), \quad 0 \leq t \leq 1, \tag{30}$$
$$w_1(t_0) = 0, \qquad w_1'(0) = w_1'(1), \qquad v_1(t_0) = 0, \qquad v_1'(0) = v_1'(1).$$

Assume further that

$$(t - t_0)(v_1(t) - w_1(t)) \leq 0, \quad 0 \leq t \leq 1, \quad (v_1'(t) - w_1'(t)) \leq 0, \quad 0 \leq t \leq 1. \tag{31}$$

Noting that $\beta > 0$, define a partial order $\succeq_1$ on $C^1[0,1]$ by

$$u \in C^1[0,1] \succeq_1 0 \Longleftrightarrow (t - t_0)u(t) \geq 0, 0 \leq t \leq 1, \text{ and } u'(t) \geq 0, 0 \leq t \leq 1.$$

In particular, assume $v_1 \succeq_1 v_1$.

Theorem 5.2 *Assume $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous and assume f satisfies the monotonicity properties (29). Assume the existence of two times continuously differentiable functions, v_1 and w_1, satisfying (30) and (31). Define the sequences of iterates $\{v_k\}_{k=1}^{\infty}$, $\{w_k\}_{k=1}^{\infty}$ by (25) and (26), respectively. Then, for each $k \in \mathbb{N}_1$,*

$$v_k \succeq_1 v_{k+1} \succeq_1 w_{k+1} \succeq_1 w_k.$$

Moreover, $\{v_k\}_{k=1}^{\infty}$ converges in $C^1[0,1]$ to a solution v of (20) and $\{w_k\}_{k=1}^{\infty}$ converges in $C^1[0,1]$ to a solution w of (20) satisfying

$$v_k \succeq_1 v_{k+1} \succeq_1 v \succeq_1 w \succeq_1 w_{k+1} \succeq_1 w_k.$$

6 Conclusion

Boundary value problems for ordinary differential equations with dependence on a real parameter β, where $\beta = 0$ is a simple eigenvalue, are studied. The concept of a maximum principle in $\beta y'$ is defined. Sufficient conditions are obtained such that if $\mathcal{A}y + \beta y' = f$ is a representation of the boundary value problem, there exists a punctured neighborhood of $\beta = 0$ such that $f \geq 0$ implies $\beta y' \geq 0$, where y is the unique solution of $\mathcal{A}y + \beta y' = f$. Two examples are provided to illustrate the main theorem and an application of a monotone method is given.

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