Nonlinear Dynamics and Systems Theory, **23** (5) (2023) 538–544

Class of Nilpotent Distributions and $\mathfrak{N}_2$-Distributions

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Received: July 9, 2023; Revised: November 26, 2023

Abstract: This paper presents a sufficient condition for two vector fields X and Y to have the squares noncommutative, i.e., $[X^2, Y^2] \neq 0$, in the case when X and Y span a 3-nilpotent distribution. And when the nilpotent disributions of class 2 or 3 are spanned from more than two vector fields, it gives the same result.

Keywords: *vector distributions; sub-Riemannian geometry; noncommutative geometry; nilpotency class; nonlinear dynamics systems.*

Mathematics Subject Classification (2010): 53A31, 53C17, 70K75, 93A10.

1 Introduction

The theory of subelliptic operators plays an important role in many applications in nonlinear dynamics and system theory, robotics and mechanical systems, optimal control of nonlinear systems, see [1, 9].

The subelliptic operator is a particular of case hypoelliptic differential equations. Hypoelliptic equations involve operators that are neither purely elliptic (like Laplace's equation) nor hyperbolic (like the wave equation), but rather fall in between. These equations often arise in the context of modeling systems with varying degrees of regularity and smoothness.

An example is the study of heat conduction in materials with varying degrees of conductivity, with a heat diffusion being non-uniform in all directions. The equation $\frac{\partial}{\partial t} - L\,u = 0$ gives a more efficient description in that direction, where $u(x,t)$ is the heat kernel and the subelliptic operator L is defined in the differential manifold. And the

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heat kernel characterizes the evolution of heat distribution over time in the context of the operator's structure.

A heat kernel of the subelliptic operator $L = X_1^2 + \cdots + X_k^2$, where $X_1, \cdots, X_k$ are vector fields of $\mathbb{R}^n$, with $k \leq n$, is an important problem. A sufficient condition for the hypo-ellipticity of the operator L is the bracket condition, see [2, 3, 6, 8]. If the squares of the aforementioned vector fields commute, i.e., $[X_i^2, X_j^2] = 0$ for all X_i, X_j, then the heat kernel of L is the product of heat kernels

$$e^{tL} = e^{tX_1^2} \cdots e^{tX_k^2}.$$

If they do not commute, the previous formula does not hold any more, and the heat kernel should be found using a different method, see [4, 7, 12].

Ovidiu Calin and Der-Chen Chang in [5] give a sufficient condition for two vector fields X and Y to have the squares noncommutative, i.e., $[X^2, Y^2] \neq 0$, in the following result.

Theorem 1.1 *[5]. Any distribution $\mathcal{D} = span\{X, Y\}$ of nilpotency class 2 is a $\mathfrak{N}_2$-Distribution, i.e., $[X^2, Y^2] \neq 0$.*

In the present work, as the first result, in Theorem 3.1, we shall prove that in the case of nilpotent distribution with the nilpotency class equal to 3, the squares of the vector fields do not commute. In the second and third results of this work, Proposition 3.1 and Proposition 3.2, we shall generalize the results of Theorem 1.1 and Theorem 3.1 for more than two vector fields.

2 Preliminaries

In an n-dimensional smooth manifold M, we recall that a smooth distribution $\mathcal{D}$ of rank m is a rank m subbundle of the tangent bundle TM [11]. We call the $\mathfrak{N}_k$-Distribution a distribution spanned by two vector fields X and Y, which satisfies the condition $[X^k, Y^k] \neq 0$ [5].

Let $\Gamma(\mathcal{D})$ be a basis of the distribution $\mathcal{D}$. We recall that the iterated commutator sets C^k of vector fields obtained by k iterated Lie brackets of horizontal vector fields are

$$\begin{aligned} C^1 =& \{[X, Y];\ X, Y \in \Gamma(\mathcal{D})\}, \\ &\vdots \\ C^n =& \{[C, Z];\ C \in C^{n-1}, Z \in \Gamma(\mathcal{D})\}. \end{aligned}$$

A distribution $\mathcal{D}$ is called nilpotent if there is an integer $k \geq 1$ such that $C^{(k)} = 0$, i.e., all the k iterated Lie brackets vanish. The smallest integer k is called the nilpotency class of $\mathcal{D}$ which is called k-nilpotent, see page 47 in [6].

Example 2.1 (*example of 3-nilpotent distribution* [11]) The Martinet distribution in $\mathbb{R}^3$ (with coordinates (x, y, z)) is the distribution generated by X and Y with

$$X = \partial x, \quad Y = \partial y + \frac{1}{2}x^2 \partial z,$$

the iterated commutators are

$$[X, Y] = x\partial z,$$

$$[X,[X,Y]] = \partial z, \quad [Y,[X,Y]] = 0,$$

and

$$\begin{aligned}[X,[X,[X,Y]]] =& 0, \quad [X,[Y,[X,Y]]] = 0,\\ [Y,[X,[X,Y]]] =& 0, \quad [Y,[Y,[X,Y]]] = 0.\end{aligned}$$

It follows that this distribution is nilpotent of class 3.

3 Main Results

Theorem 3.1 *Any distribution $\mathcal{D} = span\{X,Y\}$ of nilpotency class equal to 3 is a $\mathfrak{N}_2$-distribution, i.e., $[X^2,Y^2] \neq 0$.*

To prove this theorem, we use several lemmas, and we recall the following.
The distribution $\mathcal{D} = span\{X,Y\}$ is nilpotent of class 3 meaning that

$$[X,Y] \neq 0, \tag{1}$$

$$[Y,[X,Y]] \neq 0 \quad \text{or} \quad [X,[X,Y]] \neq 0,$$

and

$$[X,[X,[X,Y]]] = 0, \quad [Y,[Y,[X,Y]]] = 0, \tag{2}$$

$$[X,[Y,[X,Y]]] = 0, \quad [Y,[X,[X,Y]]] = 0. \tag{3}$$

Lemma 3.1 *In a distribution $\mathcal{D} = span\{X,Y\}$ of nilpotency class 3, we have*

$$[X^2,Y^2] = 0 \Longrightarrow (XY)^2 = (YX)^2. \tag{4}$$

Proof. By developing the first equation of (3), we get

$$\begin{aligned}[X,[Y,[X,Y]]] = 0 &\Longleftrightarrow X^2Y^2 - Y^2X^2 - 2(XY)^2 + 2(YX)^2 = 0\\ &\Longrightarrow X^2Y^2 - Y^2X^2 = 2((XY)^2 - (YX)^2)\end{aligned}$$

or

$$[X^2,Y^2] = 0,$$

then $(XY)^2 = (YX)^2$.

Lemma 3.2 *In a distribution $\mathcal{D} = span\{X,Y\}$ of nilpotency class 3, we have*

$$[X^2,Y^2] = 0 \Longrightarrow XYX^2Y^2 = X^2Y^2XY. \tag{5}$$

Proof. The expansion of the equations (2) gives

$$X^3Y - 3X^2YX + 3XYX^2 - YX^3 = 0, \tag{6}$$

$$Y^3X - 3Y^2XY + 3YXY^2 - XY^3 = 0. \tag{7}$$

Multiplying the right-hand side, then the left-hand side of the relation (6) by Y^2 and the relation (7) by X^2, we obtain

$$X^3Y^3 - 3X^2YXY^2 + 3XYX^2Y^2 - YX^3Y^2 = 0, \tag{8}$$

$$Y^3X^3 - 3Y^2XYX^2 + 3YXY^2X^2 - XY^3X^2 = 0, \tag{9}$$

$$Y^3X^3 - Y^2X^3Y + 3Y^2X^2YX - 3Y^2XYX^2 = 0, \tag{10}$$

$$X^3Y^3 - X^2Y^3X + 3X^2Y^2XY - 3X^2YXY^2 = 0. \tag{11}$$

By the subtractions of these equations, (8)-(9), (8)-(10), (11)-(10), we have found, respectively,

$$X^3Y^3 - Y^3X^3 = 3X^2YXY^2 - 3XYX^2Y^2 + YX^3Y^2 - 3Y^2XYX^2 + 3YXY^2X^2 - XY^3X^2, \tag{12}$$

$$X^3Y^3 - Y^3X^3 = 3X^2YXY^2 - 3XYX^2Y^2 + YX^3Y^2 - Y^2X^3Y + 3Y^2X^2YX - 3Y^2XYX^2, \tag{13}$$

$$X^3Y^3 - Y^3X^3 = X^2Y^3X - 3X^2Y^2XY + 3X^2YXY^2 - Y^2X^3Y + 3Y^2X^2YX - 3Y^2XYX^2. \tag{14}$$

Subtracting the equations (12)-(13) gives

$$-3XYX^2Y^2 + YX^3Y^2 + Y^2X^3Y - 3Y^2X^2YX - XY^3X^2 + 3YXY^2X^2 - X^2Y^3X + 3X^2Y^2XY = 0.$$

In view of the fact that $X^2Y^2 = Y^2X^2$, the last equation becomes

$$-XYX^2Y^2 + YXX^2Y^2 + X^2Y^2XY - X^2Y^2YX = 0,$$

then

$$X^2Y^2[X,Y] = [X,Y]X^2Y^2. \tag{15}$$

On the other hand, subtracting the equations (14)-(12) gives

$$-3XYX^2Y^2 + YX^3Y^2 - X^2Y^3X + 3X^2Y^2XY = 0,$$

then

$$-[X,Y]X^2Y^2 + X^2Y^2[X,Y] + 2(X^2Y^2XY - XYX^2Y^2) = 0.$$

Using the relation (15), we obtain

$$XYX^2Y^2 = X^2Y^2XY.$$

Lemma 3.3 *In a distribution $\mathcal{D} = span\{X,Y\}$ with the nilpotency class 3, we have*

$$[X^2,Y^2] = 0 \Longrightarrow X^2Y^2 = 3(XY)^2. \tag{16}$$

Proof. Multiplying the equation (6) in the proof of Lemma 3.2 by Y on two sides, we obtain

$$YX^3Y^2 - 3YX^2YXY + 3YXYX^2Y - Y^2X^3Y = 0. \tag{17}$$

Lemma 3.2 proves that

$$XYX^2Y^2 = X^2Y^2XY,$$

and interchanging X and Y, we get

$$YXX^2Y^2 = X^2Y^2YX,$$

then (17) becames

$$X^2Y^2[X,Y] - 3((XY)^3 - (YX)^3) = 0,$$

this implies that

$$(X^2Y^2 - 3(XY)^2)[X,Y] = 0$$

but $[X,Y] \neq 0$, then

$$X^2Y^2 = 3(XY)^2.$$

Proof. *(Proof of Theorem 3.1)* We shall prove this theorem by contradiction, i.e., we assume that

$$[X^2, Y^2] = 0. \tag{18}$$

By developing $[X,Y]^3$ and using Lemma 3.1, we get

$$\begin{aligned}
[X,Y]^3 &= (XY)^3 - (XY)^2(YX) - (XY)(YX)(XY) + (XY)(YX)^2 \\
&\quad - (YX)(XY)^2 + (YX)(XY)(YX) + (YX)^2(XY) - (YX)^3 \\
&= 3(XY)^3 - 3(YX)^3 - (XY)(YX)(XY) + (YX)(XY)(YX).
\end{aligned} \tag{19}$$

Using Lemma 3.3, we get

$$\begin{aligned}
(XY)(YX)(XY) &= XY^2X^2Y \\
&= 3X(YX)^2Y \\
&= 3XYXYXY = 3(XY)^3, \\
(YX)(XY)(YX) &= YX^2Y^2Y \\
&= 3Y(XY)^2X \\
&= 3YXYXYX \\
&= 3(YX)^3.
\end{aligned}$$

The equation (19) becomes

$$[X,Y]^3 = 3(XY)^3 - 3(YX)^3 - 3(XY)^3 + 3(YX)^3 = 0,$$

then $[X,Y] = 0$ is a contradiction. It turns out that (18) cannot hold. It follows that the vector fields X and Y span a $\mathfrak{N}_2$-distribution.

Example 3.1 *[11]* It is clear that the distribution $\mathcal{D} = Span\{X,Y\}$ is nilpotent of class 3 so that

$$X = \partial x, \quad Y = \partial y + \frac{1}{2}x^2\partial z,$$

and

$$[X^2, Y^2] = 4x\partial x\partial y\partial z + 2\partial y\partial z + 8x^3\partial x\partial_z^2 + 12x^2\partial_z^2 \neq 0.$$

For the second part of this paper, we need a new definition of $\mathfrak{N}_k$-distribution in the case when the distribution is spanned by more than two vector fields.

Definition 3.1 Let $\mathcal{D} = span\{X_1, X_2, \cdots, X_m\}$ with $m \leq n$. We say that $\mathcal{D}$ is a $\mathfrak{N}_k$-distribution if there exist $X_i, X_j \in \mathcal{D}$ such that $[X_i^k, X_j^k] \neq 0$.

In this proposition, we generalize Theorem 1.1 for more than two vector fields.

Proposition 3.1 *Let $\mathcal{D} = span\{X_1, \cdots, X_m\}$ be a distribution spanned by m-vector fields of $\mathbb{R}^n$ ($m \leq n$). Then if $\mathcal{D}$ is a nilpotent distribution of nilpotency class 2, then $\mathcal{D}$ is a $\mathfrak{N}_2$-distribution, i.e., $\exists\ X_i, X_j \in \mathcal{D}$ such that $[X_i^2, X_j^2] \neq 0$.*

Proof. $\mathcal{D}$ is a nilpotent distribution of nilpotency class 2, then there exist $X_i, X_j \in \mathcal{D}$ such that

$$[X_i, X_j] \neq 0$$

and

$$[X_i, [X_i, X_j] = 0 \quad \text{and} \quad [X_j, [X_i, X_j]] = 0.$$

Let us tackle a sub-distribution $\mathcal{D}' = span\{X_i, X_j\}$, from Theorem 1.1, we obtain that in $\mathcal{D}'$,

$$[X_i^2, X_j^2] \neq 0,$$

or $\mathcal{D}' \subset \mathcal{D}$, then we have in $\mathcal{D}$,

$$[X_i^2, X_j^2] \neq 0.$$

Example 3.2 *[10]* The distribution $\mathcal{D} = span\{X, Y, Z\}$ such that

$$X = \partial x, \quad Y = \partial y, \quad Z = x\partial z$$

is nilpotent of class 2,

$$[X, Y] = 0, \quad [Y, Z] = 0, \quad [X, Z] = \partial z$$

and

$$\begin{aligned} &[Y, [X, Y]] = 0, \quad [X, [X, Y]] = 0, \\ &[X, [X, Z]] = 0, \quad [Y, [Y, Z]] = 0, \\ &[Z, [X, Y]] = 0. \end{aligned}$$

On the other hand, we have

$$[X^2, Z^2] = 4x\partial x\partial^2 z + 2\partial^2 z \neq 0.$$

In the next proposition, we generalize Theorem 3.1 for more than two vector fields.

Proposition 3.2 *Let $\mathcal{D} = span\{X_1, \cdots, X_m\}$ be a distribution spanned by m-vector fields of $\mathbb{R}^n$ ($m \leq n$). Then if $\mathcal{D}$ is a nilpotent distribution of nilpotency class 3, then $\mathcal{D}$ is a $\mathfrak{N}_2$-distribution, i.e., $\exists\ X_i, X_j \in \mathcal{D}$ such that $[X_i^2, X_j^2] \neq 0$.*

Proof. $\mathcal{D}$ is a nilpotent distribution of nilpotency class 3, then there exist $X_i, X_j \in \mathcal{D}$ such that

$$[X_i, X_j] \neq 0$$

and

$$\begin{aligned} &[X_i, [X_i, [X_i, X_j]]] = 0, \quad [X_j, [X_j, [X_i, X_j]]] = 0, \\ &[X_i, [X_j, [X_i, X_j]]] = 0, \quad [X_j, [X_i, [X_i, X_j]]] = 0. \end{aligned}$$

We remark that in the proof of Theorem 3.1, we do not use the iterated brackets of degree two ($\mathcal{C}^2$). Let us tackle a sub-distribution $\mathcal{D}' = span\{X_i, X_j\}$, from Theorem 1.1, we obtain that in $\mathcal{D}'$,

$$[X_i^2, X_j^2] \neq 0,$$

or $\mathcal{D}' \subset \mathcal{D}$, then we have in $\mathcal{D}$,

$$[X_i^2, X_j^2] \neq 0.$$

4 Conclusion

In summary, the heat kernel of a subelliptic operator captures the essential dynamic behavior of heat diffusion within a system defined by that operator. It describes how the initial heat distribution changes over time due to the conduction process governed by the operator's structure. The heat kernel provides insights into the temporal evolution of heat within the context of the subelliptic operator, connecting the mathematical description of heat conduction with the dynamic behavior of the system.

Unfortunately, the only method to find the solution of this kernel, is that the square of the vector fields commutes (sufficient condition).

Ovidiu Calin and Der-Chen Chang in [5] give a sufficient condition for two vector fields X and Y to have the squares noncommutative (distribution spanned with two vectors fields).

In this work, we have given an extension for more than two vector fields, and we have proposed a new sufficient condition for the noncommutative squares of the vectors fields in the case of distribution of nilpotency class 3. This result offers less computation and excellent description for the methods of finding the heat kernel of the sum of squares operator.

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