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Study of a Penalty Method for Nonlinear Optimization Based on a New Approximate Function

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Abstract: The aim of this paper is to present a logarithmic penalty method for solving nonlinear optimization. The line search is carried out by means of an approximate function if the descent direction is determined using a classical Newton technique. Contrary to the line search method, which is costly in terms of computing volume and demands a lot of time, the proposed approximate function enables easy and quick computation of the displacement step. Numerous intriguing numerical experiments, which are presented in the last section of this work, show that our new approximate function is accurate and efficient.

Keywords: *interior point methods; logarithmic penalty method; applications; approximate functions; nonlinear optimization; quadratic optimization.*

Mathematics Subject Classification (2010): 90C25, 90C30, 90C20, 93C95, 70K75.

1 Introduction

Nonlinear optimization problems deal with the problem of optimizing an objective function in the presence of equality and inequality constraints. Furthermore, if all the functions are linear, we obviously have a linear optimization problem. Otherwise, the problem is called a nonlinear optimization problem.

This research field is motivated by the fact that several problems are collected from practice such as engineering, medicine, business administration, economics, physical sciences, and nonlinear dynamics and systems (see, e.g., [8, 9]).

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Quadratic optimization is a type of nonlinear optimization, where the objective function is quadratic. In order to solve this type, we propose a penalty approach without line search based on approximate functions, which is the efficient method for determining the displacement step. This study is supported by an important numerical simulation.

For this purpose, we consider the solution of the following quadratic programming problem:

$$(P)\begin{cases}\min \quad q(x) & = \frac{1}{2}x^tQx + c^tx\\ x \in D,\end{cases}$$

with

$$D = \{x \in \mathbb{R}^n : Ax \geq b\}.$$

The following assumptions are made.

1. $c \in \mathbb{R}^n$, Q is an $\mathbb{R}^{n\times n}$ symmetric semidefinite matrix.

2. We know a point $x_0 \in \mathbb{R}^n$ such that $Ax_0 > b$.

3. $b \in \mathbb{R}^p$, A is a $(p \times n)$ full rank matrix.

4. The set of optimal solutions of (P) is nonempty and bounded.

In this paper, the problem (P) is approximated by the problem $(P_\eta), (\eta > 0)$,

$$(P_\eta)\begin{cases}min \ \ q_\eta(x)\\ x \in \mathbb{R}^n,\end{cases}$$

where the barrier function $q_\eta : \mathbb{R}^n \to (-\infty, +\infty]$ is defined by

$$q_\eta(x) = \begin{cases} q(x) - \eta\sum_{i=1}^m \ln < e_i, Ax - b > & \text{if } \ Ax - b > 0\\ +\infty & \text{otherwise,}\end{cases}$$

where $(e_1, e_2, ..., e_m)$ is the canonical base in $\mathbb{R}^m$ and η is a strictly positive barrier parameter. Recall that the scalar product of $x, y \in \mathbb{R}^n$ is given by

$$\langle x, y\rangle = x^ty = \sum_{i=1}^n x_iy_i,$$

the Euclidean norm of y is

$$\|y\| = \sqrt{\langle x, y\rangle} = \sqrt{\sum_{i=1}^n y_i^2}.$$

A classical Newton descent approach is used to solve this problem.

In our new approach, instead of minimizing q_η, along the descent direction at a current point x, we propose an approximate function G for which the optimal solution of the displacement step α is obtained explicitly.

Let us minimize the function G so that

$$G\,(\alpha) = \frac{1}{\eta}(q_\eta(x_\eta + \alpha d) - q_\eta(x_\eta)) \geq \breve{G}(\alpha),\ \forall\alpha > 0,$$

with $G(0) = \breve{G}(0) = 0,\ G'(0) = \breve{G}'(0) < 0$. The best quality of the approximations $\breve{G}$ of G is ensured by the condition $G''(0) = \breve{G}''(0)$. The idea of this new approach consists in introducing one original process to calculate the displacement step α based on minorant functions. Then we obtain an explicit approximation which leads to reducing the objective, adding to this, it is economical and robust, contrary to the traditional methods of line search.

The paper is organized as follows. In Section 1, we prove the perturbed problem convergence to the initial one. We are interested in resolving the perturbed problem. We describe our algorithm briefly and we present our main result by introducing a new approximate function to compute efficiently the displacement step of the obtained penalty algorithm. This approach is employed to evade line search methods and expedite the algorithm's convergence.

In Section 2, we present numerical tests on some different examples to illustrate the effectiveness of the proposed approach and we compare it with the standard line search method. A conclusion and future research are given in the last Section 3.

By assumption (1), its solutions set is nonempty and bounded, and as we know, (P) is convex, consequently, in accordance with Bachir Cherif et al. [5], the strictly convex problem (P_η) has unique optimal solution x^*_η for each $\eta > 0$.

Since solving the problem (P) is similar to solving the problem (P_η) when η tends to 0, our goal is to resolve the problem (P_η).

Firstly, we need to study the convergence of (P_η) to (P).

Convergence of the Perturbed Problem (P_η) to (P)

Let the function ψ be defined on $\mathbb{R} \times \mathbb{R}^n$ by

$$\psi(\eta, x) = \begin{cases} q(x) + \sum\limits_{i=1}^{n} \xi(\eta, x_i) & \text{if} \qquad x \geq 0,\ Ax \geq b, \\ +\infty & \text{if not,} \end{cases}$$

where $\xi : \mathbb{R}^2 \longrightarrow (-\infty, +\infty]$ is a convex, lower semicontinuous and proper function given by

$$\xi(\eta, t) = \begin{cases} \eta \ln(\eta) - \eta \ln(\alpha) & \text{if} \qquad \alpha > 0 \text{ and } \eta > 0, \\ 0 & \text{if} \qquad \alpha \geq 0 \text{ and } \eta = 0, \\ +\infty & \text{otherwise.} \end{cases}$$

So, the function ψ is a convex, lower semicontinuous and proper function.

From [5], the strictly convex problem (P_η) admits a unique optimal solution x^*_η for each η. The solution of the problem (P) reduces to the solution of the series of problems (P_η). The sequence of the solutions x_η of (P_η) should converge to the solution of (P) when η tends to 0.

Now we are in a position to state the convergence result of (P_η) to (P) which is proved in Lemma 1 from [4].

Let $\eta > 0$, for all $x \in D$, we define

$$\psi(x, \eta) = q_\eta(x).$$

Lemma 1.1 *[4] Let $\eta > 0$. If x_η is an optimal solution of the problem (P_η) such that $\lim_{\eta \to 0} x_\eta = x^*$, then x^* is an optimal solution of the problem (P).*

Let $\eta > 0$, for all $x \in D$, we define $\psi(x, \eta) = q_\eta(x)$.

Resolution of the Perturbed Problem

In this section, we are interested in finding the solution of the perturbed problem $x_{k+1} = x_k + \alpha_k d_k$. For this purpose, we first use the Newton method to calculate the descent direction d_k. Then we obtain the displacement step α_k by our new minorant function. Finally, we describe a standard prototype algorithm.

1.1 Newton descent direction

The interior point methods of the logarithmic barrier type are developed for resolving this type of problems based on the optimality conditions that are necessary and sufficient since the problem (P_η) can be considered as the one without constraints.

As a result, $((x_\eta)_k = x_k)$ is an optimal solution of (P_η) such that the following condition is met:

$$\nabla q_\eta(x_\eta) = 0. \tag{1}$$

Thus, $x_{k+1} = x_k + d_k$ is the iteration of Newton, where d_k is the descent direction solution of the linear system

$$\nabla^2 q_\eta(x_\eta) d_k = -\nabla q_\eta(x_\eta). \tag{2}$$

Note 1. We insert a displacement step α_k and we write

$$x_{k+1} = x_k + \alpha_k d_k$$

to ensure the strictly feasible iterate $x_{k+1} = x_k + d_k$.

1.2 Model algorithm

In this part, we present a brief algorithm of our approach to obtain an optimal solution $\bar{x}$ of the problem (P).

Begin algorithm

Initialization

Start with x_0 being a strictly feasible solution of (P), $\eta > 0$, ε is a given precision and $k = 0$.

While $\|\nabla q_\eta(x_k)\| > \varepsilon$ **do**

– Resolve the system : $\nabla^2 q_\eta(x_k) d_k = -\nabla q_\eta(x_k)$.
– Compute the displacement step α_k.
– Take $x_{k+1} = x_k + \alpha_k d_k$ and $k = k + 1$.
– Put $\eta = \sigma\eta,\ 0 < \sigma < 1$.

End While

We have obtained a good approximate solution of the problem (P).

End algorithm.

1.3 Computation of the displacement step

There are two main techniques used for computing the displacement step α_k.

(1) **Line search methods**: The method of Goldstein-Armijo, Fibonacci, Wolfe, etc. They are based on the unidimensional function's minimization:

$$\phi(\alpha) = \min_{\alpha>0} q_\eta(x_\eta + \alpha d).$$

They are time-consuming and unfortunately very sensitive.

(2) **Minorant function:** The technique of the minorant function was first proposed by Leulmi [10] for the positive semidefinite programming. This technique relies on approximating the function

$$G(\alpha) = \frac{1}{\eta}(q_\eta(x_\eta + \alpha d) - q_\eta(x_\eta))$$

by another function whose minimum can be easily computed, which permits the computation of the displacement step at each iteration in a relatively short time and with a smaller number of instructions in contrast to the line search technique.

We start with the following lemma, and in the rest of the paper, we consider x instead of x_η.

Lemma 1.2 *[14] The function G can be written as follows:*

$$G(\alpha) = \frac{1}{\eta}(\frac{1}{2}\alpha^2 d^t Q d - \alpha d^t Q d) + \alpha(\sum_{i=1}^{m} y_i - ||y||^2) - \sum_{i=1}^{m} \ln(1 + \alpha y_i) \tag{3}$$

for all $\alpha \in [0, \widehat{\alpha}]$ such that $\widehat{\alpha} = \min_{i\in I_-}\{\frac{-1}{y_i}\}$ and $I = \{i : y_i < 0\}$, where

$$y_i = \frac{< e_i, Ad >}{< e_i, Ax - b >}, i \in \{1, ..., m\}.$$

Now, we give the main result of the paper.

1.4 New approximate function

To introduce our new majorant function, we use the following well known inequality:

$$\left(\|y\| - \sum_{i=1}^{n} y_i\right)\alpha - \ln(1 + \alpha\|y\|) + \sum_{i=1}^{n} \ln(1 + \alpha y_i) \leq 0. \tag{4}$$

Replacing by the precedent inequality in (3), we obtain $\breve{G}(\alpha) \leq G(\alpha)$, then

$$\breve{G}(\alpha) = \delta\alpha - \ln(1 + \beta\alpha) + \frac{1}{2\eta}\widehat{\alpha}^2 d^t Q d, \ \alpha \in [0, \widehat{\alpha}[\,,$$

with $\delta = -\|y\|(\|y\| - 1)$ and $\beta_2 = \|y\|$.

Lemma 1.3 *For $\alpha \in I_\alpha = [0, \widehat{\alpha}[$, we have*

$$\breve{G}(\alpha) \leq G(\alpha).$$

Proof. From inequality (4) and for $\alpha \in I_\alpha$, we have

$$\alpha \sum_{i=1}^{m} y_i - \sum_{i=1}^{m} \ln(1+\alpha y_i) - \alpha\|y\|^2 \geq -\alpha\|y\| - \alpha\|y\|^2 - \ln(1+\alpha\|y\|).$$

We have $(-\alpha d^t Q d) > 0$ and

$$\frac{1}{2\eta}\alpha^2 d^t Q d < \frac{1}{2\eta}\widehat{\alpha}^2 d^t Q d, \ \forall \alpha \in I_\alpha.$$

This produces

$$\begin{aligned} G(\alpha) &= \alpha \sum_{i=1}^{n} y_i - \alpha\|y\|^2 - \sum_{i=1}^{n} \ln(1+\alpha y_i) + \frac{1}{\eta}\left(\frac{1}{2}\alpha^2 d^t Q d - \alpha d^t Q d\right) \\ &\geq -\alpha(\|y\|^2 - \|y\|) - \ln(1+\alpha\|y\|) + \frac{1}{2\eta}\widehat{\alpha}^2 d^t Q d = \breve{G}(\alpha). \end{aligned}$$

Then

$$\forall \alpha \in I_\alpha : G(\alpha) \geq \breve{G}(\alpha).$$

Remark 1.1 We note that

$$\breve{G}''(\alpha) = \frac{\|y\|^2}{(1-\|y\|^2)^2} \geq 0, \forall \alpha \in [0, \widehat{\alpha}[,$$

hence $\breve{G}$ is convex, and if it admits a minimum, this minimum is global.

Minimization of the minorant function $\breve{G}$ is defined and convex on $[0, \widehat{\alpha}[$, then its global minimum is reached when $\breve{G}'(\alpha) = 0$, therefore finding the minimum of the function $\breve{G}$ is equivalent to solving the equation $\breve{G}'(\alpha) = 0$. The solution of the later is the root of the equation

$$\alpha(\|y\|^2 - \|y\|^3) - \|y\|^2 = 0. \tag{5}$$

The root of the equation (5) is

$$\alpha^* = -\left(\|y\| - 1\right)^{-1} \in I_\alpha,$$

which is the global minimum of the function $\breve{G}$.

The following lemma indicates that the interior point x_{k+1} generated in each iteration k of the algorithm ensures the decreases of the function q_η.

Lemma 1.4 *The function q_η significantly decrease from the iteration k to the iteration $k+1$, that is, if x_k and x_{k+1} are two feasible solutions obtained at the iteration k and $k+1$, respectively, then*

$$q_\eta(x_{k+1}) < q_\eta(x_k). \tag{6}$$

Proof. Let x_k and x_{k+1} be two feasible solutions obtained at the iteration k and $k+1$, respectively, we have

$$q_\eta(x_{k+1}) \simeq q_\eta(x_k) + \langle \nabla q_\eta(x_k), x_{k+1} - x_k \rangle$$

and

$$x_{k+1} = x_k + \alpha_k d_k, \tag{7}$$

then

$$\begin{aligned} q_\eta(x_{k+1}) - q_\eta(x_k) &\simeq \langle \nabla q_\eta(x_k), \alpha_k d_k \rangle \\ &\simeq -\alpha_k \left\langle \nabla^2 q_\eta(x_k) d_k, d_k \right\rangle < 0. \end{aligned}$$

Hence,

$$q_\eta(x_{k+1}) < q_\eta(x_k),$$

which implies the claimed result.

2 Numerical Tests

We evaluate our algorithm's efficiency based on our approximate function. We conducted comparative numerical tests between our new two approximate functions (minorant function) and Armijo-Goldstein's line search method.

For this, in this part, we present a comparative numerical tests on different examples taken from the literature [1–3].

In the below tables, we reported the results obtained by implementing the algorithm in MATLAB R2013a on I5, 8350 (3.6 GHz) with 8 Go RAM.

We have taken $\varepsilon = 1.0e - 005$.

We use the following designations:

- (itrat) represents the number of iterations necessary to obtain an optimal solution.
- (time) represents the time of computation in seconds (s).
- (stmin) represents the strategy of approximate functions introduced in this paper.
- (LS) represents the classical Armijo-Goldstein line search.

We consider the following quadratic problem:

$$\alpha = \min[q(x) : x \geq 0,\ Ax \geq b],$$

where $q(x) = \frac{1}{2} x^t Q x + c^t x$.

2.1 Examples

Example 01: The matrix Q is defined by

$$Q[i,j] = \begin{cases} 2j-1 & \text{if } i > j, \\ 2i-1 & \text{if } i < j, \\ i(i+1)-1 & \text{if } i = j,\ i,j = 1,..,n, \end{cases}$$

$$A[i,j] = \begin{cases} 1 & \text{if } i = j \text{ or } j = i+m,\ i = 1,..,m \text{ and } j = 1,..,n \\ 0 & \text{otherwise.} \end{cases}$$

$$c[i] = -1,\ c[i+m] = 0 \text{ and } b[i] = 2,\ \forall i = 1,..,m,$$

with $n = 2m$. We test this example for different values of n. The following table resumes the obtained results:

ex(m,n)	stmin		LS	
	itrat	time	itrat	time
200×400	10	5.99012	26	22.52401
300×600	15	50.03129	35	97.10345
600×1200	25	71.66481	48	224.32120
1000×2000	30	122.27613	51	497.01165
1500×3000	39	320.79313	78	1321.03278

Example 02: We defined the matrix Q by

$$Q[i,j] = \left\{ \frac{1}{i+j} \quad \text{for} \quad i,j = 1,..,n, \right.$$

$$A[i,j] = \begin{cases} 1 & \text{if } i = j \text{ or } j = i+m, \ i = 1,..,m \text{ and } j = 1,..,n, \\ 0 & \text{otherwise,} \end{cases}$$
$$c[j] = 2j \text{ and } b[i] = i^2, \ \forall i = 1,..,m,$$

with $n = 2m$. We test this example for different values of n.
The following table resumes the obtained results:

ex(m,n)	stmin		LS	
	itrat	time	itrat	time
200×400	9	9.01214	16	29.12331
300×600	11	25.03119	27	84.15001
600×1200	33	74.06481	55	153.92210
1000×2000	39	198.07613	68	2213.11431
1500×3000	58	401.09313	124	3121.11303

Example 03: Let us define the matrix Q by

$$\begin{cases} Q[1,1] = 1, \\ Q[i,i] = i^2 + 1, \\ Q[i,i-1] = Q[i-1,i] = i, \quad i = 2,..,n, \end{cases}$$

$$A[i,j] = \begin{cases} 1 & \text{if } i = j \text{ or } j = i+m, \ i = 1,..,m \text{ and } j = 1,..,n \\ 0 & \text{otherwise,} \end{cases}$$
$$c[j] = j \text{ and } b[i] = \tfrac{i+1}{2}, \ \forall i = 1,..,m,$$

with $n = 2m$. We test this example for different values of n.
The following table resumes the obtained results:

ex(m,n)	stmin		LS	
	itrat	time	itrat	time
200×400	23	10.22544	38	22.52401
300×600	32	41.02385	45	88.11235
600×1200	39	91.10519	66	148.62103
1000×2000	50	148.47512	70	2004.11257
1500×3000	75	322.10134	101	2453.92312

Example 04: Let Q be the matrix define by

$$\begin{cases} Q[1,1] = 1, \\ Q[i,i] = 4, \quad i = 2,...,n-1, \\ Q[i,i-1] = Q[i-1,i] = 1, \quad i = 2,..,n, \end{cases}$$

$$A[i,j] = \begin{cases} 1 & \text{if } i=j \text{ or } j=i+m,\ i=1,..,m \text{ and } j=1,..,n, \\ 0 & i \neq j \text{ or } (i+1) \neq j, \end{cases}$$
$$c[j] = \frac{i+1}{2} \text{ and } b[i]=4,\ \forall i=1,..,m \text{ and } j=1,..,n,$$

with $n = 2m$. We test this example for different values of n.
The following table resumes the obtained results:

ex(m,n)	stmin		LS	
	itrat	time	itrat	time
200×400	12	6.22544	21	34.32215
300×600	25	81.02385	31	172.32550
600×1200	36	122.10519	53	503.44316
1000×2000	39	148.47512	67	2033.50062
1500×3000	55	352.10134	132	3121.11303

Commentary. These experiments demonstrate clearly the impact of our approach on the numerical behavior of the algorithm, expressed by the reduction of the number of iterations and computation time. The number of iterations and the computing time are considerably reduced in the approximate approaches in comparison with the line search method. Always, in the problems of linear dynamics, we arrive to the problem of optimization. Then we solve this problem by our approach. This is what we look forward to in future.

3 Conclusion

In order to solve a quadratic optimization problem, this study provides a logarithmic penalty method based on new approximate functions (minorant). As anticipated, the minorant function strategy for computing the displacement step demonstrates its effectiveness by lowering the computational cost when compared to the line search method. This effectiveness is a result of the nature of the mentioned functions. The numerical results demonstrate that our strategy reduces the cost of iteration for the quadratic optimization compared to the line search method. Proposing some other new majorant [5] and minorant functions seems to be an interesting topic in the future in the different class of optimization and we will apply our important results in different problems of nonlinear dynamics problems.

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