Nonlinear Dynamics and Systems Theory, **24** (1) (2024) 1–11

Sequential Initial Value Problems with Delay

A. Adoui *, A. Guezane-Lakoud and R. Khaldi

Laboratory of Advanced Materials,
Faculty of Sciences, Badji Mokhtar-Annaba University,
P.O. Box 12, 23000 Annaba, Algeria.

Received: May 6, 2023; Revised: January 21, 2024

Abstract: In this paper, we discuss the solvability of a nonlinear Riemann-Liouville sequential initial value problem with infinite delay. We give sufficient conditions for the existence, uniqueness and stability of solutions. Proofs are carried out employing fixed point theory.

Keywords: *sequential fractional derivative; initial value problem; delay; existence of solution; fixed point theorem; stability.*

Mathematics Subject Classification (2010): 26A33, 34A08, 34K37, 93D05.

1 Introduction

In the last few centuries, non-integer order derivatives were widely expanded as a useful theoretical concept and numerous books were devoted to this field, see monographs [21, 24, 26]. Due to their nonlocal nature, fractional derivatives play a significant role in describing physical phenomena with memory effect and hereditary processes. Hence, they give better accuracy when compared to classical derivatives, the evidence of which has been provided for instance in [3] by virtue of numerical simulations. Consequently, more study has been conducted on new classes of fractional differential equations. In particular, fractional differential equations with time delay were capable to attract the attention of many researchers over the last few years, see [1, 4, 6, 7, 9], and the references therein.

Very recently, sequential fractional differential equations have been the subject of many investigations. Sequential fractional derivatives were introduced for the first time by Miller and Ross in their book [24]. As a matter of fact, they appear often in physics, where the substitution of formulas containing derivatives for one another is

* Corresponding author: mailto:ahlem.a.adoui@gmail.com, ahlem.adoui@univ-annaba.org

very common. Many recent works are devoted to sequential boundary value problems of Caputo, Hadamard, mixed, Caputo-Hadamard, and Hilfer type fractional derivatives, see [27], [23], [5, 13, 14, 19, 20], [2] and [25], respectively. In the following interesting papers, the authors established existence results for fractional differential equations, these results are pertinent to the topic of this work.

In [12], the following Riemann-Liouville sequential fractional differential equation is studied:

$$\mathbb{D}^{\upsilon}_{0^+}[(t-a)^r \mathbb{D}^{\varrho}_{0^+} x(t)] = f(t,x),\ t \in (0,b].$$

In [11], the authors studied a general Basset-Boussinesq-Oseen fractional equation and proved the global existence, uniqueness and regularity of solutions in a partially ordered Banach space for the following problem:

$$\begin{aligned}
&\mathbb{D}^{\upsilon}_{0+}(\mathbb{D}^{\varrho}_{0+} + A)x(t) + Bx(t) = f(t), \ \ 0 < t \leq 1,\ 0 < \upsilon, \varrho \leq 1,\\
&x(0) = a,\\
&\mathbb{D}^{\varrho}_{0+} x(0) = b.
\end{aligned}$$

Nonetheless, the analysis of fractional sequential initial value problems is still not sufficiently enriched. To the best of our knowledge, sequential fractional differential equations involving the Riemann-Liouville fractional derivatives associated with infinite delay have not been considered yet. The primary focus of this paper is the investigation of sufficient criteria for nonlinearity that ensure the existence and uniqueness of solutions for the following initial value problem:

$$\begin{aligned}
&\mathbb{D}^{\varrho}_{0+}\mathbb{D}^{\upsilon}_{0+} y(t) = f(t, y_t),\ t \in (0,b],\\
&y(t) = \chi(t), t \in (-\infty, 0],\\
&\mathbb{D}^{\upsilon}_{0+} y(0) = 0,
\end{aligned} \tag{1}$$

where $0 < \upsilon, \varrho < 1,\ f : [0,b] \times \mathbb{B} \to \mathbb{R},\ \chi \in \mathbb{B},\ \chi(0) = 0,\ x_t(\theta) = x(t+\theta),\ \theta \leq 0.$
The phase space $\mathbb{B}$ is a semi-normed linear space of functions mapping $(-\infty, 0]$ into $\mathbb{R}$ and characterized by the following axioms.
If $y : (-\infty, b] \to \mathbb{R}$ and $y_0 \in \mathbb{B}$, then for any $t \in [0,b]$,

1. $y_t \in \mathbb{B}$ and we have
$$\|y_t\|_{\mathbb{B}} \leq K(t) \sup_{s\in[0,t]} |y(s)| + M(t)\|y_0\|_{\mathbb{B}},$$
$$|y(t)| \leq H\|y_t\|_{\mathbb{B}},$$
where $H \geq 0$ is a constant, $K : [0,b] \to [0,\infty[$ is continuous and $M : [0,\infty[\to [0,\infty[$ is locally bounded. H, K, M are independent of $y(.)$.

2. y_t is a $\mathbb{B}-$valued function on $[0,b]$.

3. $\mathbb{B}$ is complete.

For more details on the theory of delay differential equations, we refer the interested reader to [15,17,22]. The main motivation for our paper is to continue the quest of broadening the study of fractional operators for much larger classes of differential equations. We emphasize that our results are novel in the aforementioned context.

The rest of this paper is structured as follows. In Section 2, we recall some preliminary results which are relevant to our work. Section 3 is devoted to the existence and uniqueness results. In Section 4, in order to enhance the physical meaning of our main outcomes, we study the stability of the problem under consideration. We exhibit an example in Section 5 to illustrate the applicability of our results. Finally, a conclusion is given.

2 Preliminaries

In this section, we recall the basic definitions and properties needed in our proofs.

Definition 2.1 [26] If $f \in L^1(a,b)$ and $\upsilon > 0$, then the left-sided Riemann-Liouville fractional integral is defined by

$$\mathbb{I}^{\upsilon}_{a+}f(x) = \frac{1}{\Gamma(\upsilon)}\int_a^x \frac{f(t)}{(x-t)^{1-\upsilon}}dt,\ x > a.$$

Lemma 2.1 [21] *The operator $\mathbb{I}^{\upsilon}_{a+}$ maps continuous functions into continuous functions.*

Definition 2.2 [26] Let $0 < \upsilon < 1$. The left-sided Riemann-Liouville fractional derivative is defined by

$$\mathbb{D}^{\upsilon}_{a+}f(x) = \frac{1}{\Gamma(1-\upsilon)}\frac{d}{dt}\int_a^x \frac{f(t)}{(x-t)^{\upsilon}}dt,\ x > a.$$

Moreover, if $\mathbb{I}^{1-\upsilon}f \in AC[a,b]$, then $\mathbb{D}^{\upsilon}_{a+}f$ exists almost everywhere on $[a,b]$.

The following lemma gives some properties of the composition of the Riemann-Liouville fractional integral and derivative.

Lemma 2.2 [21] *Let $\upsilon, \varrho > 0$ and $f \in L^1(a,b)$, then*

$$\mathbb{I}^{\upsilon}_{a+}\mathbb{I}^{\varrho}_{a+}f = \mathbb{I}^{\upsilon+\varrho}_{a+}f \tag{2}$$

is satisfied at almost every point $x \in [a,b]$.
Let $\upsilon > 0$ and $f \in L^1(a,b)$. Then

$$\mathbb{D}^{\upsilon}_{a+}\mathbb{I}^{\upsilon}_{a+}f(x) = f(x) \tag{3}$$

at almost every $x \in [a,b]$.
Let $0 < \upsilon < 1$, $f \in L^1(a,b)$ and $\mathbb{I}^{1-\upsilon}_{a+}f \in AC[a,b]$. Then

$$\mathbb{I}^{\upsilon}_{a+}\mathbb{D}^{\upsilon}_{a+}f(x) = f(x) - \frac{\mathbb{I}^{1-\upsilon}_{a+}f(a)}{\Gamma(\upsilon)}(x-a)^{\upsilon-1} \tag{4}$$

holds almost everywhere on $[a,b]$.

Let $a = 0$. In the following, we denote $\mathbb{I}^{\upsilon}_{0+}$ by $\mathbb{I}^{\upsilon}$, and $\mathbb{D}^{\upsilon}_{0+}$ by $\mathbb{D}^{\upsilon}$, for simplicity.

Proposition 2.1 *Let $0 < \upsilon < 1$, $F : [0, b] \to \mathbb{R}$ be a continuous function. Then y is a solution of the IVP*

$$\begin{aligned} &\mathbb{D}^{\upsilon} y(t) = F(t),\ t \in (0, b], \\ &y(0) = 0 \end{aligned} \tag{5}$$

if and only if $y \in C[0, b]$ is a solution of the integral equation

$$y(t) = \frac{1}{\Gamma(\upsilon)} \int_0^t (t-s)^{\upsilon-1} F(s) ds. \tag{6}$$

Proof. Let $y \in C[0, b]$ be such that $\mathbb{D}^{\upsilon} y = F$, i.e., $\mathbb{D}\mathbb{I}^{1-\upsilon} y = F$. Integrating, we get $\mathbb{I}^{1-\upsilon} y(t) = \mathbb{I}^{1-\upsilon} y(0) + \mathbb{I}^1 F$ so that $\mathbb{I}^{1-\upsilon} y$ is absolutely continuous. Apply operator $\mathbb{I}^{\upsilon}$ to the differential equation in (5), then by virtue of (4), we obtain $y(t) = \frac{c}{\Gamma(\upsilon)} t^{\upsilon-1} + \mathbb{I}^{\upsilon} F(t)$. Employing the initial condition, we find $c = 0$. Hence, (6) holds.

Conversely, if $y = \mathbb{I}^{\upsilon} F$, then, taking into account Lemma 2.1, we see that y is continuous. Moreover, $y(0) = \mathbb{I}^{\upsilon} F(0) = 0$ because F is continuous. Then, applying operator $\mathbb{I}^{1-\upsilon}$ and (2), we obtain $\mathbb{I}^{1-\upsilon} y = \mathbb{I}^1 F$. Deriving, we get $\mathbb{D}^{\upsilon} y = F$.

3 Existence Results

In this section, we give an existence and uniqueness result based upon Banach's fixed point theorem. Moreover, we retrieve an existence result by means of the nonlinear alternative of Leray-Schauder for a larger class of functions $f(t, y_t)$.

The following space will be considered hereafter. Let Ω be the Banach space of all continuous functions $y : (-\infty, b] \to \mathbb{R}$ such that $y_0 \in \mathbb{B}$ and $y\big|_{[0,b]}$ is continuous.

Definition 3.1 A function $y \in \Omega$ is said to be a solution of (1) if y satisfies the fractional differential equation $\mathbb{D}^{\varrho}\mathbb{D}^{\upsilon} y(t) = f(t, y_t)$ on $(0, b]$, the initial condition $\mathbb{D}^{\upsilon} y(0) = 0$, and $y(t) = \chi(t)$ on $(-\infty, 0]$.

Lemma 3.1 *Let $F(t) = f(t, y_t) \in C[0, b]$. Then y is a solution of the IVP*

$$\begin{aligned} &\mathbb{D}^{\varrho}\mathbb{D}^{\upsilon} y(t) = f(t, y_t),\ t \in (0, b], \\ &\mathbb{D}^{\upsilon} y(0) = 0, \\ &y(0) = 0 \end{aligned} \tag{7}$$

if and only if y is a solution of the fractional integral equation

$$y(t) = \frac{1}{\Gamma(\upsilon + \varrho)} \int_0^t (t-s)^{\upsilon+\varrho-1} f(s, y_s) ds.$$

Proof. Since F is continuous, applying Proposition 2.1, we obtain that

$$\begin{aligned} &\mathbb{D}^{\upsilon} y(t) = \frac{1}{\Gamma(\varrho)} \int_0^t (t-s)^{\varrho-1} f(s, y_s) ds, \\ &y(0) = 0. \end{aligned}$$

$\mathbb{I}^{\varrho}F$ is continuous. Hence, using Proposition 2.1 again, we get

$$\begin{aligned} y(t) &= \frac{1}{\Gamma(\upsilon+\varrho)}\int_0^t (t-s)^{\upsilon+\varrho-1} f(s,y_s)ds, \\ y(0) &= 0. \end{aligned}$$

Similarly, the converse is easily shown.

Theorem 3.1 *Let $f : [0,b]\times\mathbb{B}\to \mathbb{R}$ be a continuous function. Suppose that there exists $L > 0$ such that*

$$|f(t,u) - f(t,v)| \leq L\|u-v\|_{\mathbb{B}}, \quad t \in [0,b], \text{ for every } u,v \in \mathbb{B}.$$

Then the IVP (1) has a unique solution on $[0,b]$ provided that $\frac{b^{\upsilon+\varrho}K_bL}{\Gamma(\upsilon+\varrho+1)} < 1$, where $K_b = \sup\limits_{t\in[0,b]} |k(t)|$.

Proof. Using the previous lemma, we show that solving the initial value problem is equivalent to proving that the operator $S : \Omega \to \Omega$ has a unique fixed point, where

$$(Sy)(t) = \begin{cases} \chi(t), \ t \in (-\infty, 0], \\ \frac{1}{\Gamma(\upsilon+\varrho)}\int_0^t (t-s)^{\upsilon+\varrho-1} f(s,y_s)ds, \ t \in [0,b]. \end{cases} \tag{8}$$

Consider the following decomposition.
Let $x(.) : (-\infty, b] \to \mathbb{R}$ be the function defined by

$$x(t) = \begin{cases} 0, \ t \in [0,b], \\ \chi(t), \ t \in (-\infty, 0]. \end{cases} \tag{9}$$

Then take $z(.) : (0,b] \to \mathbb{R}$ given by $z = y\big|_{[0,b]}$, denote by $\overline{z}$ the function defined by

$$\overline{z}(t) = \begin{cases} z(t), \ t \in [0,b], \\ 0, \ t \in (-\infty, 0]. \end{cases} \tag{10}$$

Thus, $y(t) = \overline{z}(t) + x(t), \ t \in [0,b]$, then $y_t = \overline{z}_t + x_t, \ t \in [0,b]$.
In addition, set $C_0 = \{z \in C([0,b]) : z_0 = 0\}$ equipped with the semi-norm in C_0 defined by $\|z\|_b = \sup\limits_{t\in[0,b]} |z(t)|$. Consider now the operator $\mathbb{T} : C_0 \to C_0$ given by

$$(\mathbb{T}z)(t) = \begin{cases} 0, \text{ if } t \in (-\infty, 0], \\ \frac{1}{\Gamma(\upsilon+\varrho)}\int_0^t (t-s)^{\upsilon+\varrho-1} f(\tau, \overline{z}_s + x_s)ds, \text{ if } t \in [0,b]. \end{cases} \tag{11}$$

Then the operator S has a fixed point is equivalent to $\mathbb{T}$ has a fixed point.
Indeed, $\mathbb{T}$ is a contraction mapping.

Consider $z, z^* \in C_0$. Then we have for every $t \in [0,b]$,

$$\begin{aligned}
&|(\mathbb{T}z)(t) - (\mathbb{T}z^*)(t)| \\
&\quad \leq \frac{1}{\Gamma(\upsilon+\varrho)} \int_0^t (t-s)^{\upsilon+\varrho-1} |f(s, \overline{z}_s + x_s) - f(s, \overline{z_s^*} + x_s)| ds \\
&\quad \leq \frac{1}{\Gamma(\upsilon+\varrho)} \int_0^t (t-s)^{\upsilon+\varrho-1} L \|\overline{z}_s - \overline{z_s^*}\|_{\mathbb{B}} ds \\
&\quad \leq \frac{LK_b}{\Gamma(\upsilon+\varrho)} \int_0^t (t-s)^{\upsilon+\varrho-1} \sup_{\tau\in[0,s]} |z(\tau) - z^*(\tau)| ds \\
&\quad \leq \frac{LK_b \|z - z^*\|_b}{\Gamma(\upsilon+\varrho)} \int_0^t (t-s)^{\upsilon+\varrho-1} ds \leq \frac{LK_b b^{\upsilon+\varrho}}{\Gamma(\upsilon+\varrho+1)} \|z - z^*\|_b.
\end{aligned}$$

Therefore, $\mathbb{T}$ is a contraction mapping. Hence, applying Banach's fixed point theorem, we see that $\mathbb{T}$ has a unique fixed point.

Now, we give an existence result based upon the nonlinear alternative of Leray-Schauder.

Lemma 3.2 [16] *Let $v : [0,b] \to [0,\infty)$ be a real function and $w(.)$ be a nonnegative, locally integrable function on $[0,b]$ and there exist constants $a > 0$ and $0 < \upsilon < 1$ such that*

$$v(t) \leq w(t) + a \int_0^t \frac{v(s)}{(t-s)^\upsilon} ds.$$

Then there exists a constant $K = K(\upsilon)$ such that $v(t) \leq w(t) + Ka \int_0^t \frac{w(s)}{(t-s)^\upsilon} ds$, $t \in [0,b]$.

Theorem 3.2 *Under the following assumptions:*

1. *f is a continuous function,*
2. *there exists $p, q \in C([0,b], \mathbb{R}_+)$ such that*

$$|f(t,u)| \leq p(t) + q(t)\|u\|_{\mathbb{B}}, \quad t \in [0,b],\ u \in \mathbb{B},$$

the IVP (1) has at least one solution on $[0,b]$.

Proof. Let $\mathbb{T} : C_0 \to C_0$ be defined as in (11). We will show that $\mathbb{T}$ is a continuous and a completely continuous operator.

Step 1: $\mathbb{T}$ is continuous.

Let (z_n) be a sequence in C_0 such that $z_n \to z$ in C_0. Then

$$\begin{aligned}
&|(\mathbb{T}z_n)(t) - (\mathbb{T}z)(t)| \\
&\quad \leq \frac{1}{\Gamma(\upsilon+\varrho)} \int_0^b (t-s)^{\upsilon+\varrho-1} |f(s, \overline{z}_{n_s} + x_s) - f(s, \overline{z}_s + x_s)| ds \\
&\quad \leq \frac{b^{\upsilon+\varrho}}{\Gamma(\upsilon+\varrho+1)} \|f(., \overline{z}_{n_{(.)}} + x_{(.)}) - f(., \overline{z}_{(.)} + x_{(.)})\|_\infty,
\end{aligned}$$

which tends to 0 as $n \to \infty$.

Step 2: $\mathbb{T}$ maps bounded sets into bounded sets in C_0.

Let $z \in B_\eta := \{z \in C_0 : \|z\|_b \leq \eta\}$, since f is a continuous function, we have for each $t \in [0, b]$,

$$\begin{aligned}|(\mathbb{T}z)(t)| &\leq \frac{1}{\Gamma(\upsilon+\varrho)} \int_0^b (t-s)^{\upsilon+\varrho-1}(p(s) + q(s)\|\overline{z}_s + x_s\|_{\mathbb{B}})ds \\ &\leq \frac{b^{\upsilon+\varrho}}{\Gamma(\upsilon+\varrho+1)}\|p\|_\infty + \frac{b^{\upsilon+\varrho}}{\Gamma(\upsilon+1)\Gamma(\varrho+1)}\|q\|_\infty \eta_* =: l,\end{aligned}$$

where $\|\overline{z}_s + x_s\|_{\mathbb{B}} \leq \|\overline{z}_s\|_{\mathbb{B}} + \|x_s\|_{\mathbb{B}} \leq K_b\eta + M_b\|\chi\|_{\mathbb{B}} := \eta_*$. Hence, $\|\mathbb{T}z\|_\infty \leq l$.

Step 3: $\mathbb{T}$ maps bounded sets into equicontinuous sets of C_0.

Let $t_1, t_2 \in [0, b], t_1 < t_2$ and let B_η be a bounded set of C_0 as in Step 2. Let $z \in B_\eta$.

$$\begin{aligned}&|(\mathbb{T}z)(t_2) - (\mathbb{T}z)(t_1)| \\ &\leq \frac{1}{\Gamma(\upsilon+\varrho)}(\int_0^{t_1} |(t_2-s)^{\upsilon+\varrho-1} - (t_1-s)^{\upsilon+\varrho-1}||f(s, \overline{z}_s + x_s)|ds \\ &+ \int_{t_1}^{t_2} |(t_2-s)^{\upsilon+\varrho-1} f(s, \overline{z}_s + x_s)|ds) \\ &\leq \frac{\|p\|_\infty + \|q\|_\infty\eta_*}{\Gamma(\upsilon+\varrho)} \int_0^{t_1} |(t_2-s)^{\upsilon+\varrho-1} - (t_1-s)^{\upsilon+\varrho-1}|ds \\ &+ \frac{\|p\|_\infty + \|q\|_\infty\eta_*}{\Gamma(\upsilon+\varrho)} \int_{t_1}^{t_2} (t_2-s)^{\upsilon+\varrho-1}ds.\end{aligned}$$

As $t_1 \to t_2$, the right-hand side of the above inequality tends to zero. By virtue of Steps 1-3, along with the Arzelà-Ascoli theorem, we infer that $\mathbb{T} : C_0 \to C_0$ is continuous and completely continuous.

Step 4: A priori bounds.

It is sufficient to show that there exists an open set $U \subseteq C_0$ with $z \neq \lambda\mathbb{T}z$, for $\lambda \in (0, 1)$ and $z \in \partial U$.

Take $z \in C_0$ and $z = \lambda\mathbb{T}(z)$ for some $0 < \lambda < 1$, then for each $t \in [0, b]$,

$$\begin{aligned}|z(t)| &\leq \frac{1}{\Gamma(\upsilon+\varrho)} \int_0^t (t-\tau)^{\upsilon+\varrho-1}(p(\tau) + q(\tau)\|\overline{z}_\tau + x_\tau\|_{\mathbb{B}})d\tau \\ &\leq \frac{b^{\upsilon+\varrho}}{\Gamma(\upsilon+\varrho+1)}\|p\|_\infty + \frac{1}{\Gamma(\upsilon+\varrho)} \int_0^t (t-\tau)^{\upsilon+\varrho-1} q(\tau)\|\overline{z}_\tau + x_\tau\|_{\mathbb{B}}d\tau.\end{aligned}$$

Then $\|\overline{z}_\tau + x_\tau\|_{\mathbb{B}} \leq K_b \sup_{s\in[0,\tau]} |z(s)| + M_b\|\chi\|_B := w(\tau)$, which implies that

$$|z(t)| \leq \frac{1}{\Gamma(\upsilon+\varrho)} \int_0^t (t-\tau)^{\upsilon+\varrho-1} q(\tau)w(\tau)d\tau + \frac{b^{\upsilon+\varrho}}{\Gamma(\upsilon+\varrho+1)}\|p\|_\infty, \ t \in [0, b].$$

By inserting the above in w, we get for each $t \in [0, b]$,

$$w(t) \leq M_b\|\chi\|_{\mathbb{B}} + \frac{K_b b^{\upsilon+\varrho}}{\Gamma(\upsilon+\varrho+1)}\|p\|_\infty + \frac{K_b\|q\|_\infty}{\Gamma(\upsilon+\varrho)} \int_0^t (t-\tau)^{\upsilon-1}(t-\tau)^{\varrho}w(\tau)d\tau.$$

To this end, applying Lemma 3.2 yields that there exists a constant $K = K(\upsilon)$ such that for each $t \in [0, b]$,

$$|w(t)| \leq M_b \|\chi\|_{\mathbb{B}} + \frac{K_b b^{\upsilon+\varrho}}{\Gamma(\upsilon+\varrho+1)} \|p\|_\infty + K(\upsilon) \frac{K_b \|q\|_\infty b^\varrho}{\Gamma(\upsilon+\varrho)} \int_0^t (t-\tau)^{\upsilon-1} R \, d\tau,$$

where

$$R = M_b \|\chi\|_{\mathbb{B}} + \frac{K_b b^{\upsilon+\varrho}}{\Gamma(\upsilon+\varrho+1)} \|p\|_\infty.$$

Hence,

$$\|w\|_\infty \leq R + R \frac{K(\upsilon) K_b b^{\upsilon+\varrho}}{\upsilon \Gamma(\upsilon+\varrho)} \|q\|_\infty := r.$$

Then $\|z\|_\infty \leq r \|\mathbb{I}^{\upsilon+\varrho} q\|_\infty + \frac{b^{\upsilon+\varrho}}{\Gamma(\upsilon+\varrho+1)} \|p\|_\infty := r^*$.
Set $U = \{z \in C_0 : \|z\|_b < r^* + 1\}$. $\mathbb{T} : \overline{U} \to C_0$ is continuous and completely continuous. From the choice of U, there is no $z \in \partial U$ such that $z = \lambda \mathbb{T}(z)$, for $\lambda \in (0, 1)$. Consequently, the nonlinear alternative of Leray-Schauder is applicable. It follows that $\mathbb{T}$ has a fixed point z in U.

4 Stability

In this section, we provide sufficient conditions that ensure the Hyers-Ulam stability of our problem.

Definition 4.1 Problem (1) is said to be Hyers-Ulam stable if there exists a constant $\lambda > 0$ such that for each $\epsilon > 0$ and for each $u \in \Omega$,

$$\begin{aligned} &|\mathbb{D}^\varrho \mathbb{D}^\upsilon u - f(t, u_t)| \leq \epsilon, \ t \in [0, b], \\ &u(t) = \chi(t), \ t \in (-\infty, 0], \end{aligned} \tag{12}$$

there exists a solution $v \in \Omega$ of (1) such that $|u(t) - v(t)| \leq \lambda \epsilon, \ t \in [0, b]$.

Theorem 4.1 *Let $f : [0, b] \times \to \mathbb{R}$ be continuous and satisfy the Lipschitz condition with respect to the second variable, i.e., there exists $L > 0$ such that*

$$|f(t, u) - f(t, v)| \leq L \|u - v\|_{\mathbb{B}}, \quad t \in [0, b] \ u, v \in \mathbb{B}.$$

Then (1) is Hyers-Ulam stable provided that $\frac{b^{\upsilon+\varrho} K_b L}{\Gamma(\upsilon+\varrho+1)} < 1$.

Proof. Since f satisfies the assumptions of Theorem 3.1 , the solution of (1) can be written as

$$v(t) = \frac{1}{\Gamma(\upsilon+\varrho)} \int_0^t (t-s)^{\upsilon+\varrho-1} f(s, v_s) ds.$$

Now, let u be a solution to (12), then there exists a function γ such that $|\gamma(t)| \leq \epsilon$ and

$$\mathbb{D}^\varrho \mathbb{D}^\upsilon u = f(t, u_t) + \gamma(t).$$

Proceeding as in Section 2, we find

$$u(t) = \frac{1}{\Gamma(\upsilon+\varrho)} \int_0^t (t-s)^{\upsilon+\varrho-1} f(s, u_s) ds + \frac{1}{\Gamma(\upsilon+\varrho)} \int_0^t (t-s)^{\upsilon+\varrho-1} \gamma(s) ds,$$

which gives $|u(t) - \frac{1}{\Gamma(\upsilon+\varrho)}\int_0^t (t-s)^{\upsilon+\varrho-1} f(s,u_s)ds| \le \frac{\epsilon}{\Gamma(\upsilon+\varrho+1)} b^{\upsilon+\varrho} := \Lambda\epsilon$ so that

$$\begin{aligned}
|u(t)-v(t)| &\le |u(t) - \frac{1}{\Gamma(\upsilon+\varrho)}\int_0^t (t-s)^{\upsilon+\varrho-1} f(s,u_s)ds| \\
&+ \frac{1}{\Gamma(\upsilon+\varrho)}|\int_0^t (t-s)^{\upsilon+\varrho-1} f(s,u_s) - \frac{1}{\Gamma(\upsilon+\varrho)}\int_0^t (t-s)^{\upsilon+\varrho-1} f(s,v_s)ds| \\
&\le \Lambda\epsilon + \frac{L}{\Gamma(\upsilon+\varrho)}\int_0^t (t-s)^{\upsilon-1}(t-s)^{\varrho}\|u_s - v_s\|_{\mathbb{B}} ds \\
&\le \Lambda\epsilon + \frac{Lb^{\varrho}K_b}{\Gamma(\upsilon+\varrho)}\int_0^t (t-s)^{\upsilon-1}\sup_{\tau\in[0,s]}|u(\tau)-v(\tau)|ds.
\end{aligned} \tag{13}$$

Set $\psi(t) = \sup_{\tau\in[0,t]}|u(\tau)-v(\tau)|$, so $\psi(t) \le \Lambda\epsilon + \frac{Lb^{\varrho}K_b}{\Gamma(\upsilon+\varrho)}\int_0^t (t-s)^{\upsilon-1}\psi(s)ds$. Applying the Gronwall lemma, we find

$$\begin{aligned}
\psi(t) &\le \Lambda\epsilon + \frac{Lb^{\varrho}K_b}{\Gamma(\upsilon+\varrho)}\int_0^t (t-s)^{\upsilon-1}\Lambda\epsilon ds \\
&\le \Lambda\epsilon + K\frac{LK_b}{\upsilon}\frac{b^{\upsilon+\varrho}}{\Gamma(\upsilon+\varrho)}\Lambda\epsilon = \Lambda(1 + K\frac{LK_b}{\upsilon}\frac{b^{\upsilon+\varrho}}{\Gamma(\upsilon+\varrho)})\epsilon.
\end{aligned} \tag{14}$$

5 Example

In this section, we provide a numerical example to show the viability of our outcomes. Take $\upsilon = \varrho = \frac{1}{2}, b = 1, \gamma > 0$. The nonlinearity $f : [0,1]\times\mathbb{B}_\gamma \to \mathbb{R}$ is given by

$$f(t,x) = e^{-\gamma t}(\frac{1}{\sqrt{t+4}}\frac{x^2+2|x|}{1+|x|} + \sin(t))$$

and $\chi \in \mathbb{B}_\gamma$ which is defined by

$$\mathbb{B}_\gamma = \{\chi : C((-\infty,0],\mathbb{R}) : \lim_{s\to-\infty} e^{\gamma s}|\chi(s)| \text{exists in } \mathbb{R}\}$$

and endowed with the norm $\|\chi\|_{\mathbb{B}_\gamma} = \sup_{s\in(-\infty,0]} e^{\gamma s}|\chi(s)|$. It is easily verified that $\mathbb{B}_\gamma$ is an admissible phase space, i.e., it is a Banach space and it fulfills the phase space axioms with $K(t) = 1, M(t) = e^{-\gamma t}$ and $H = 1$.
Moreover, for any $t \in [0,1],\ x,y \in \mathbb{B}_\gamma$, we have

$$\begin{aligned}
|f(t,x) - f(t,y)| &= e^{-\gamma t}\frac{1}{\sqrt{t+4}}|\frac{x^2+2|x|}{1+|x|} - \frac{y^2+2|y|}{1+|y|}| \\
&\le e^{-\gamma t}\frac{1}{\sqrt{t+4}}|\frac{x^2-y^2}{(1+|x|)(1+|y|)}| \\
&\le e^{-\gamma t}\frac{1}{\sqrt{t+4}}|x-y| \le \frac{1}{2}\|x-y\|_{\mathbb{B}_\gamma}
\end{aligned}$$

so that f satisfies the Lipschitz condition with $L = \frac{1}{2}$ and $\frac{b^{\upsilon+\varrho}K_bL}{\Gamma(\upsilon+\varrho+1)} = \frac{1}{4} < 1$, so the IVP has exactly one solution by virtue of Theorem 3.1. Also, the hypothesis from Theorem

4.1 is satisfied, and therefore the IVP is shown to be Hyers-Ulam stable.
Furthermore, for any $t \in [0,1]$ and $x \in \mathbb{B}_\gamma$, we have

$$|f(t,x)| \leq \frac{e^{-\gamma t}}{\sqrt{t+4}} \frac{2+|x|}{1+|x|}|x| + e^{-\gamma t}\sin(t) \leq p(t)\|x\|_{\mathbb{B}_\gamma} + q(t),$$

where $p(t) = \frac{2}{\sqrt{t+4}}$ and $q(t) = e^{-\gamma t}\sin(t)$. We see that f satisfies the conditions of Theorem 3.2. Thus, the existence of solutions follows immediately.

6 Conclusion

In this work, we gave sufficient conditions for the existence, uniqueness and stability of solutions of nonlinear sequential IVPs with delay. The novelty of our findings resides in the usefulness of our existence results in proving the controllability of a more generalized system by the aid of semigroup techniques. Namely, when we extend the differential equation to a fractional differential evolution equation with control, i.e., when the nonlinear part takes the form $Ax(t)+Bu(t)+f(t,x_t)$, where x takes values in a Banach space X, A is a generator of a strongly continuous semigroup of bounded linear operators, B is a bounded linear operator and the control function u is given in $L^2([0,b],U)$, where U is a Banach space. This will make the subject of a future publication.

Acknowledgment

The authors would like to thank the referee and editors for their helpful comments.

References

[1] M. Abdo and S. Panchal. Initial value problem for a fractional neutral differential equation with infinite delay. *Int. J. of Nonlinear Anal. Appl.* **12** (1) (2021) 1195–1206.

[2] B. Ahmad and S. Ntouyas. Existence and uniqueness of solutions for Caputo-Hadamard sequential fractional order neutral functional differential equations. *Electron. J. Differ Equ.* **2017** (36) (2017) 1–11.

[3] R. Almeida, N. R. O. Bastos and M. T. T. Teresa. Modeling some real phenomena by fractional differential equations. *Math. Methods Appl. Sci.* **39** (16) (2015) 4846–4855.

[4] M. Benchohra, J. Henderson, S. K. Ntouyas and A. Ouahab. Existence results for fractional order functional differential equations with infinite delay. *J. Math. Anal. Appl.* **338** (2) (2008) 1340–1350.

[5] A. Cabada and R. Khaldi. Lyapunov-type inequality for higher order left and right fractional p-Laplacian problems. *Proyecciones* **40** (4) (2021) 1031–1040.

[6] C. Chen and Q. Dong. Existence and Hyers-Ulam stability for a multi-term fractional differential equation with infinite delay. *Mathematics* **10** (7) (2022).

[7] J. Deng and H. Qu. New uniqueness results of solutions for fractional differential equations with infinite delay. *Comput. Math. Appl.* **60** (8) (2010) 2253–2259.

[8] S. Dhar and J. T. Neugebauer. Lyapunov-Type Inequalities for a Fractional Boundary Value Problem with a Fractional Boundary Condition. *Nonlinear Dyn. Syst. Theory* **22** (2) (2022) 133–143.

[9] Q. Dong, C. Liu and Z. Fan. Weighted fractional differential equations with infinite delay in Banach spaces. *Open Math.* **14** (1) (2016) 370–383.

[10] C. Duque, H. Leiva and A. Tridane. Existence and Uniqueness of Solutions for a Semilinear Functional Dynamic Equation with Infinite Delay and Impulses. *Nonlinear Dyn. Syst. Theory* **22** (2) (2022) 155–168.

[11] H. Fazli, F. Bahrami and J. Nieto. General Basset-Boussinesq-Oseen equation: existence, uniqueness, approximation and regularity of solutions. *Int. J. Comput. Math. Comput. Syst. Theory* **97** (9) (2020) 1792–1805

[12] K. M. Furati. A Cauchy-type problem involving a weighted sequential derivative in the space of integrable functions. *Comput. Math. Appl.* **66** (5) (2013) 883–891.

[13] A. Guezane-Lakoud and R. Rodríguez-López. Positive solutions to mixed fractional p-Laplacian boundary value problems. *J. Appl. Anal.* (2022).

[14] A. Guezane-Lakoud and A. Ashyralyev. Existence of solutions for weighted p(t)-Laplacian mixed Caputo fractional differential equations at resonance. *Filomat* **36** (1) (2022) 231–241.

[15] J. Hale and J. Kato. Phase Space for Retarded Equations with Infinite Delay. *Funkcial. Ekvac.* **21** (1978) 11–44.

[16] D. Henry. *Geometric Theory of Semilinear Parabolic Partial Differential Equations.* Springer-Verlag, Berlin, 1989.

[17] Y. Hino, S. Murakami and T. Naito. *Functional Differential Equations with Infinite Delay.* Springer, Berlin, Heidelberg, 2006.

[18] E. Kenef and A. Guezane-Lakoud. Impulsive mixed fractional differential equations with delay. *Prog. Fract. Differ. Appl.* **7** (3) (2017) 203–215.

[19] R. Khaldi and A. Guezane-Lakoud. On generalized nonlinear Euler-Bernoulli Beam type equations. *Acta Univ. Sapientiae, Math* **10** (1) (2018) 90–100.

[20] R. Khaldi and A. Guezane-Lakoud. Higher order fractional boundary value problem for mixed type derivatives. *J. Nonlinear Funct. Anal.* 2017:Article ID 30 (2017).

[21] A. A. Kilbas, H. Srivastava and J. Trujillo. *Theory and Applications Of Fractional Differential Equations* Elsevier, Amsterdam, 2006.

[22] V. Lakshmikantham, L. Wen and B. Zhang. *Theory of Differential Equations with Unbounded Delay.* Springer, New York, 1994.

[23] N. Mehmood and N. Ahmad. Existence results for fractional order boundary value problem with nonlocal non-separated type multi-point integral boundary conditions. AIMS Mathematics **1** (2019) 385–398.

[24] K. S. Miller and B. Ross. *An Introduction to the Fractional Calculus and Fractional Differential Equations.* Wiley, New York, 1993.

[25] N. Phuangthong, S. Ntouyas, J. Tariboon and K. Nonlaopon. Nonlocal Sequential Boundary Value Problems for Hilfer Type Fractional Integro-Differential Equations and Inclusions. *Mathematics* **9** (6) (2021).

[26] S. G. Samko, A. A. Kilbas and O. I. Marichev. *Fractional Integrals and Derivatives: Theory and Applications.* Gordon and Breach, Yverdon, 1993.

[27] H. Ye and R. Huang. On the nonlinear fractional differential equations with Caputo sequential fractional derivative. *Adv. Math. Phys.* **2015** (2015) 1–9.