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A New Feedback Control for Exponential and Strong Stability of Semi-Linear Systems with General Decay Estimates

M. Chqondi [1*], S. Chqondi [2], K. Tigma [3] and Y. Akdim [1]

[1] *Laboratory LAMA (Analyse Mathématique et Applications), Department of Mathematics and Informatics, Sidi Mohamed Ben Abdellah University, Faculty of Science, Dhar El Mahraz - FES, Morocco.*
[2] *Laboratory ISTM (Innovation in Sciences, Technologies and Modeling) Department of Physics, Faculty of Science, Chouaïb Doukkali University, El Jadida, Morocco.*
[3] *Higher Institute of Maritime Studies, Casablanca, Morocco.*

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Abstract: In this paper, to study the stabilization for the inhomogeneous nonlinear Schrödinger equation, we will explore the general form of semilinear control systems in Hilbert state space and apply the obtained results to the particular case of the nonlinear Schrödinger equation. We propose a new output feedback control approach that achieves strong and exponential stabilization if certain approximate observability assumptions are met. We demonstrate the existence and uniqueness of solutions and provide an estimate of convergence speed in the case of strong stabilization.

Keywords: *control systems; stabilization of systems by feedback; semilinear systems; exponential stability.*

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* Corresponding author: mailto:chminfo@gmail.com

1 Introduction

The inhomogeneous nonlinear Schrödinger equation is a natural occurrence in nonlinear optics when it comes to the propagation of laser beams. A preliminary laser beam can be sent to create a channel with reduced electron density to achieve stable high-power propagation in plasma. This ultimately reduces the non-linearity within the channel. Gill [1] and Tripathi and Liu [2] provide examples of this approach. In this scenario, the propagation of the beam can be explained by the equation

$$I\partial_t u + \Delta u + K(x)|u|^\alpha u = 0,$$

which represents the inhomogeneous non-linear Schrödinger equation and is a typical example of a semi-linear control system in a Hilbert space $\mathcal{H}$ given by the following evolution equation:

$$\begin{cases} y'(t) = Ay(t) + u(t)Ny(t), \\ y(0) = y_0 \in \mathcal{H}, \end{cases} \tag{1}$$

where A is an unbounded operator with domain $\mathcal{D}(A) \subset \mathcal{H}$ and generates a strongly continuous semigroup of contraction $(S(t))_{t\geq 0}$ on an infinite-dimensional real Hilbert space $\mathcal{H}$ (state space) whose norm and scalar product are denoted, respectively, by $\|.\|$ and $\langle .,. \rangle$, N is a nonlinear operator from $\mathcal{H}$ into $\mathcal{H}$, which is locally Lipschitz and sequentially continuous operator such that $N(0) = 0$, the control function $u(.)$ denotes the scalar control.
When trying to stabilize a control system, one of the main approaches is to look for a feedback control that can guarantee that the system is well-posed and that the solution converges to zero over time. This feedback control is usually represented by $u(y(t))$ and must be carefully chosen to achieve the desired stabilization. By formally computing the time rate of change of the energy $\frac{d}{dt}\|y(t)\|^2$ and using the fact that the semigroup is of contraction so that $\langle Ay, y\rangle \leq 0$ for all $y \in D(A)$, we get

$$\frac{d}{dt}\|y(t)\|^2 \leq 2u(t) < y(t); Ny(t) >; \forall t \in [0, T].$$

To make the energy nonincreasing, we consider the family of controls:

$$\text{for } r \geq 0, \quad u_r\left(y(t)\right) = -\frac{\langle y(t), Ny(t)\rangle \mid \langle y(t), Ny(t)\rangle \mid^r}{\|y(t)\|^r}, \quad \forall t \in [0, T]. \tag{2}$$

By using this control, we can guarantee the dissipation of energy while adhering to the following inequality:

$$\frac{d}{dt}\|y(t)\|^2 \leq -2\frac{|\langle y(t), Ny(t)\rangle|^{r+2}}{\|y(t)\|^r}; \forall t \in [0, T]. \tag{3}$$

In our control family, there is a particular case when $r = 0$, and it is called the quadratic feedback control $u_0(y(t)) = -\langle y(t), By(t)\rangle$. Various works have extensively studied this control to achieve weak or strong stabilizability. In Ball and Slemrod (see [3] and [4], p. 175), it has been shown that if $N = B$ is a compact linear operator and $S(t)$ is a semigroup of contractions such that

$$\langle BS(t)y, S(t)y\rangle = 0 \forall t \geq 0 \Longrightarrow y = 0, \tag{4}$$

then the feedback $u_0\left(y(t)\right)$ weakly stabilizes the system (1).
In Ouzahra (see [5], p. 511 and [6], p. 814), it has been established that if (4) is replaced by the following:

$$\int_0^T |\langle BS(t)y, S(t)y\rangle| dt \geq \delta \|y\|^2 \forall y \in H$$

(for some $T, \delta > 0$), then the control $u_0(y(t))$ strongly stabilizes the system (1). More precisely, the state satisfies the estimate

$$\|z(t)\|^2 = O\left(\frac{1}{t}\right), \text{ as } t \longrightarrow +\infty.$$

In Berrahmoune (see [7]), a strong stabilization result has been obtained using the control $u_0(t)$, and the following estimate $\|y(t)\| = O\left(\frac{1}{\sqrt{t}}\right)$ has been obtained.

In this paper, we study the strong and exponential stabilizability of the system (1) using the control (2) for all $r \geq 0$. By implementing control (2), we can enhance the estimate provided by $u_0\left(y(t)\right)$ in [3], [4], [5], [6] and [7]. This document is structured in the following manner. Section 2 demonstrates the existence and uniqueness of the solution in the semilinear case using control (2) for all $t \in [-2; +\infty[$. In Section 3, we explore strong stabilization and decay estimates, while Section 4 focuses on the exponential stabilization problem using the selected control $u_r(y(t))$ for all $t \in [-2; +\infty[$. In the last section, we give examples governed by the nonlinear Schroedinger and heat equations to illustrate our findings.

2 Well-Possedness

With the control (2), system (1) becomes

$$\begin{cases} y'(t) = Ay(t) + F_r(t, y(t)), \\ y(0) = y_0, \end{cases} \tag{5}$$

where

$$F_r(t, y(t)) = -\frac{\langle y(t), Ny(t)\rangle |\langle y(t), Ny(t)\rangle|^r}{\|y(t)\|^r} Ny(t) \text{ if } y \neq 0.$$

In this section, we will discuss the existence and uniqueness of the solution of the system (5).

Firstly, we need to demonstrate that the system's state is decreasing. To do so, integrate the inequality (3) over the interval [s, t], it follows that

$$\|y(t)\|^2 - \|y(s)\|^2 \leq -2\int_s^t \frac{|\langle y(t), Ny(t)\rangle|^{r+2}}{\|y(t)\|^r}, \quad \forall t \geq s \geq 0,$$

therefore $\|y(t)\| \leq \|y_0\|, \forall t \geq 0$.

Theorem 2.1 *Let A generate a semigroup of contractions $S(t)$, let N be a locally Lipschitz and sequentially continuous operator, then for all $r \in [-2, +\infty[$ and $y_0 \in \mathcal{H}$, the system (5) possesses a unique global mild solution $y(t)$ defined on the infinite interval $[0, +\infty[$, which is given by the following variation of constants formula:*

$$y(t) = S(t)y_0 - \int_0^t F_r(s, y(s))S(t-s)Ny(s)ds.$$

Proof. We will consider the system (5) and demonstrate that the map $F_r : y \mapsto F_r(t, y(t))$ is locally Lipschitz from $\mathcal{H}$ to $\mathcal{H}$. Let $x \in \mathcal{H}$ and let $R > 0$, $L_N > 0$ such that for all $z, y \in \mathcal{H}$ such that $\|x-y\| \leq R$ and $\|x-z\| \leq R$, we have $\|Nz - Ny\| \leq L_N\|z-y\|$. From the development below, it will be clear that we can suppose that $x = 0$ and $0 < \|y\| \leq \|z\|$.

Let us consider two functions: $f_1(y) = \frac{|\langle y,Ny\rangle|^r}{\|y\|^r}$ and $f_2(y) = \langle y, Ny\rangle Ny$. We can then conclude that

$$\begin{aligned}\|F_r(t,y) - F_r(t,z)\| &= \|f_1(y)f_2(y) - f_1(z)f_2(z)\| \\ &\leq L_N^r\|y\|^r\|f_2(y) - f_2(z)\| + L_N^2\|z\|^3\|f_1(y) - f_1(z)\|.\end{aligned}$$

It is easy to increase the value of $L_N^r\|y\|^r\|f_2(y) - f_2(z)\|$, in fact,

$$\begin{aligned}\|f_2(y) - f_2(z)\| &= \|\langle y, Ny\rangle Ny - \langle z, Nz\rangle Nz\| \\ &\leq (L_N\|y\| + 2L_N\|z\|)\, L_N\|y\|\|y-z\| \\ &\leq 3L_N^2\|y\|^2\|y-z\|.\end{aligned}$$

Therefore

$$L_N^r\|y\|^r\|f_2(y) - f_2(z)\| \leq 3L_N^{r+2}\|y\|^{r+2}\|y-z\|. \tag{6}$$

There are two cases to increase the value of $\|f_1(y) - f_1(z)\|$. This can be achieved by utilizing the real function $t \mapsto t^r$, which satisfies the following conditions.

If $r \geqslant 1$, then $|\|z\|^r - \|y\|^r| \leq r\|z\|^{r-1}|\|z\| - \|y\||$.
If $r < 1$, then $|\|z\|^r - \|y\|^r| \leq | r \mid \|y\|^{r-1}|\|z\| - \|y\||$.

Case 1 : if $r \geqslant 1$. In this case, we have

$$\|f_1(y) - f_1(z)\| \leq r\left|\frac{|\langle z, Nz\rangle|}{\|z\|}\right|^{r-1}\left|\frac{|\langle y, Ny\rangle|}{\|y\|} - \frac{|\langle z, Nz\rangle|}{\|z\|}\right|$$

and since

$$\begin{aligned}\left| \frac{\langle y, Ny\rangle}{\|y(t)\|} - \frac{\langle z, Nz\rangle}{\|z(t)\|} \right| &= \left| \frac{\|z(t)\|\langle y, Ny\rangle - \|y(t)\|\langle z, Nz\rangle}{\|y(t)\|\|z(t)\|} \right| \\ &\leq \frac{\|y\|(L_N\|y\|) \mid \|z\| - \|y\| \mid + \|y(t)\|(L_N\|z\| + L_N\|y\|)\, \|z-y\|}{\|y(t)\|\|z(t)\|} \\ &\leq \frac{L_N\|y\|^2 \mid \|z\| - \|y\| \mid + 2L_N\|y(t)\|^2\|z-y\|}{\|y(t)\|\|z(t)\|} \\ &\leq \frac{3L_N\|y\|}{\|z\|}\|z-y\|,\end{aligned}$$

we find

$$\begin{aligned}\|f_1(y) - f_1(z)\| &\leq r\left|\frac{|\langle z, Nz\rangle|}{\|z\|}\right|^{r-1}\frac{3L_N\|y\|}{\|z\|}\|z-y\| \\ &\leq r(L_N\|z\|)^{r-1}\frac{3L_N\|y\|}{\|z\|}\|z-y\| \\ &\leq \frac{3rL_N^r\|y\|^r}{\|z\|}\|z-y\|.\end{aligned}$$

Therefore

$$L_N^2\|z\|^3\|f_1(y)-f_1(z)\|\leq 3rL_N^{2+r}\|y\|^{r+2}\|z-y\|. \tag{7}$$

Based on the inequalities (6) and (7) presented above, we can conclude that

$$\|F_r(t,y)-F_r(t,z)\|\leq 3L_N^{r+2}\,(1+r)\,\|y_0\|^{r+2}\|y-z\|.$$

Case 2 : If $-2\leqslant r\leqslant 1$.

We have $\|f_1(y)-f_1(z)\|\leq \dfrac{3\mid r\mid L_N^r\|y\|^r}{\|z\|}\|z-y\|$, therefore

$$L_N^2\|z\|^3\|f_1(y)-f_1(z)\|\leq 3\mid r\mid L_N^{r+2}\|y\|^{r+2}\|z-y\|. \tag{8}$$

By utilizing the inequalities (6) and (8), we can come to the conclusion that

$$\|F_r(t,y)-F_r(t,z)\|\leq 3L_N^{r+2}(1+\mid r\mid)\|y_0\|^{r+2}\|y-z\|.$$

We have proven that for all $r\geqslant -2$, $F_r(t;y(t))$ satisfies a local Lipschitz condition in y, uniformly in t on bounded intervals. It follows (see [8], p.185) that the system (5) possesses a unique mild solution $y(t)$ defined on a maximal interval $[0,t_{\max}[$, which is given by the following variation of constants formula:

$$y(t)=S(t)y_0-\int_0^t F_r(s,y(s))S(t-s)Ny(s)ds.$$

To show that $t_{\max}=+\infty$, it is sufficient to prove that for each $T>0$, the mild solution $y(t)$ is bounded by a constant independent of T. To do this, we discuss two cases:

- For $y_0\in\mathcal{D}(A)$, the solution $y(t)$ is differentiable (see [8], p.189), and since $S(t)$ is a semigroup of contractions (so that A is dissipative), we can write

  $$\frac{d}{dt}\|y(t)\|^2\leq -2\frac{|\langle y(t),Ny(t)\rangle|^{r+2}}{\|y(t)\|^r};\forall t\in[0,T].$$

  Integrate this last inequality over the interval [s, t], it follows that

  $$\|y(t)\|^2-\|y(s)\|^2\leq -2\int_s^t\frac{|\langle y(t),Ny(t)\rangle|^{r+2}}{\|y(t)\|^r},\forall t\geq s\geq 0.$$

  Therefore,

  $$\|y(t)\|\leq\|y_0\|\,,\forall t\geq 0.$$

- If $y_0\notin D(A)$, we can find a sequence $(y_0^n)_n$ of elements in $D(A)$ converging to y_0 in $\mathcal{H}$ since $\overline{D(A)}=\mathcal{H}$. For all $t\in[0,T]$ and all integer n, we know from the first case that $\|y^n(t)\|\leq\|y_0^n\|$. We conclude that $\|y(t)\|\leq\|y_0\|\,,\quad\forall t\in[0,t_{\max}[$, hence $t_{\max}=+\infty$.

3 Strong Stabilization and Decay Estimate

Proposition 3.1 *Let A generate a semigroup of contractions $S(t)$ and N be locally Lipschitz. Then, for all r $\geqslant 0$, the state of the system (4) satisfies the decay estimate as $t\to+\infty$,*

$$\int_0^T|\langle NS(s)y(t),S(s)y(t)\rangle|\mathrm{d}s=O\left(\|y(\tau)\|^{\frac{r}{r+2}}\left(\int_0^T\frac{|\langle y(s+\tau),Ny(s+\tau)+c\rangle|^{r+2}}{\|y(s+\tau)\|^r}\,\mathrm{d}s\right)^{\frac{1}{r+2}}\right). \tag{9}$$

Proof. Using Proposition 3.1, we deduce that the system (9) admits a unique global mild solution given by the following formula of variation of constants:

$$y(t) = S(t)y_0 - \int_0^t \frac{\langle y(t), Ny(t)\rangle \mid \langle y(t), Ny(t)\rangle \mid^r}{\|y(s)\|^r} S(t-s)Ny(s)\mathrm{d}s,$$

and due to the fact that $S(t)$ is a semigroup of contractions, for all $t \in [0,T]$,

$$\begin{aligned}\|y(t) - S(t)y_0\| &\leqslant \int_0^T \frac{|\langle y(s), Ny(s)\rangle|^{r+1}}{\|y(s)\|^r}(\|Ny(s))\|\mathrm{d}s \\ &\leqslant L_N \int_0^T \frac{|\langle y(s), Ny(s)\rangle|^{r+1}}{\|y(s)\|^r}\|y(s)\| \ \mathrm{d}s.\end{aligned}$$

We can apply the Hölder inequality

$$p = \frac{r+2}{r+1} \quad , q = r+2 \quad \left(\frac{1}{p} + \frac{1}{q} = 1\right), \quad r \in]-1+\infty[$$

$$\begin{aligned}\|y(t) - S(t)y_0\| &\leqslant L_N T^{\frac{1}{r+2}} \left(\int_0^T \left(\frac{|\langle y(s), Ny(s)\rangle|^{r+1}}{\|y(s)\|^r}\|y(s)\|\right)^{\frac{r+2}{r+1}} \mathrm{d}s\right)^{\frac{r+1}{r+2}} \\ &\leqslant L_N T^{\frac{1}{r+2}} \left(\int_0^T \frac{|\langle y(s), Ny(s)\rangle|^{r+2}}{\|y(s)\|^r}\|y(s)\|^{\frac{2}{r+1}}\mathrm{d}s\right)^{\frac{r+1}{r+2}}.\end{aligned}$$

Since $r \geqslant 0$, we have $\frac{2}{r+1} > 0$ and the fact that $\|y(t)\| \leq \|y_0\|$. We get $\|y(t)\|^{\frac{2}{r+1}} \leq \|y_0\|^{\frac{2}{r+1}}$. So

$$\|y(t) - S(t)y_0\| \leqslant L_N T^{\frac{1}{r+2}} \|y_0\|^{\frac{2}{r+2}} \left(\int_0^T \frac{|\langle y(s), Ny(s) +\rangle|^{r+2}}{\|y(s)\|^r}\mathrm{d}s\right)^{\frac{r+1}{r+2}}. \tag{9}$$

From the relation

$$\langle NS(t)y_0, S(t)y_0\rangle = \langle NS(t)y_0, S(t)y_0 - y(t)\rangle + \langle NS(t)y_0 - Ny(t), y(t)\rangle + \langle Ny(t), y(t)\rangle,$$

when using $\|y(t)\| \leq \|y(0)\|, \forall t \in [0, t_{\max}[$, the fact that $S(t)$ is a semigroup of contraction, N is locally Lipschitz, and Schwartz's inequality, it comes

$$|\langle NS(s)y_0, S(s)y_0\rangle| \leq 2L_N \|y_0\| \, \|y(t)) - S(t)y_0\| + | \langle Ny(s), y(s)\rangle | . \tag{10}$$

Using (9), we have

$$|\langle NS(s)y_0, S(s)y_0\rangle| \leq 2L_N^2 T^{\frac{1}{r+2}} \|y_0\|^{\frac{r+4}{r+2}} \left(\int_0^T \frac{|\langle y(s), Ny(s)\rangle|^{r+2}}{\|y(s)\|^r}\mathrm{d}s\right)^{\frac{r+1}{r+2}} + | \langle Ny(s), y(s)\rangle | . \tag{11}$$

Replacing y_0 by $y(\tau)$ in (11), we get

$$\begin{aligned}&|\langle NS(s+\tau)y(\tau), S(s+\tau)y(\tau)\rangle| \\ &\leq 2L_N^2 T^{\frac{1}{r+2}} \|y(\tau)\|^{\frac{r+4}{r+2}} \left(\int_0^T \frac{|\langle y(s+\tau), Ny(s+\tau) + c\rangle|^{r+2}}{\|y(s+\tau)\|^r}\mathrm{d}s\right)^{\frac{r+1}{r+2}} \\ &+ |\langle Ny(s+\tau), y(s+\tau)\rangle |.\end{aligned}$$

Integrate the last inequality over the interval $[0,T]$

$$\begin{aligned}&\int_0^T |\langle NS(s+\tau)y(\tau), S(s+\tau)y(\tau)\rangle| \ \mathrm{d}s \\ &\leq \int_0^T \langle Ny(s+\tau), y(s+\tau)\rangle \\ &\mathrm{d}s + 2L_N^2 T^{\frac{1}{r+2}+1}\|y(\tau)\|^{\frac{r+4}{r+2}} \left(\int_0^T \frac{|\langle y(s+\tau), Ny(s+\tau)\rangle|^{r+2}}{\|y(s+\tau)\|^r}\mathrm{d}s\right)^{\frac{r+1}{r+2}}.\end{aligned} \tag{12}$$

Due to the fact that $\|y(s+\tau)\| \leq \|y(\tau)\| \, \forall t \geq 0$, and Hölder's inequality

$$\int_0^T \langle Ny(s+\tau), y(s+\tau)\rangle \ \mathrm{d}s \leq T^{\frac{r+1}{r+2}}\|y(\tau)\|^{\frac{r}{r+2}}\left(\int_0^T \frac{\langle Ny(s+\tau), y(s+\tau)\rangle^{r+2}}{\|y(s+\tau)\|^r} \ \mathrm{d}s\right)^{\frac{1}{r+2}},$$

(12) becomes

$$\begin{aligned}&\int_0^T |\langle NS(s+\tau)y(\tau), S(s+\tau)y(\tau)\rangle| \ \mathrm{d}s \\ &\leq 2L_N^2 T^{\frac{1}{r+2}+1}\|y(\tau)\|^{\frac{r+4}{r+2}} \left(\int_0^T \frac{|\langle y(s+\tau), Ny(s+\tau)\rangle|^{r+2}}{\|y(s+\tau)\|^r}\mathrm{d}s\right)^{\frac{r+1}{r+2}} \\ &\quad + T^{\frac{r+1}{r+2}}\|y(\tau)\|^{\frac{r}{r+2}}\left(\int_0^T \frac{\langle Ny(s+\tau), y(s+\tau)\rangle^{r+2}}{\|y(s+\tau)\|^r} \ \mathrm{d}s\right)^{\frac{1}{r+2}}.\end{aligned} \tag{13}$$

We have

$$\left(\int_0^T \frac{|\langle y(s+\tau), Ny(s+\tau)\rangle|^{r+2}}{\|y(s+\tau)\|^r} \ \mathrm{d}s\right)^{\frac{r+1}{r+2}} = \left(\int_0^T \frac{|\langle y(s+\tau), Ny(s+\tau)\rangle|^{r+2}}{\|y(s+\tau)\|^r} \ \mathrm{d}s\right)^{\frac{1}{r+2}+\frac{r}{r+2}}.$$

And by Schwartz's inequality, it comes

$$\begin{aligned}&\int_0^T \frac{|\langle y(s+\tau), Ny(s+\tau)\rangle|^{r+2}}{\|y(s+\tau)\|^r} \ \mathrm{d}s \leq TL_N^{r+2}\|y(\tau)\|^{r+4} \\ &\left(\int_0^T \frac{|\langle y(s+\tau), Ny(s+\tau)\rangle|^{r+2}}{\|y(s+\tau)\|^r} \ \mathrm{d}s\right)^{\frac{r+1}{r+2}} \\ &\qquad \leq T^{\frac{r}{r+2}} L_N^{\frac{r}{r+2}} |y(\tau)\|^{\frac{r^2+4r}{r+2}} \left(\int_0^T \frac{|\langle y(s+\tau), Ny(s+\tau)\rangle|^{r+2}}{\|y(s+\tau)\|^r} \ \mathrm{d}s\right)^{\frac{1}{r+2}}.\end{aligned}$$

Finally, using (13), we have

$$\begin{aligned}&\int_0^T |\langle NS(s+\tau)y(\tau), S(s+\tau)y(\tau)\rangle| \mathrm{d}s \\ &\qquad \leq \mathcal{M}_{\|y_0\|}\|y(\tau)\|^{\frac{r}{r+2}}\left(\int_0^T \frac{|\langle y(s+\tau), Ny(s+\tau)\rangle|^{r+2}}{\|y(s+\tau)\|^r} \ \mathrm{d}s\right)^{\frac{1}{r+2}},\end{aligned} \tag{14}$$

where $\mathcal{M}_{\|y_0\|} = T^{\frac{r+1}{r+2}}\left(2L_N^2 T L_N^{\frac{r}{r+2}}\|y(\tau)\|^{r+2}+1\right)$.

Theorem 3.1 *Let A generate a C_0-semigroup $S_u(t)$, and suppose that the following conditions hold:*
1. $S_u(t)$ is a contraction semigroup;
2. there exist $\delta, T > 0$ such that

$$\int_0^T |\langle NSs)y(t), S(s)y(t)\rangle|\, ds \geq \delta \|y(t)\|^2, \quad \forall y \in H. \tag{15}$$

Then the feedback (2) for all $r \geqslant 0$, strongly stabilizes the system (1) with the following decay estimate:

$$\|y(t)\| = O\left(t^{-\frac{r+2}{4}}\right) \text{ as } t \to +\infty.$$

Proof. Let us consider the sequence $s_k = \|y(kT)\|^2$, $k \in IN$.
Integrating the inequality (3) over the interval $[kT, (k+1)T]$, we get

$$\int_{KT}^{(k+1)T} \frac{d}{dt}\|y(t)\|^2 dt \leq -2\int_{KT}^{(k+1)T)} \frac{(\langle Ny(t), y(t)\rangle)^{r+2}}{\|y(t)\|^r} dt$$

$$\|y((K+1)T)\|^2 - \|y(KT)\|^2 \leq -2\int_{KT}^{(k+1)T)} \frac{(\langle Ny(t), y(t)\rangle)^{2+r}}{\|y(t)\|^r} dt.$$

Using now the estimate (14), we deduce that

$$\int_0^T |\langle NS(s+\tau)y(\tau) + c, S(s+\tau)y(\tau)\rangle| \text{ d}s$$

$$\leq \left(\mathcal{M}_{\|y_0\|}\|y(\tau)\|^{\frac{r}{2+r}}\right)\left(\int_\tau^{T+\tau} \frac{|\langle Ny(s)+c, y(s)\rangle|^{2+r}}{\|y(s)\|^r} \text{ d}s\right)^{\frac{1}{2+r}}$$

$$\|y((K+1)T)\|^2 - \|y(KT)\|^2 \leq \frac{-2}{\mathcal{M}_{\|y_0\|}\|y(\tau)\|^{\frac{r}{2+r}}} \int_0^T |\langle NS(s+\tau)y(\tau), S(s+\tau)y(\tau)\rangle| \text{ d}s$$

and according to the inequality (15), we have

$$\|y((K+1)T)\|^2 - \|y(KT)\|^2 \leq \frac{-2\rho}{\mathcal{M}_{\|y_0\|}}\|y(KT)\|^{\frac{-r}{2+r}}\|y(KT)\|^2$$

$$\|y((K+1)T)\|^2 - \|y(KT)\|^2 \leq \frac{-2\rho}{\mathcal{M}_{\|y_0\|}}\|y(KT)\|^{1+\frac{2}{2+r}}.$$

Letting $s_k = \|y(kT)\|^2, k \in \mathbb{N}$, the last inequality can be written as

$$s_{k+1} \leq s_k - \frac{2\rho}{\mathcal{M}_{\|y_0\|}} s_k^{1+\frac{2}{2+r}}, \quad \forall k \geq 0.$$

Using the fact that $t \mapsto \|y(t)\|$ is a decreasing function on $[0, +\infty[$, we get

$$s_{k+1} \leq s_k - \frac{2\rho}{\mathcal{M}_{\|y_0\|}} s_{k+1}^{1+\frac{2}{2+r}}, \quad \forall k \geq 0.$$

$$s_{k+1} + \frac{2\rho}{\mathcal{M}_{\|y_0\|}} s_{k+1}^{1+\frac{2}{2+r}} \leq s_k, \quad \forall k \geq 0.$$

The last inequality can be written as follows: $s_{k+1} + Cs_{k+1}^{2+\alpha} \leq s_k, \quad \forall k \geq 0,$
where $C = \frac{2\rho}{\mathcal{M}_{\|y_0\|}} > 0$ and $\alpha = \frac{-r}{r+2} > -1 \quad \forall r \in \mathbb{R}/(-1,-2)$.
Now, to obtain the decay rate for the solutions of (1), we recall the following lemma, see [9].

Lemma 3.1 *(Lasiecka and Tataru, 1993) Let $(s_k)_{k\geq 0}$ be a sequence of positive real numbers satisfying the relation $s_{k+1} + Cs_{k+1}^{2+\alpha} \leq s_k, \quad \forall k \geq 0$, where $C > 0$ and $\alpha > -1$ are constants. Then there exists a positive constant M_2 (depending on α and C) such that $s_k \leq \frac{M_2}{(k+1)^{\frac{1}{\alpha+1}}}, \quad k \geq 0$.*

So, from the lemma (Lasiecka & Tataru, 1993), we have $s_k \leq \frac{M_2}{(k+1)^{\frac{r+2}{2}}}, \quad k \geq 0$.
For $k = E\left(\frac{t}{T}\right), \left(E\left(\frac{t}{T}\right)\right.$ designed the integer part of $\left.\frac{t}{T}\right)$, we obtain $s_k \leq \frac{M_3}{t}$, $(M_3 > 0)$, which gives $\|y(t)\|^2 \leq \frac{M_3}{t^{\frac{r+2}{2}}}$.
Hence, $\|y(t)\| = \mathbf{O}\left(t^{-\frac{r+2}{4}}\right)$ as $t \to +\infty$.

Remark 3.1 *For $r = 0$, we find the estimate guaranteed by $u_0(t)$ in [5] and [6].*

4 Exponential Stabilization

Theorem 4.1 *Let A generate a C_0-semigroup $S(t)$, and suppose that the following conditions hold:*
1. $S(t)$ is a contraction semigroup;
2. there exist $\delta, T > 0$ such that

$$\int_0^T |\langle NS(s)y(t), S(s)y(t)\rangle|\, ds \geq \delta \|y(t)\|^{2+\frac{r}{r+2}}, \quad \forall y \in \mathcal{H}. \tag{16}$$

Then the feedback (2) for all $r \geqslant 0$, exponentially stabilizes the system (5).
More precisely, there exists $\beta > 0$ such that $\|y(t)\| \leq e^{-\beta} \|y_0\| e^{-\frac{\beta}{T}(t)}, \quad \forall t > 0$.

Proof. From

$$\|y((K+1)T)\|^2 - \|y(KT)\|^2 \leq \frac{-2}{\mathcal{M}_{\|y_0\|} \|y(\tau)\|^{\frac{r}{2+r}}} \int_0^T |\langle NS(s+\tau)y(\tau), S(s+\tau)y(\tau)\rangle| \mathrm{d}s$$

and according to the inequality (16), we have

$$\|y((K+1)T)\|^2 - \|y(KT)\|^2 \leq \frac{-2\rho}{\mathcal{M}_{\|y_0\|}} \|y(KT)\|^{\frac{-r}{2+r}} \|y(KT)\|^{2+\frac{r}{2+r}}$$

$$\|y((K+1)T)\|^2 - \|y(KT)\|^2 \leq \frac{-2\delta}{\mathcal{M}_{\|y_0\|}} \|y(KT)\|^2. \tag{17}$$

Letting $s_k = \|y(kT)\|^2, k \in \mathbb{N}$, the inequality (17) can be written as

$$s_{k+1} - s_k \leq \frac{-2\delta}{\mathcal{M}_{\|y_0\|}} s_k, \quad \forall k \geq 0,$$

$$s_{k+1} \leq Cs_k, \quad \forall k \geq 0,$$

where $C = \left(1 - \frac{2\delta}{\mathcal{M}_{\|y_0\|}}\right) < 1$, which gives $s_k \leq e^{-k \ln \frac{1}{C}} s_0$, i.e, $\|y(kT)\|^2 e^{-k \ln \frac{1}{C}} \|y_0\|$.
For $k = E\left(\frac{t}{T}\right), \left(E\left(\frac{t}{T}\right)\right.$ designed the integer part of $\frac{t}{T}$), and using the fact that $E\left(\frac{t}{T}\right) T \leq t$ and $t \mapsto \|y(t)\|$ is a decreasing function on $[0, +\infty[$,
we get $\|y(t)\| \leq e^{-\frac{\ln\left(\frac{1}{C}\right)}{2}} \|y_0\| e^{-\frac{\ln\left(\frac{1}{C}\right)}{2T} t}$ for all $t \geq 0$,
$\|y(t)\| \leq e^{-\beta} \|y_0\| e^{-\frac{\beta}{T} t}$ for all $t \geq 0$, where $\beta = \frac{\ln\left(\frac{1}{C}\right)}{2} > 0$.

5 Applications

Example 5.1 Let $\Omega \subset \mathbb{R}^N, H = L^2(\Omega)$. This example will study the nonlinear Schrödinger equation in $\Omega \subset \mathbb{R}^N$. Let us consider the system defined on an open and bounded domain Ω with $\mathcal{C}^\infty$ boundary $\partial\Omega$ by the equation

$$\begin{cases} i\partial_t \psi = \Delta\psi + K(x)|\psi|^\alpha \psi, & \text{in } \Omega \times]0, \infty[, \quad \alpha > 0, \\ \psi(\cdot, t) = 0, & \text{on } \partial\Omega \times]0, \infty[, \\ \psi(\cdot, 0) = \varphi. & \end{cases} \tag{18}$$

For $\alpha = 2$, it is the Gross-Pitaevskii equation describing the evolution of a Bose-Einstein condensate.

The equation (18) has garnered much interest recently. Bergé conducted a formal study on the stability condition of soliton solutions in [10]. Towers-Malomed, in [11], discovered that a specific type of time-dependent nonlinear medium generates fully stable beams through variational approximation and direct simulations. Merle in [12] and Raphaël-Szeftel in [13] studied (18) for $k_1 < K(x) < k_2$ with $k_1, k_2 > 0$. Fibich-Wang in [14] investigated (18) with $K(x) := K(\epsilon|x|)$, where $\epsilon > 0$ is small and $K \in C^4\left(\mathbb{R}^d\right) \cap L^\infty\left(\mathbb{R}^d\right)$.

In this example, we will study the stability of (18) by considering $K(x) = u_r(\psi(x,t))$, where

$$u_r(\psi(x,t)) = -\frac{\langle \psi(x,t), N\psi(x,t)\rangle |\langle \psi(x,t), N\psi(x,t)\rangle|^r}{\|\psi(x,t)\|^r}; \quad r \geqslant 0.$$

We are considering the control operator $N\psi(x,t) = |\psi(x,t)|^\alpha \psi(x,t)$ in order to express the system (18) in the following form:

$$\begin{cases} i\frac{\partial \psi(x,t)}{\partial t} = A\psi(x,t) + u_r(\psi(x,t))N\psi(x,t), & \text{in } \Omega \times]0, \infty[, \\ \psi(x,t) = 0, & \text{on } \partial\Omega \times]0, \infty[, \\ \psi(x,0) = \varphi. & \end{cases} \tag{19}$$

Here, the state space $H = L^2(\Omega)$ is endowed with its natural complex inner product.

- The operator A is defined by $A\psi(x,t) = -i\Delta\psi(x,t), \forall \psi(x,t) \in \mathcal{D}(A)$, where $\mathcal{D}(A) = H^2(\Omega) \cap H_0^1(\Omega)$.
  It is well known that the operator $A = -i\Delta$ with domain $\mathcal{D}(A)$ generates a semi-group of isometries $\{\mathcal{T}(t)\}_{t \in \mathbb{R}} \in \mathcal{L}\left((D(A))^*\right)$.
- The operator of control is given by $N\psi = |\psi|^\alpha \psi$.
  We pose $N(\psi(x,t)) = f(t, \psi(x,t)) = \frac{\psi(x,t)}{|\psi(x,t)|} f(t, |\psi(x,t)|) \quad \forall \psi \in \mathcal{D}(A), \psi \neq 0$.
  We deduce that

$$|\psi_1||\psi_2|(f(t,\psi_1) - f(t,\psi_2)|$$

$$= |\psi_1|\psi_2|[f(t,|\psi_1|) - f(t,|\psi_2|)] + [\psi_1(|\psi_2| - |\psi_1| + |\psi_1|(\psi_1 - \psi_2))]\, f(t,|\psi_2|)|.$$

We have $f(t,|\psi_1|) = |\psi_1|^{\alpha+1}$, so for every $K > 0$ and $0 < \psi_1, \psi 2 \leqslant K$, there exists $L_K < \infty$ such that

$$|f(t,|\psi_1|) - f(t,|\psi_2|)| \leq L_K|\psi_1 - \psi_2|.$$

So

$$\begin{aligned}|\psi_1\|\psi_2\|f(t,\psi_1) - f(t,\psi_2)| &\leq |\psi_1\|\psi_2\|f(t,|\psi_1|) - f(t,|\psi_2|)| + 2|\psi_1||\psi_1 - \psi_2\|f(t,|\psi_2|)| \\ &\leq 3|\psi_1\|\psi_2|L_K|\psi_1 - \psi_2|,\end{aligned}$$

$$\begin{aligned}&|f(t,\psi_1) - f(t,\psi_2)| \leq 3L_K|\psi_1 - \psi_2|,\\ &||\psi_1|^\alpha\, \psi_1 - |\psi_2|^\alpha \psi_2| \leq L_K|\psi_1 - \psi_2\,|\,.\end{aligned}$$

Therefore N is a nonlinear and locally Lipschitz operator such that $N(0) = 0$, applying Theorem 2.1, we deduce that the system (18) possesses a unique global mild solution $\psi(x,t))$ defined on the infinite interval $[0,+\infty[$, which is given by the following variation of constants formula:

$$\psi(x,t) = \mathcal{T}(t)\varphi + i\int_0^t \mathcal{T}(t-s)u_r(\psi(x,s))N\psi(x,s)ds, \quad \forall t \geq 0.$$

Let us show that the condition of Theorem 3.1 is verified,

$$\begin{aligned}\langle N\mathcal{T}(t)\psi(x,t), \mathcal{T}(t)\psi(x,t)\rangle_H &= \langle |\mathcal{T}(t)\psi(x,t)|^\alpha \mathcal{T}(t)\psi(x,t), \mathcal{T}(t)\psi(x,t)\rangle_H \\ &= |\mathcal{T}(t)\psi(x,t)|^\alpha \langle \mathcal{T}(t)\psi(x,t), \mathcal{T}(t)\psi(x,t)\rangle_H \\ &= |\mathcal{T}(t)\psi(x,t)|^\alpha |\mathcal{T}(t)\psi(x,t)|^2 \\ &= |\mathcal{T}(t)\psi(x,t)|^{\alpha+2}.\end{aligned}$$

We know that $|\mathcal{T}(t)\psi(x,t)| = |\psi(x,t)|$, therefore,

$$\int_0^T \langle N\mathcal{T}(t)\psi(x,t), \mathcal{T}(t)\psi(x,t)\rangle_H dt = \int_0^T |\psi(x,t)|^{\alpha+2} dt.$$

Applying the Holder inequality, we obtain

$$\begin{aligned}\int_0^T |\psi(x,t)|^2 \,\mathrm{d}t &\leq T^{\frac{\alpha}{\alpha+2}} \left(\int_0^T |\psi(x,t)|^{\alpha+2}\,\mathrm{d}t\right)^{\frac{2}{\alpha+2}} \\ &\leq T^{\frac{\alpha}{\alpha+2}} \left(\int_0^T |\psi(x,t)|^{\alpha+2}\,\mathrm{d}t\right)^{\frac{2}{\alpha+2}} \\ \sqrt{\int_0^T |\psi(x,t)|_H^2\,\mathrm{d}t} &\leq T^{\frac{\alpha}{2\alpha+4}} \left(\int_0^T |\psi(x,t)|^{\alpha+2}\,\mathrm{d}t\right)^{\frac{1}{\alpha+2}} \\ T^{\frac{-\alpha}{2}}\|\psi(x,t)\|^{\alpha+2}_{L^2(]0,T[;H)} &\leq \int_0^T |\psi(x,t)|^{\alpha+2}\,\mathrm{d}t \end{aligned}$$

$$\int_0^T \langle N\mathcal{T}(t)\psi(x,t), \mathcal{T}(t)\psi(x,t)\rangle_H \,\mathrm{d}t \geqslant T^{\frac{-\alpha}{2}}\|\psi(x,t)\|^{\alpha+2}_{L^2(]0,T[;H)}. \tag{20}$$

- For $\alpha = 0$, the system (18) becomes

$$\begin{cases} i\partial_t \psi = \Delta\psi + K(x)\psi, & \text{in } \Omega\times]0,\infty[, \\ \psi(\cdot,t) = 0, & \text{on } \partial\Omega\times]0,\infty[, \\ \psi(\cdot,0) = \varphi. \end{cases} \tag{21}$$

Using (20), we can deduce that for $\delta = 1$ and $T > 0$, we have the inequality

$$\int_0^T \langle N\mathcal{T}(t)\psi(x,t), \mathcal{T}(t)\psi(x,t)\rangle_H \, \mathrm{d}t \geqslant \delta \|\psi(x,t)\|^2_{L^2(]0,T[;H)}.$$

Based on the verification of condition (15), we can apply Theorem 3.1 to confirm that the feedback $u_r(\psi(x,t))$ stabilizes system (21) for all $r \geqslant 0$ with the following decay estimate. Moreover, we can estimate solution decay using the following formula:

$$\|\psi(x,t)\| = O\left(t^{-\frac{r+2}{4}}\right) \text{ as } t \to +\infty.$$

- For $0 < \alpha < 1$, when we set $r = \frac{2\alpha}{1-\alpha}$, we ensure that $r > 0$. Solving for α using the equation $\alpha = \frac{r}{r+2}$ and for $\delta = \frac{1}{\sqrt{T^\alpha}}$, we can then substitute this value into equation (20) to obtain the modified expression

$$\int_0^T \langle N\mathcal{T}(t)\psi(x,t), \mathcal{T}(t)\psi(x,t)\rangle_H \, \mathrm{d}t \geqslant \delta \|\psi(x,t)\|^{2+\frac{r}{r+2}}.$$

Based on the verification of condition (16), we can apply Theorem 4.1 to confirm that the feedback $u_r(\psi(x,t))$ exponentially stabilizes the system (20) for $0 < \alpha < 1$ and $r = \frac{2\alpha}{1-\alpha}$.

More precisely, there exists $\beta > 0$ such that $\|\psi(x,t)\| \leq e^{-\beta} \|\varphi\| \, e^{-\frac{\beta}{T}(t)}, \quad \forall t > 0,$ where $\beta = \frac{\ln\left(\frac{1}{C}\right)}{2} > 0$, $C = \left(1 - \frac{2\delta}{M_{|\varphi|}}\right) < 1$,
$\mathcal{M}_{\|\varphi\|} = T^{\frac{\alpha+1}{2}+1}\left(2T\,(\alpha+1)^{\frac{3+5\alpha}{2}} + \mid \varphi \mid^{\frac{\alpha(3+5\alpha)}{2}+\frac{2}{1-\alpha}} + 1\right).$

6 Conclusion

This paper introduces a new type of control to enhance the stability of a semilinear system. The suggested control results in strong and exponential stability of the closed-loop system, particularly the inhomogeneous nonlinear Schrödinger equation. Moreover, an estimate of the decay can be achieved through approximate observation assumptions depending on r. Therefore, implementing the new control results in a significantly faster response time than quadratic control. This research raises questions about applying the same family of controls to stabilize inhomogeneous semi-linear systems in physics, including the non-homogeneous case considered in [15].

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