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A New Efficient Step-Size in Karmarkar's Projective Interior Point Method for Optimization Problems

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Abstract: In this paper, we are concerned with the optimization problem by using the logarithmic penalty method via new upper bound and lower bound functions in the Karmarkar's algorithm to find the solution. Then, we establish the direction by Newton's method. Also, we propose a new approach based on new upper bound and lower bound functions to determine the step size. To lend further support to our theoretical results, a simulation study is carried out to illustrate the good accuracy of the studied method.

Keywords: *Karmarkar's projective method; logarithmic barrier method; upper and lower bound functions; averaging of perturbations; step size; mathematical modeling; interior point methods.*

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1 Introduction

The appearance, rapid evolution and success of interior point methods since their revival by Karmarkar (1984) in the field of linear optimization problems, have prompted researchers around the world to develop a whole arsenal of methods (software) allowing to properly deal with several classes of problems once considered difficult to solve, including nonlinear programming, semi-definite programming, etc.

Linear optimization is a general mathematical framework for modelling and solving some optimization problems and it appears in many areas of applications such as agriculture, finance, economics, geometric problems and optimal control.

Mathematically, the problem is to optimize a linear function under linear constraints on the variables.

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Historically, linear programming has been developed by George Bernard Danzig since 1947.

Our work concerns the linear optimization (LO) problem which has become a much coveted research topic since the revival of interior point methods from new investigations brought by Karmarkar (1984) and others.

These methods are recognized as formidable competitors of the famous simplex method. Their major drawback (at about 80% of the computational value) is the calculation of the projection which dominates the cost of the iteration.

Our work is devoted to an approach inspired by the interior point logarithmic barrier method, and to avoid the calculation of the projection in Karmarkar's projective method, we propose an original procedure for the calculation of the step size based on the idea of the upper and lower bound functions and we introduce new functions. This study is supported by interesting numerical tests.

To our knowledge, our new upper and lower bound functions have not been studied in the linear optimization literature. These approximate functions have the advantage that they allow computing the step size easily and without consuming much time, contrarily to the line search method, which is time-consuming and expensive to identify the step size. For this, our objective is to optimize a linear problem based on prior efforts, and we propose a straightforward and effective logarithmic barrier method based on new upper and lower bound functions.

The rest of the paper is built as follows. We first present in Section 2 of our paper, a linear optimization problem formulation. We introduce in Section 3, a brief description of the algorithm. In Sections 4 and 5, we establish new upper and lower bound functions and the algorithm. Two lemmas are proved in Section 6 to show the convergence results. A simulation study is carried out to show the good behaviour of our approach in Section 7. Finally, a conclusion is summarized in the last section.

2 Posing of the Problem

We consider the following linear optimization problem:

$$(ka)\begin{cases} \min\langle c, x\rangle = 0, \\ Ax = 0, \\ x \geq 0, \\ x \neq 0, \end{cases}$$

which was applied to the Karmarkar algorithm. Here, $c \in \mathbb{R}^n$ and A is an $m \times n$ matrix of *rank* m.

In all that follows, the following conventions are adopted: the vector $e \in \mathbb{R}^n$ is the vector whose components are all equal to 1, given a vector $x \in \mathbb{R}^n$, X is the diagonal matrix whose diagonal elements are the components of x (*i.e.*, $X = diag\{x\}$).

The following assumptions are made:

1. $Ax = 0$ and $x \geq 0 \Rightarrow \langle c, x\rangle \geq 0$.
2. We know a point $x > 0$ such that $Ax = 0$ and $\langle c, x\rangle > 0$.
3. We know that the problem (ka) has solutions. We put

$$C = \{x : Ax = 0, x \geq 0, x \neq 0\}.$$

If $\bar{x}$ is a solution of (ka), then $k\bar{x}$ with $k > 0$ is also a solution.

We can thus proceed to a normalization of x and consider, for example, the following linear problem:

$$(pk)\begin{cases} \min c^t x = 0, \\ Ax = 0, \\ \langle e, x\rangle = n, x \geq 0. \end{cases}$$

We note that the set of optimal solutions of this problem is a convex polyhedron contained in the relative boundary of the set of feasible points.

The potential function.

The convergence of the algorithm is based on the following function, called the "multiplicative potential function", defined for all $x \in C, x > 0$, by

$$f(x) = \frac{\langle c, x\rangle^n}{\prod\limits_{i=1}^{n} x_i},$$

which we extend by semi-continuity on C.

One can also consider the function called the "logarithmic potential function" defined by

$$q(x) = \ln f(x) = n\ln(\langle c, x\rangle) - \sum_{i=1}^{n} \ln(x_i).$$

The function f has the following properties:

1) $0 < f(x) < +\infty$ if $x > 0$ and $Ax = 0$.

2) $f(x) = +\infty$ if x belongs to the relative boundary of C without being a solution of (ka).

3) $f(x) = 0$ if x is a solution of (ka) or if $x = 0$.

4) $f(kx) = f(x)$ for all $x \in C$ and all $k > 0$.

Thus, problem (ka) consists of finding the optimal solutions to the problem

$$(km)\begin{cases} \min f(x) = 0, \\ Ax = 0, \\ x \geq 0, x \neq 0. \end{cases}$$

3 Description of the Algorithm

Starting from the point $x \in C$ which is known, the Karmarkar algorithm is a descent method which generates, due to the barrier character of the objective function f, a sequence of points all contained in the relative interior of C, hence the name of the method of interior points. We will describe the transition from the initial iterate x to the next iterate $\tilde{x}$.

It is assumed that the iterate $\tilde{x}$ verifies $\tilde{x} > 0$ and $A\tilde{x} = 0$.

3.1 Normalisation

We normalize x by the relation

$$x = \sqrt{\frac{n}{\langle x, x\rangle}}x$$

so that $\langle x, x\rangle = n$.

3.2 Direction of descent

It is easy to see that we have

$$\frac{f(\tilde{x})}{f(x)} = g(z)$$

with

$$z = X^{-1}\tilde{x}, g(z) = \frac{\langle b, z\rangle^n}{\prod_{i=1}^{n} z_i} \text{ and } b = \frac{1}{\langle c, x\rangle} Xc.$$

The conditions $A\tilde{x} = 0,\ \tilde{x} \geq 0$ and $\tilde{x} \neq 0$ transpose to

$$AXz = 0, z \geq 0 \text{ and } z \neq 0.$$

Let $B = AX$. Problem (km) is equivalent to the problem

$$(kmz) \begin{cases} \min g(z) = 0, \\ Bz = 0, \\ z \geq 0, \\ z \neq 0. \end{cases}$$

e is a feasible solution of this problem and we have $g(e) = \langle b, e\rangle = 1$.

Since we have $g(kz) = g(z)$ for all $z \geq 0$ and all $k > 0$, we will work on the following normalized problem:

$$(kz) \begin{cases} \min g(z) = 0, \\ Bz = 0, \\ \langle e, z\rangle = n, \\ z \geq 0. \end{cases}$$

It is easy to see that the matrix (A^t, x) is of rank $m+1$, so is the matrix (B^t, e). The Newtonian descent direction at point e for the problem (kz) is obtained by solving the quadratic problem

$$(PQ) \begin{cases} \min \frac{1}{2} \left\langle \nabla^2 g(e)\, d, d \right\rangle + \langle \nabla g(e), d\rangle, \\ Bd = 0, \\ \langle e, d\rangle = 0. \end{cases}$$

To do this, let us introduce the matrix

$$P = I - \left(B^t, e\right) \left[\left(B^t, e\right)^t \left(B^t, e\right)\right]^{-1} \left(B^t, e\right)^t,$$

which corresponds to the projection on the linear subspace:

$$E = \{d : Bd = 0, \langle e, d\rangle = 0\}.$$

We have

$$P^2 = P = P^t, PB^t = 0 \text{ and } Pe = 0.$$

It is easy to see that we have

$$P\nabla g(e) = Pb \text{ and } P\nabla^2 g(e)P = I + n(n-1)\, Pbb^tP,$$

the quadratic problem is equivalent to

$$\begin{cases} \min \frac{1}{2} \left\langle \nabla^2 g\left(e\right) d, d \right\rangle + \left\langle \nabla g\left(e\right), d \right\rangle, \\ Pd = d, \end{cases}$$

whose optimal solution is collinear with

$$d = -Pb = -P\nabla g\left(e\right).$$

The direction d thus coincides with the direction given by the projected gradient. We are now interested in some properties of d.

First of all, we have

$$\langle d, e \rangle = -\langle Pb, e \rangle = -\langle b, Pe \rangle = 0,$$

we then observe that on the one hand, we have

$$\left\{ \begin{array}{c} z : \langle e, z \rangle = n \\ \|z - e\|^2 \leq \frac{n}{n-1} \end{array} \right\} \subset \left\{ \begin{array}{c} z : \langle e, z \rangle = n \\ z \geq 0 \end{array} \right\} \subset \left\{ \begin{array}{c} z : \langle e, z \rangle = n \\ \|z - e\|^2 \leq n\left(n-1\right) \end{array} \right\}.$$

and on the other hand,

$$\begin{cases} \min \langle b, z - e \rangle = -1, \\ Bz = 0, \\ \langle e, z \rangle = n, \\ z \geq 0, \end{cases}$$

and since $P\left(e - \bar{z}\right) = e - \bar{z}$,

$$\langle b, z - e \rangle = \langle Pb, z - e \rangle,$$

we get

$$\|Pb\| \sqrt{\frac{n}{n-1}} \leq 1 \leq \|Pb\| \sqrt{n\left(n-1\right)}.$$

So, in summary,

$$\langle d, b \rangle = -\|d\|^2 = -\|Pb\|^2, \langle d, e \rangle = 0 \text{ and } \frac{1}{n\left(n-1\right)} \leq \|d\|^2 \leq \frac{n-1}{n}.$$

In the following, we denote by $\bar{d}$ and σ the mean and standard deviations of $\left(d_1, d_2, ..., d_n\right)$. We have

$$\bar{d} = \frac{1}{n}\sum_{i=1}^{n} d_i = 0 \text{ and } \frac{1}{n^2\left(n-1\right)} \leq \sigma^2 = \frac{\|d\|^2}{n} - \bar{d}^2 \leq \frac{n-1}{n^2}. \tag{1}$$

4 Calculation of the Step Size

The calculation of the step size consists in obtaining a value $t > 0$ such that we have $e + td > 0$ and which gives a significant decrease of $\mu_0(t) = g(e + td)$ or, equivalently, of $\omega_0(t) = ln(g(e + td))$.

Since the equation $\omega_0'(t) = 0$ cannot be solved explicitly in a large class of optimization problems, it is normal to think of iterative methods of solution, one can also apply to ω_0

an Armijo-Golstein-Price type method. In both cases, this requires several evaluations of the function ω_0 and its derivative and is therefore expensive. Our approach consists in minimizing an upper bound ω_{MAJ} and lower bound ω_{MIN} of the function ω_0 whose minimum can be obtained explicitly. Recall that we have

$$\omega_0(t) = n \ln\left(1 - t\|d\|^2\right) - \sum_{i=1}^{n} \ln(1 + td_i),$$

it is clear that $\omega_0(0) = 0$.

We need the following theorem to find the upper bound and lower bound functions of $\omega(t)$.

Theorem 4.1 *[1] Suppose that $x_i > 0$ for all $i = 1, 2, \ldots, n$, then*

$$n \ln\left(\bar{x} - \sigma_x\sqrt{n-1}\right) \leq A \leq \sum_{i=1}^{n} \ln(x_i) \leq B \leq n \ln(\bar{x}),$$

with

$$A = (n-1)\ln\left(\bar{x} + \frac{\sigma_x}{\sqrt{n-1}}\right) + \ln\left(\bar{x} - \sigma_x\sqrt{n-1}\right)$$

and

$$B = \ln\left(\bar{x} + \sigma_x\sqrt{n-1}\right) + (n-1)\ln\left(\bar{x} - \frac{\sigma_x}{\sqrt{n-1}}\right),$$

such that $\bar{x}$ and σ_x are, respectively, the mean and standard deviations of a statistical series $\{x_1, x_2, ..., x_n\}$ of n real numbers. These quantities are defined as follows:

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i \text{ and } \sigma^2{}_x = \frac{1}{n}\sum_{i=1}^{n} x^2{}_i - \bar{x}^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x}^2).$$

In the following, we take $x_i = 1 + td_i, i = 1, ..., m$, we have $\bar{x} = 1 + td$ and $\sigma_x = t\sigma_d$.

4.1 Upper bound functions

4.1.1 First upper bound function

From the previous theorem, we have

$$A \leq \sum_{i=1}^{n} \ln(x_i),$$

then

$$\begin{aligned}
\sum_{i=1}^{n} \ln(1 + td_i) &\geq (n-1)\ln\left(\bar{x} + \frac{\sigma_x}{\sqrt{n-1}}\right) + \ln(\bar{x} - \sigma_x\sqrt{n-1}), \\
\sum_{i=1}^{n} \ln(1 + td_i) &\geq (n-1)\ln\left(1 + \frac{\sigma t}{\sqrt{n-1}}\right) + \ln\left(1 - \sigma t\sqrt{n-1}\right),
\end{aligned}$$

multiplying by (-1), we find

$$-\sum_{i=1}^{n}\ln(1+td_i) \leq -(n-1)\ln\left(1+\frac{\sigma t}{\sqrt{n-1}}\right) - \ln(1-\sigma t\sqrt{n-1}),$$

and therefore

$$\omega_0\left(t\right) \leq \omega_{MAJ}\left(t\right) = n\ln\left(1-nt\sigma^2\right) - (n-1)\ln\left(1+\frac{\sigma t}{\sqrt{n-1}}\right) - \ln\left(1-\sigma t\sqrt{n-1}\right).$$

From this, we can deduce that the function ω_{MAJ} reaches its minimum at the point

$$\bar{t}_{MAJ} = \frac{n\sqrt{n-1}}{\sqrt{n-1}+n\left(n-2\right)\sigma}.$$

4.1.2 Second upper bound function

Let us consider a function that contains only a logarithm and is simpler than the function ω_{MAJ}. Then, we consider the following function:

$$\omega_{MAJ1}(t) = -2n\left\|d\right\|^2 t - \ln\left(1-\sigma t\sqrt{n-1}\right).$$

Lemma 4.1 *$\omega_{MAJ1}(t)$ is strictly convex for all $t \geq 0$; and we have*

$$\omega_0(t) \leq \omega_{MAJ1}(t) \leq +\infty.$$

Proof. We have

$$\omega_0\left(t\right) = n\ln\left(1-t\left\|d\right\|^2\right) - \sum_{i=1}^{n}\ln\left(1+td_i\right).$$

We pose

$$g(t) = \omega_{MAJ1}(t) - \omega_0(t),$$

we have $g(0) = 0$ and

$$g''(t) = \frac{\sigma^2(n-1)}{(1-\sigma t\sqrt{n-1})^2} + \frac{n\left\|d\right\|^2}{(1-t\left\|d\right\|^2)^2} + \sum_{i=1}^{n}\frac{d_i^2}{(1+td_i^2)^2} \geq 0$$

for all $t \geq 0$. This gives $g(t) \geq 0, \forall t \geq 0$.
Then

$$\omega_0(t) \leq \omega_{MAJ1}(t).$$

From this, we can deduce that the function ω_{MAJ1} reaches its minimum at the point

$$\tilde{t} = \frac{1}{\sigma\sqrt{n-1}} - \frac{1}{2n^2\sigma^2}.$$

The new iterate is

$$\tilde{x} = X\left(e+\bar{t}d\right) = x + \bar{t}Xd.$$

By construction, we have $\tilde{x} > 0$ and $A\tilde{x} = 0$.

4.1.3 The decrease

Replacing $\tilde{t}$ with its value gives

$$\begin{aligned}\omega_{MAJ1}(\tilde{t}) &= -2n^2\sigma^2\tilde{t} - \ln(1-\sigma\tilde{t}\sqrt{n-1}) \\ &= 1 - \frac{2n^2\sigma}{\sqrt{n-1}} - \ln\left(1 - \frac{2n^2\sigma - \sqrt{n-1}}{2n^2\sigma}\right) \\ &= 1 - \frac{2n^2\sigma}{\sqrt{n-1}} - \ln\left(\frac{\sqrt{n-1}}{2n^2\sigma}\right).\end{aligned}$$

The quantity $\omega_{MAJ1}(\tilde{t}) - \omega_{MAJ1}(0) = \omega_{MAJ1}(\tilde{t})$ depends on σ.
Since we have

$$\frac{1}{n\sqrt{n-1}} \leq \sigma \leq \frac{\sqrt{n-1}}{n},$$

we obtain

$$1 \leq \sigma n\sqrt{n-1} \leq n-1.$$

It is useful to put $u = n\sigma\sqrt{n-1}$, then we get $1 \leq u \leq n-1$ and

$$\begin{aligned}\omega_{MAJ1}(\tilde{t}) &= 1 - \frac{2n^2\sigma}{\sqrt{n-1}} - \ln\left(\frac{\sqrt{n-1}}{2n^2\sigma}\right) \\ &= 1 - \frac{2nu}{n-1} - \ln\left(\frac{n-1}{2nu}\right) \\ &= 1 - \frac{2nu}{n-1} - \ln(n-1) + \ln(2nu) \\ &= \xi(u).\end{aligned}$$

The function ξ is concave (a sum of two concave functions) and therefore

$$\omega_{MAJ1}(\tilde{t}) = \xi(u) < \xi(1) + (u-1)\xi'(1).$$

This leads to

$$\omega_{MAJ1}(\tilde{t}) = \xi(u) < \ln\left(\frac{2ne}{(n-1)e^{\frac{2n}{n-1}}}\right) - (u-1)\left(\frac{2n}{n-1} - 1\right).$$

We deduce that in the worst case (where $u = 1$), we have

$$\omega_{MAJ1}(\tilde{t}) = \xi(u) < \ln\left(\frac{2ne}{(n-1)e^{\frac{2n}{n-1}}}\right) \leq 0,$$

then we obtain

$$f(\tilde{x}) \leq \left(\frac{2ne}{(n-1)e^{\frac{2n}{n-1}}}\right) f(x_0).$$

4.2 Lower bound function

4.2.1 First lower bound function

From the previous theorem, we have

$$\sum_{i=1}^{n} \ln(x_i) \leq B,$$

then

$$\begin{aligned}\sum_{i=1}^{n} \ln(1+td_i) &\leq (n-1)\ln\left(\bar{x} - \frac{\sigma_x}{\sqrt{n-1}}\right) + \ln(\bar{x} + \sigma_x\sqrt{n-1}),\\ \sum_{i=1}^{n} \ln(1+td_i) &\leq (n-1)\ln\left(1 - \frac{\sigma t}{\sqrt{n-1}}\right) + \ln(1 + \sigma t\sqrt{n-1}).\end{aligned}$$

Multiplying by (-1), we find

$$-\sum_{i=1}^{n} \ln(1+td_i) \geq -(n-1)\ln(1 - \frac{\sigma t}{\sqrt{n-1}}) - \ln(1 + \sigma t\sqrt{n-1}),$$

and therefore

$$\omega_0(t) \geq \omega_{MIN}(t) = n\ln(1 - n\sigma^2 t) - (n-1)\ln\left(1 - \frac{\sigma t}{\sqrt{n-1}}\right) - \ln(1 + \sigma t\sqrt{n-1}).$$

From this we can deduce that the function ω_{MIN} reaches its minimum at the point

$$\bar{t}_{MIN} = \frac{n\sqrt{n-1}}{\sqrt{n-1} - n\,(n-2)\,\sigma}.$$

4.2.2 Second lower bound function

We can consider a function that contains only a logarithm and is simpler than the function ω_{MIN}. Then let us consider the following function:

$$\omega_{MIN1}(t) = -2n\left\|d\right\|^2 t - (n-1)\ln\left(1 - \frac{\sigma t}{\sqrt{n-1}}\right).$$

Lemma 4.2 *$\omega_{MIN1}(t)$ is strictly convex for all $t \geq 0$; and we have*

$$-\infty \leq \omega_{MIN1}(t) \leq \omega_{MIN}(t).$$

Proof. We have

$$\omega_{MIN}(t) = n\ln(1 - n\sigma^2 t) - (n-1)\ln\left(1 - \frac{\sigma t}{\sqrt{n-1}}\right) - \ln(1 + \sigma t\sqrt{n-1}).$$

We put

$$g(t) = \omega_{MIN1}(t) - \omega_{MIN}(t).$$

We obtain $g(0) = 0$ and

$$g''(t) = -\frac{\sigma^2(n-1)}{(1+\sigma t\sqrt{n-1})^2} \leq 0$$

for all $t \geq 0$. This gives $g(t) \leq 0, \forall t \geq 0$, then

$$\omega_{MIN}(t) \geq \omega_{MIN1}(t).$$

We deduce that the function ω_{MIN1} reaches its minimum at the point

$$\tilde{t} = \frac{2n^2\sigma\sqrt{n-1} - (n-1)}{2n^2\sigma^2}.$$

4.2.3 The decrease

Replacing $\tilde{t}$ with its value gives

$$\begin{aligned} \omega_{MIN1}(\tilde{t}) &= -2n^2\sigma^2\tilde{t} - (n-1)\ln\left(1 - \frac{\sigma}{\sqrt{n-1}}\tilde{t}\right) \\ &= (n-1) - 2n^2\sigma\sqrt{n-1} - (n-1)\ln\left(\frac{n-1}{2n^2\sigma\sqrt{n-1}}\right). \end{aligned}$$

The quantity $\omega_{MIN1}(\tilde{t}) - \omega_{MIN1}(0) = \omega_{MIN1}(\tilde{t})$ depends on σ.
Since we have

$$\frac{1}{n\sqrt{n-1}} \leq \sigma \leq \frac{\sqrt{n-1}}{n},$$

we get

$$1 \leq \sigma n\sqrt{n-1} \leq n-1,$$

it is useful to set $u = n\sigma\sqrt{n-1}$, then we obtain $1 \leq u \leq n-1$ and

$$\begin{aligned} \omega_{MIN1}(\tilde{t}) &= (n-1) - 2n^2\sigma\sqrt{n-1} - (n-1)\ln\left(\frac{n-1}{2n^2\sigma\sqrt{n-1}}\right) \\ &= (n-1) - 2nu - (n-1)\ln\left(\frac{n-1}{2nu}\right) \\ &= \xi(u). \end{aligned}$$

The function ξ is concave (a sum of two concave functions) and therefore

$$\omega_{MIN1}(\tilde{t}) = \xi(u) < \xi(1) + (u-1)\xi'(1).$$

We can deduce

$$\omega_{MIN1}(\tilde{t}) = \xi(u) < \ln\left(\left(\frac{2ne}{n-1}\right)^{n-1} \times \frac{1}{e^{2n}}\right) - (u-1)(n+1).$$

We deduce that in the worst case (where $u = 1$), we have

$$\omega_{MIN1}(\tilde{t}) = \xi(u) < \ln\left(\left(\frac{2ne}{n-1}\right)^{n-1} \times \frac{1}{e^{2n}}\right) \leq 0,$$

so, we obtain

$$f(\tilde{x}) \leq \left(\left(\frac{2ne}{n-1}\right)^{n-1} \times \frac{1}{e^{2n}}\right) f(x_0).$$

5 The Algorithm

Karmarkar's algorithm via the upper bound and lower bound functions.

Initialization: We start from $x > 0$ such that $Ax = 0$, ϵ is a given precision.

Result: x^*.

Iteration:

While $c^t x > \epsilon$ *do:*

1- Normalization:

$$x = \sqrt{\frac{n}{\langle x, x \rangle}} x.$$

2- Descent direction: We take b and B as follows:

$$b = \frac{1}{\langle c, x \rangle} Xc, \;\; B = AX.$$

We determine d projection of b onto the linear subspace:

$$\{d : B_k d = 0, \langle e, d \rangle = 0\}.$$

Finally, the descent direction is:

$$\delta = Xd.$$

3- The step size: We calculate:

$$\sigma = \frac{\|d\|}{\sqrt{n}}.$$

Case 1: upper bound function:

$$\bar{t}_{MAJ} = \frac{2n^2\sigma - \sqrt{n-1}}{2n^2\sigma\sqrt{n-1}}.$$

Case 2: lower bound function:

$$\bar{t}_{MIN} = \frac{2n^2\sigma\sqrt{n-1} - (n-1)}{2n^2\sigma^2}.$$

4- The new iterate: is $\tilde{x} = x + t\delta$.

5- Taking $k = k + 1$ and returning to (1).

End While.

$x^* = \tilde{x}$.

End Algorithm.

6 The Convergence

Upper bound function.

Lemma 6.1 *At the* k^{th} *iteration, we have*

$$f(x_k) < \left(\frac{2ne}{(n-1)e^{\frac{2n}{n-1}}} \right)^k f(x_0).$$

Therefore $f(x_k)$ *converges linearly to* 0. *We deduce that any membership value of the sequence* $\{x_k\}$ *is an optimal solution of problem* (ka).

Lower bound function.

Lemma 6.2 *At the k^{th} iteration, we have*

$$f(x_k) < \left(\left(\frac{2ne}{n-1}\right)^{n-1} \times \frac{1}{e^{2n}}\right)^k f(x_0).$$

Therefore $f(x_k)$ converges linearly to 0. We deduce that any membership value of the sequence $\{x_k\}$ is an optimal solution of problem (ka).

7 Numerical Tests

To evaluate our algorithm's efficiency based on our upper and lower bound functions, we conducted comparative numerical tests between our two new approximate functions (the upper bound function (**TUF**)and the lower bound function (**TLF**) and Wolfe's line search method (**LSW**)). The algorithm is described in our work using Matlab10 software. The examples tested are taken from the literature, see for example [4, 5].

We have taken ϵ between $(10^{-4}$ et $10^{-6})$. We denote by

TUF: The upper bound function technique.

TLF: The technique of the lower bound function.

LSW: Wolfe's line search method.

iter: The number of iterations required to obtain an optimal solution.

time (s): The calculation time in seconds.

7.1 Examples

Example 1:

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 3 & 0 \end{bmatrix}, \; b = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \text{ and } c = \begin{bmatrix} 2 & 1 & 0 \end{bmatrix}^t.$$

The optimal value is $z^* = \frac{1}{3}$.

The exact optimal solution is $x^* = \begin{bmatrix} 0 & \frac{1}{3} & \frac{5}{6} \end{bmatrix}^t$.

Comparative table:

Method	**iter**	**time (s)**
TUF1	03	0:0:0:1
TLF1	04	0:0:0:19
LSW	11	0:0:01:31

Example 2:

$$A = \begin{bmatrix} 2 & 3 & 1 & 2 \\ 3 & 0 & -2 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \text{ and } c = \begin{bmatrix} 4 & 1 & 2 & 0 \end{bmatrix}^t.$$

The optimal value is $z^* = 0.67$.

The exact optimal solution is $x^* = \begin{bmatrix} 0 & 0.67 & 0 & 0 \end{bmatrix}^t$.

Comparative table:

Method	**iter**	**time (s)**
TUF1	06	0:0:0:1
TLF1	09	0:0:0:26
LSW	13	0:0:10:01

Example 3:

$$A = \begin{bmatrix} 2 & 3 & 1 & 0 & 3 \\ 1 & 2 & 5 & 0 & 1 \\ 5 & -1 & 2 & 3 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \text{ and } c = \begin{bmatrix} 1 & 2 & 3 & 5 & 4 \end{bmatrix}^t.$$

The optimal value is $z^* = \frac{22}{9}$.
The exact optimal solution is $x^* = \begin{bmatrix} \frac{1}{3} & 0 & \frac{1}{3} & \frac{2}{9} & 0 \end{bmatrix}^t$.
Comparative table:

Method	**iter**	**time (s)**
TUF	09	0:0:0:01
TLF	11	0:0:0:02
LSW	32	0:0:11:09

Example 4:

$$A = \begin{bmatrix} 2 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \text{ and } c = \begin{bmatrix} 3 & -1 & 1 & 0 & 0 & 0 \end{bmatrix}^t.$$

The optimal value is $z^* = -0.5$.
The exact optimal solution is $x^* = \begin{bmatrix} 0 & 0.5 & 0 & 0.5 & 0 & 0 \end{bmatrix}^t$.
Comparative table:

Method	**iter**	**time (s)**
TUF1	10	0:0:0:01
TLF1	13	0:0:0:01
LSW	33	0:0:12:08

Example 5:

$$A = \begin{bmatrix} 1 & 0 & -4 & 3 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 5 & 3 & 1 & 0 & -1 & 3 & 0 & 1 & 0 & 0 & 0 & 0 \\ 4 & 5 & -3 & 3 & -4 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 2 & 1 & -5 & 0 & 0 & 0 & 1 & 0 & 0 \\ -2 & 1 & 1 & 1 & 2 & 2 & 0 & 0 & 0 & 0 & 1 & 0 \\ 2 & -3 & 2 & -1 & 4 & 5 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 4 \\ 4 \\ 5 \\ 7 \\ 5 \end{bmatrix}$$

$$\text{and } c = \begin{bmatrix} -4 & -5 & -1 & -3 & 5 & -8 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^t.$$

The optimal value is $z^* = -17$.
The exact optimal solution is $x^* = \begin{bmatrix} 0 & 0 & 2.5 & 3.5 & 0 & 0.5 & 0 & 0 & 0.5 & 0.5 & 0 & 1 \end{bmatrix}^t$.
Comparative table:

Method	**iter**	**time (s)**
TUF1	09	0:0:0:01
TLF1	07	0:0:0:01
LSW	20	0:0:17:12

Example 6: The matrix A is

$$\begin{bmatrix}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 3 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 6 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 4 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}$$

The vectors c and b are

$$\begin{aligned}
c &= \begin{bmatrix} 2 & -1 & -3 & 5 & -2 & 0 & 4 & 1 & 2 & -1 & 1 & -1 & 0 & 2 \\ & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & \end{bmatrix}^t, \\
b &= \begin{bmatrix} 8 & 4 & 6 & 2 & 5 & 1 & 2 & 6 & 3 & 9 & 4 \end{bmatrix}^t.
\end{aligned}$$

The optimal value is $z^* = -13.25$.
The exact optimal solution is

$$x^* = \begin{bmatrix} 0 & 1 & 2 & 0 & 0.5 & 1.33 & 0 & 0 & 0 & 1.5 & 0 & 3.48 & 0 \\ 0 & 7 & 0 & 0 & 0 & 1 & 2.33 & 2 & 0.18 & 4.5 & 0 & 0.15 & \end{bmatrix}^t.$$

Comparative table:

Method	**iter**	**time (s)**
TUF1	05	0:0:0:01
TLF1	06	0:0:0:01
LSW	22	0:0:57:09

Example 7: (with a variable size)
We consider the following linear problem of variable size:

$$\zeta = \min[c^T x : x \geq 0, \;\; Ax = b],$$

where A is the $m \times 2m$ matrix defined by

$$\begin{aligned}
A[i,j] &= \begin{cases} 1 & \text{if } i = j \quad \text{or } j = i + m, \\ 0 & \text{if not.} \end{cases} \\
c[i] &= -1, \;\; c[i+m] = 0 \text{ and } b[i] = 2, \;\; \forall i = 1, ...m,
\end{aligned}$$

where the vectors $c \in \mathbb{R}^{2m}$ and $b \in \mathbb{R}^{2m}$.
The optimal value is $z^* = -2m$. The exact optimal solution is

$$x_i^* = \begin{cases} 2 & \text{if} \quad i = 1, ..., m, \\ 0 & \text{if} \quad i = m+1, ..., n. \end{cases}$$

Comparative table:

size	Method	iter	time (s)
5×10	**TUF**	4	0:0:0:01
	TLF	6	0:0:0:01
	LSW	90	0:02:40:12
25×50	**TUF**	9	0:0:0:09
	TLF	14	0:0:0:15
	LSW	89	0:02:00:07
50×100	**TUF**	8	0:0:0:26
	TLF	10	0:0:0:33
	LSW	89	0:02:33:40

Comments: The numerical tests carried out show that our approach of upper bound and lower bound functions that we have proposed leads to a very significant reduction in the cost of calculation and an improvement in the result. The number of iterations and the computing time are considerably reduced in the upper bound and lower bound functions in comparison with the line search method.

8 Conclusion

Despite the mathematical development in the field of linear programming, many problems remain to be developed.

For this, in this study, we used a logarithmic barrier method for solving this problem. We proposed a new upper and lower bound functions to compute the displacement step, and we showed that our new technique is efficient in reducing the computational cost in Karmarkar's projective algorithm. This method has made it possible to significantly reduce the number of iterations and the time of their calculation. Numerical simulations confirm the efficiency of our approach.

Our future exciting work is to further improve the computational time of the logarithmic barrier algorithm by proposing more efficient upper and lower bound functions. But extensions would be envisaged to the nonlinear, not necessarily to the linear programming problem.

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