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Thermo-Electroelastic Contact Problem with Temperature Dependent Friction Law

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Abstract: In this work, we consider a dynamical unilateral contact problem with Coulomb's friction and thermo-electroelastic effects. We focus here on the dynamical effects such as frictional heating and thermal softening at the contact interface. The thermo-electro-elastic constitutive law is assumed to be linear and the foundation is thermally and electrically conductive. We derive a variational formulation of the problem and establish the existence of a weak solution. The proof is based on a suitable combination of the penalty method, standard arguments of variational equations and fixed point theorem.

Keywords: *thermo-electro-elastic materials; dynamic contact problem; frictional heating; variational analysis.*

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1 Introduction

A piezoelectric material is a substance that generates electrical charges when mechanical pressure is applied and mechanically deforms when an electric field is applied. As a result, the piezoelectric material performs the function of a transducer, converting electrical energy into mechanical energy and vice versa. These so-called smart materials, among other things, are used as switches in radio-logic, electric-acoustic, and measuring devices. Piezoelectric materials have been extensively studied, and one natural extension of these coupled electro-mechanical models is to include temperature as an additional state variable to account for thermal effects as well as piezoelectric effects.

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General mathematical models on piezoelectricity were studied in [3,9,13]. Results on static frictional contact problems for piezoelectric materials under the assumption that the foundation is insulated can be found in [5,11,16], and these results were extended in [7,8,12] in the case of an electrically conductive foundation. In the quasi-static case, we refer to [6,10] and references therein. Moreover, the theory of thermo-piezoelectricity was first proposed by [14], the physical laws and the governing equations for thermo-piezoelectric materials have been explored in [7,14,15,17] and for some recent results on the thermo-piezoelectric contact problem, we refer to [1,4].

Here, we seek to apply the static/quasi-static instances of our previous studies to a dynamical contact problem with temperature-dependent friction. Heat is produced as a result of the body and foundation sliding against one another through friction. This fact serves as the inspiration for our expansion of the dynamic thermo-electroelastic contact issue, which takes into consideration the effects of thermal softening and frictional heating at the contact surface.

The remainder of the paper is organized as follows. The model of the dynamical frictional contact process between a thermo-electro-viscoelastic body and a conductive deformable foundation is described in Section 2. Section 3 introduces some notations, lists the data assumptions, and derives the variational formulation of the model. The main existence and uniqueness result of the model's weak solution is stated in Theorem 4.1. This theorem is proved in several steps in Section 4, the proof is based on the arguments of compactness, time discretization, and the Banach fixed point theorem.

2 Preliminaries

In this section, we recall some useful definitions and lemmas which will be used in the sequel. Let X be a reflexive Banach space and $\langle\cdot,\cdot\rangle$ denote the duality of X and X^*, we have the following interesting results (see e.g., [2]).

Definition 2.1 A single-valued operator $A : X \to X^*$ is pseudomonotone if

1. A is a bounded, i.e., it maps the bounded sets in X into the bounded sets in X^*,
2. for every sequence $\{x_n\} \subset X$ converging weakly to x of X such that $\limsup_{n\infty}\langle Ax_n, x_n - x\rangle \leq 0$, we have $\langle Ax, x-y\rangle \leq \liminf_{n\infty}\langle Ax_n, x_n - y\rangle$ for all $y \in X$.

Definition 2.2 A multi-valued operator $T : X \to 2^{X^*}$ is pseudo-monotone if

1. for every $v \in V$, the set $Tv \subset X^*$ is nonempty, closed and convex,
2. the operator T is upper semi-continuous from each finite-dimensional subspace of X to X^* endowed with weak topology,
3. for any sequences $\{u_n\} \subset X$ and $\{u_n^*\} \subset X^*$ such that $u_n \to u$ weakly in X, $u_n^* \in Tu_n$ for all n, and $\limsup_{n\infty}\langle u_n^*, u_n - u\rangle \leq 0$, we have that for every $v \in X$, there exists $u^*(v) \in Tu$ such that $\langle u^*(v), u - v\rangle \leq \liminf_{n\infty}\langle u_n^*, u_n - v\rangle$.

Let $(F, \|\cdot\|_F) \subset (G, \|\cdot\|_G)$ be the reflexive Banach spaces such that $\|\cdot\|_F \geq \|\cdot\|_G$ and $\overline{F} = G$, thus, we may write $F \subset G \equiv G' \subset F'$. Suppose that $\mathcal{B}$ is a linear, bounded, positive and symmetric operator from G to G'. Let $\mathbb{F} = L^2([a,b],F)$, $\mathbb{G} = L^2([a,b],G)$ and define $\mathbb{X} = \{w \in \mathbb{F} : (\mathcal{B}w)' \in \mathbb{F}'\}$ which is a reflexive Banach space for the norm

$$\|w\|_{\mathbb{X}} = \|w\|_F + \|(\mathcal{B}w)'\|_{F'}.$$

Moreover, let $\mathcal{A}(t,\cdot)$ be an operator from F to F', and denote by $\mathcal{A} : \mathbb{F} \to \mathbb{F}'$ its natural extension, given by $\mathcal{A}w(t) = A(t, w(t))$. Assume that

$$\begin{aligned} &\mathcal{A} : \mathbb{X} \to \mathbb{X}' \text{ is pseudo-monotone,} \\ &\mathcal{A} : \mathbb{F} \to \mathbb{F}' \text{ is a bounded operator,} \end{aligned} \tag{1}$$

and for some $\lambda \in \mathbb{R}$, one has

$$\lim_{\|w\|_{\mathbb{F}}+\infty} \frac{\lambda \langle \mathcal{B}w, w\rangle_{\mathbb{G}'\times\mathbb{G}} + \langle \mathcal{A}w, w\rangle_{\mathbb{F}'\times\mathbb{F}}}{\|w\|_{\mathbb{F}}} = \infty. \tag{2}$$

Then the following existence theorem holds, see [18, Theorem 3.1].

Theorem 2.1 *Let $\mathcal{A}$ and $\mathcal{B}$ be defined above. Then, for each $w_0 \in G$ and $\ell \in \mathbb{F}'$, there exists a $w \in \mathbb{X}$ such that*

$$\begin{cases} (\mathcal{B}w)' + \mathcal{A}w = \ell \quad in \ \ \mathbb{F}', \\ \mathcal{B}w(0) = \mathcal{B}w_0 \quad in \ \ G'. \end{cases}$$

3 Problem Statement and Variational Formulation

We consider an elastic body in the reference configuration $\Omega \subset \mathbb{R}^d$ with the dimension $d = 2, 3$. We are interested in the displacement field $u(x;t)$, the electrical potential $\varphi(x,t)$ and the temperature $\theta(x;t)$ for $(x;t) \in \Omega \times (0;T)$, where $(0;T)$ is the given time interval. For the sake of simplicity, we will omit the dependence of various functions on the spatial variable $x \in \bar{\Omega}$. Hence, the local momentum of balance for stress, electric displacement and heat conduction are given as follows:

$$\begin{aligned} &\ddot{u} - \operatorname{Div}\sigma = f_0 && \text{in } \ \Omega \times (0,T), \\ &\operatorname{div} D = \phi_0 && \text{in } \ \Omega \times (0,T), \\ &\dot{\theta} + \operatorname{div} q = -\mathcal{M}\,\varepsilon(\dot{u}) - \mathcal{P}\,E(\varphi) + q_0 && \text{in } \ \Omega \times (0,T). \end{aligned}$$

Here, the quantities f_0, ϕ_0 and q_0 describe the given body forces, volume electric charge and heat source term, acting on Ω. In the case of linear thermo-visco-piezoelectricity, the stress tensor is given by

$$\sigma = \mathcal{A}\,\varepsilon(\dot{u}) + \mathfrak{F}\,\varepsilon(u) - \mathcal{E}^* E(\varphi) - \mathcal{M}\,\theta \ \text{ in } \ \Omega \times (0,T),$$

where $\mathcal{A}$ is the linear viscosity operator, $\mathfrak{F} = (f_{ijkl})$ is the linear elasticity operator, $\varepsilon(u)$ is the linearized strain tensor, $E(\varphi) = -\nabla\varphi$ is the electric field, $\mathcal{E} = (e_{ijk})$ is the third-order piezoelectric tensor and $\mathcal{E}^* = (e_{kij})$ is its transpose, $\mathcal{M} = (m_{ij})$ is the thermal expansion tensor. Moreover, the electric displacement and the heat flux are defined by

$$\begin{aligned} &D = \mathcal{E}\,\varepsilon(u) + \beta\,E(\varphi) + \mathcal{P}^*\,\theta && \text{in } \ \Omega \times (0,T), \\ &q = -\mathcal{K}\,\nabla\theta && \text{in } \ \Omega \times (0,T), \end{aligned}$$

where $\beta = (\beta_{ij})$ is the electric permittivity tensor, $\mathcal{P} = (p_i)$ and $\mathcal{K} = (k_{ij})$ are the thermal expansion and thermal conductivity tensors.

Recall that $\mathbb{S}^d$ is the space of second order symmetric tensors on $\mathbb{R}^d$. The canonical inner products and associated norms on $\mathbb{R}^d$ and $\mathbb{S}^d$ are given by

$$\begin{array}{lll} \forall\, u,\, v \in \mathbb{R}^d, \;\; u \cdot v = u_i\, v_i & ; & \forall\, \sigma,\, \tau \in \mathbb{S}^d, \;\; \sigma \cdot \tau = \sigma_{ij}\, \tau_{ij}, \\ \forall\, u,\, v \in \mathbb{R}^d, \;\; \|v\| = (v \cdot v)^{\frac{1}{2}} & ; & \forall\, \sigma,\, \tau \in \mathbb{S}^d, \;\; \|\tau\| = (\tau \cdot \tau)^{\frac{1}{2}}. \end{array}$$

If ν is the outward unit normal vector on the boundary $\Gamma = \partial\Omega$, then the normal and tangential components of the displacement vector v and the stress field σ on Γ are

$$v_\nu = v \cdot \nu \;,\; v_\tau = v - v_\nu\, \nu \quad \text{and} \quad \sigma_\nu = \sigma\, \nu \cdot \nu \;,\; \sigma_\tau = \sigma\, \nu - \sigma_\nu\, \nu.$$

In order to formulate the boundary conditions and the initial boundary values, we divide the boundary Γ into three disjoint open subsets Γ_D, Γ_N and Γ_C such that $\overline{\Gamma}_D \cup \overline{\Gamma}_N \cup \overline{\Gamma}_C = \overline{\Gamma}$. We also assume that $\Gamma_D \cup \Gamma_N$ is partitioned into two disjoint open parts Γ_a and Γ_b of nonzero measure such that $\overline{\Gamma}_D \cup \overline{\Gamma}_N = \overline{\Gamma}_a \cup \overline{\Gamma}_b$. We assume that the body is clamped on $\Gamma_D \times (0,T)$, the surface traction of density f_N acts on $\Gamma_N \times (0,T)$, the electrical potential vanishes on $\Gamma_a \times (0,T)$, the surface electric charge of density ϕ_b acts on $\Gamma_b \times (0,T)$ and the temperature is assumed to be zero on $\Gamma_D \cup \Gamma_N \times (0,T)$. Therefore, we have

$$\begin{array}{ll} u = 0 & \text{on } \;\Gamma_D \times (0,T), \\ \sigma\, \nu = f_N & \text{on } \;\Gamma_N \times (0,T), \\ \varphi = 0 & \text{on } \;\Gamma_a \times (0,T), \\ D \cdot \nu = \phi_b & \text{on } \;\Gamma_b \times (0,T), \\ \theta = 0 & \text{on } \;\Gamma_D \cup \Gamma_N \times (0,T). \end{array}$$

On the contact surface Γ_C, the body is supposed to be in unilateral contact with a rigid foundation by Coulomb's friction law

$$\sigma_\nu(u,\varphi,\theta) \le 0\,,\; (u_\nu - g) \le 0\,,\; \sigma_\nu(u,\varphi,\theta)\,(u_\nu - g) = 0 \quad \text{on } \;\Gamma_C \times (0,T),$$

$$\left\{ \begin{array}{ll} \|\sigma_\tau\| \le \mu(\theta)\, |R\sigma_\nu| & \\ \|\sigma_\tau\| < \mu(\theta)\, |R\sigma_\nu| \;\;\Longrightarrow\;\; [\dot{u}]_\tau = 0 & \quad \text{on } \;\Gamma_C \times (0,T), \\ \|\sigma_\tau\| = \mu(\theta)\, |R\sigma_\nu| \;\;\Longrightarrow\;\; \exists\, \lambda \in \mathbb{R},\; \sigma_\tau = -\lambda^2\, \dot{u}_\tau, & \end{array} \right.$$

where for the temperature dependent coefficient of friction $\mu(\theta) \ge 0$, we use

$$\mu(\theta) = \mu_0\, \frac{[\theta - \theta_d]^2}{[\theta_d - \theta_f]^2},$$

where μ_0 is the static coefficient of friction at the given reference temperature θ_f and θ_d is a damage temperature on the interface. Temperature θ_d is related to the temperature at which frictional stress is no longer due to the solid shearing effects, but is generated by the viscous shear of a molten film on the contact interface. It can be taken as the lowest melting temperature of the body and the foundation in contact. Since $\theta < \theta_d$, we have $\mu'(\theta_d) \le 0$ and $\lim\limits_{\theta_d} \mu(\theta) = 0$. Therefore this equation shows a thermal softening effect. Moreover, the thermal and electrical flow conditions on the contact zone are given by

$$|D \cdot \nu| \le k\,, \quad |D \cdot \nu| = k\,\frac{\varphi}{|\varphi|} \;\; \text{if} \;\; \varphi \ne 0 \quad \text{on } \;\Gamma_C \times (0,T).$$

This condition represents the electric condition on the contact surface and we assume them by analogy with Tresca's friction law, where k is a given positive function, the electric conductivity coefficient

$$q \cdot \nu = k_c(u_\nu - g)\, \phi_L(\theta - \theta_F) \quad \text{on} \quad \Gamma_C \times (0,T),$$

describes the heat balance on the contact interface, where ϕ_L is the truncation function, k_c represents the thermal conductance function that is supposed such that

$$\phi_L(s) = \begin{cases} -L & \text{if } s < -L \\ s & \text{if } -L \le s \le L \\ L & \text{if } s > L \end{cases}, \quad k_c(r) = \begin{cases} k_c(r) = 0 & \text{if } r < 0, \\ \\ k_c(r) > 0 & \text{if } r \ge 0, \end{cases}$$

where $L > 0$ is a sufficiently large constant. Finally, we denote by u_0, v_0, φ_0 and θ_0 the initial displacement, initial velocity, initial potential and initial temperature, respectively. We collect the above relations to obtain the following mathematical model.

Problem (P) : *Find a displacement* $u : \Omega \times [0,T] \to \mathbb{R}^d$*, an electric potential* φ : $\Omega \times [0,T] \to \mathbb{R}$ *and a temperature* $\theta : \Omega \times [0,T] \to \mathbb{R}$ *such that*

$$\sigma = \mathcal{A}\,\varepsilon(\dot{u}) + \mathcal{F}\,\varepsilon(u) - \mathcal{E}^* E(\varphi) - \mathcal{M}\,\theta \quad \text{in } \Omega \times (0,T), \tag{3}$$

$$D = \mathcal{E}\,\varepsilon(u) + \beta\, E(\varphi) + \mathcal{P}\,\theta \quad \text{in } \Omega \times (0,T), \tag{4}$$

$$q = -\mathcal{K}\,\nabla\theta \quad \text{in } \Omega \times (0,T), \tag{5}$$

$$\ddot{u} - \operatorname{Div}\sigma = f_0 \quad \text{in } \Omega \times (0,T), \tag{6}$$

$$\operatorname{div} D = \phi_0 \quad \text{in } \Omega \times (0,T), \tag{7}$$

$$\dot{\theta} + \operatorname{div} q = -\mathcal{M}^*\,\varepsilon(\dot{u}) - \mathcal{N}\,E(\varphi) + q_0 \quad \text{in } \Omega \times (0,T), \tag{8}$$

$$u = 0 \quad \text{on } \Gamma_D \times (0,T), \tag{9}$$

$$\sigma\nu = f_2 \quad \text{on } \Gamma_N \times (0,T), \tag{10}$$

$$\sigma_\nu(u,\varphi,\theta) \le 0\ ,\ (u_\nu - g) \le 0\ ,\ \sigma_\nu(u,\varphi,\theta)\,(u_\nu - g) = 0 \quad \text{on } \Gamma_C \times (0,T), \tag{11}$$

$$\begin{cases} \|\sigma_\tau\| \le \mu(\theta)\,|R\sigma_\nu| \\ \|\sigma_\tau\| < \mu(\theta)\,|R\sigma_\nu| \implies [\dot{u}]_\tau = 0 \\ \|\sigma_\tau\| = \mu(\theta)\,|R\sigma_\nu| \implies \exists\,\lambda \in \mathbb{R},\ \sigma_\tau = -\lambda^2\,[\dot{u}]_\tau \end{cases} \quad \text{on } \Gamma_C \times (0,T), \tag{12}$$

$$\varphi = 0 \quad \text{on } \Gamma_a \times (0,T), \tag{13}$$

$$D \cdot \nu = q_2 \quad \text{on } \Gamma_b \times (0,T), \tag{14}$$

$$|D \cdot \nu| \le k\ ,\quad |D \cdot \nu| = k\frac{\varphi}{|\varphi|} \ \text{ if } \ \varphi \ne 0 \quad \text{on } \Gamma_C \times (0,T), \tag{15}$$

$$\theta = 0 \quad \text{on } \Gamma_D \cup \Gamma_N \times (0,T), \tag{16}$$

$$q \cdot \nu = k_c(u_\nu - g)\,\phi_L(\theta - \theta_F) \quad \text{on } \Gamma_C \times (0,T), \tag{17}$$

$$u(\cdot,0) = u_0\ ,\quad \dot{u}(\cdot,0) = v_0\ ,\quad \theta(\cdot,0) = \theta_0 \quad \text{in } \Omega. \tag{18}$$

To derive the weak formulation of Problem P, we introduce the following spaces:

$$H = L^2(\Omega)^d\,,\quad \mathcal{H} = \{\tau = (\tau_{ij})\,,\ \tau_{ij} = \tau_{ji} \in L^2(\Omega)\},$$
$$H_1 = H^1(\Omega)^d\,,\quad \mathcal{H}_1 = \{\sigma \in \mathcal{H}\,,\ \operatorname{Div}\sigma \in H\},$$

which are the real Hilbert spaces for the associated Euclidean norms to the inner products

$$(u,v)_H=\int_\Omega u_i v_i\,dx\ ,\quad (u,v)_{H_1}=(u,v)_H+(\varepsilon(u),\varepsilon(v))_{\mathcal{H}},$$
$$(\sigma,\tau)_{\mathcal{H}}=\int_\Omega \sigma_{ij}\tau_{ij}\,dx\ ,\quad (\sigma,\tau)_{\mathcal{H}_1}=(\sigma,\tau)_{\mathcal{H}}+(\mathrm{Div}\,\sigma,\mathrm{Div}\,\tau)_{\mathcal{H}}.$$

Keeping in mind (9), (13) and (16), we define the following variational subspaces:

$$V=\{v\in H_1\ ,\ v=0\ \text{ on }\ \Gamma_D\},$$
$$W=\{\psi\in H^1(\Omega)\ ,\ \psi=0\ \text{ on }\ \Gamma_a\},$$
$$Q\ =\{\eta\in H^1(\Omega),\ \eta=0\ \text{ on }\ \Gamma_D\cup\Gamma_N\}.$$

Over spaces V, Q and W, we use the following inner products and norms given by

$$(u,v)_V=(\varepsilon(u),\varepsilon(v))_{\mathcal{H}}\ ,\quad \|u\|_V=(u,u)_V^{1/2},\quad \forall\,u,\,v\in V, \tag{19}$$
$$(\varphi,\psi)_W=(\nabla\varphi,\nabla\psi)_H\ ,\quad \|\varphi\|_W=(\varphi,\varphi)_W^{1/2},\quad \forall\,\varphi,\,\psi\in W, \tag{20}$$
$$(\theta,\eta)_Q=(\nabla\theta,\nabla\eta)_H\ ,\quad \|\theta\|_Q=(\theta,\theta)_Q^{1/2},\quad \forall\,\theta,\,\eta\in Q. \tag{21}$$

Since V is a closed subspace of the Hilbert space H_1, and $meas(\Gamma_1)>0$, Korn's inequality holds, then there exists a constant $c_k>0$ depending only on Ω and Γ_1 such that

$$\|\varepsilon(v)\|_{\mathcal{H}}\geq\ c_k\,\|v\|_{H_1},\quad \forall\,v\in V. \tag{22}$$

Then the norms $\|\cdot\|_{H_1}$ and $\|\cdot\|_V$ are equivalent on V and therefore $(V,\|\cdot\|_V)$ is a real Hilbert space. Moreover, by the Sobolev trace theorem, there exists a constant $c_0>0$ depending only on Ω, Γ_C and Γ_D such that

$$\|v\|_{L^2(\Gamma)^d}\leq c_0\,\|v\|_V,\quad \forall\,v\in V. \tag{23}$$

Since $meas(\Gamma_a)>0$, the Friedrichs-Poincaré inequality holds and thus

$$\|\nabla\psi\|_H\geq c_F\|\psi\|_{H^1(\Omega)},\quad \forall\,\psi\in W, \tag{24}$$

where $c_F>0$ is a constant which depends only on Ω and Γ_a. It follows from (20) and (24) that $\|\cdot\|_W$ and $\|\cdot\|_{H^1(\Omega)}$ are equivalent norms on W and then $(W,\|\cdot\|_W)$ is a real Hilbert space. The Sobolev trace theorem implies that there exists $c_1>0$ depending on Ω, Γ_a and Γ_C such that

$$\|\xi\|_{L^2(\Gamma_C)}\leq c_1\,\|\xi\|_W,\quad \forall\,\xi\in W. \tag{25}$$

In an analogous way, we can get that $\|\cdot\|_Q$ and $\|\cdot\|_{H^1(\Omega)}$ are equivalent norms on Q and then $(Q,\|\cdot\|_Q)$ is a real Hilbert space. Using the Sobolev trace theorem, we obtain that there exists a constant $c_2>0$ depending only on Ω, Γ_D, Γ_N and Γ_C such that

$$\|\eta\|_{L^2(\Gamma)}\leq c_2\,\|\eta\|_Q,\quad \forall\,\eta\in Q. \tag{26}$$

For a real Banach space $(X,\|\cdot\|_X)$, we denote by $X^{'}$ the dual space of X and by $\langle\cdot,\cdot\rangle_{X'\times X}$ the duality pairing between $X^{'}$ and X. We consider the following standard Bochner-Lebesgue function spaces:

$$\mathbb{H}=L^2([0,T],H)\,,\ \mathbb{V}=L^2([0,T],V)\,,\ \mathbb{W}=L^2([0,T],W)\,,\ \mathbb{Q}=L^2([0,T],Q). \tag{27}$$

The notations $\|.\|_{\mathbb{H}}$, $\|.\|_{\mathbb{V}}$, $\|.\|_{\mathbb{W}}$ and $\|.\|_{\mathbb{Q}}$ stand for the norms of $\mathbb{H}$, $\mathbb{V}$, $\mathbb{Q}$ and $\mathbb{W}$, respectively. We also denote by $\langle .,. \rangle$ the duality pairing between V' and V or W' and W or Q' and Q , as the meaning is evident from the context.

To solve the mechanical problem (3)-(18), we need the following assumptions.

$\underline{A_1}$: (a) The elasticity and the viscosity tensors $\mathfrak{F}$, $\mathcal{A} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d$, the electric permittivity and the thermal conductivity tensors β, $\mathcal{K} : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ satisfy

$$\mathfrak{F}_{ijkl} = \mathfrak{F}_{klij} = \mathfrak{F}_{jikl} \in L^\infty(\Omega) \;, \quad \beta_{ij} = \beta_{ji} \in L^\infty(\Omega),$$
$$\mathcal{A}_{ijkl} = \mathcal{A}_{jikl} = \mathcal{A}_{lkij} \in L^\infty(\Omega) \;, \quad \mathcal{K}_{ij} = \mathcal{K}_{ji} \in L^\infty(\Omega),$$

(b) There exist positive constants $m_{\mathcal{F}}$, $m_{\mathcal{A}}$, m_β and $m_{\mathcal{K}}$ such that

$$\mathcal{F}_{ijkl}(x)\, \xi_{ij}\xi_{kl} \geq m_{\mathcal{F}}\, \|\xi\|^2 \;, \quad \mathcal{A}_{ijkl}(x)\, \xi_{ij}\xi_{kl} \geq m_{\mathcal{A}}\, \|\xi\|^2, \quad \forall\, \xi = (\xi_{ij}) \in \mathbb{S}^d,$$
$$\beta_{ij}(x)\, \zeta_i\zeta_j \geq m_\beta\, \|\zeta\|^2 \;, \quad \mathcal{K}_{ij}(x)\, \zeta_i\zeta_j \geq m_{\mathcal{K}}\, \|\zeta\|^2, \quad \forall\, \zeta = (\zeta_i) \in \mathbb{R}^d.$$

Under the previous assumptions, the following constants are well-defined:

$$M_{\mathfrak{F}} = \sup_{ijkl} \|\mathfrak{F}_{ijkl}\|_{L^\infty(\Omega)} \;, \quad M_\beta = \sup_{ij} \|\beta_{ij}\|_{L^\infty(\Omega)},$$
$$M_{\mathcal{A}} = \sup_{ijkl} \|\mathcal{A}_{ijkl}\|_{L^\infty(\Omega)} \;, \quad M_{\mathcal{K}} = \sup_{ij} \|\mathcal{K}_{ij}\|_{L^\infty(\Omega)}.$$

$\underline{A_2}$: The piezoelectric tensor $\mathcal{E} : \Omega \times \mathbb{S}^d \to \mathbb{R}^d$, the pyroelectric tensor $\mathcal{P} : \Omega \times \mathbb{R} \to \mathbb{R}^d$, the thermal expansion tensor $\mathcal{M} : \Omega \times \mathbb{R} \to \mathbb{R}^d$ and the tensors $\mathcal{N} : \Omega \times \mathbb{R}^d \to \mathbb{R}$ satisfy the following properties:

$$\mathcal{E}_{ijk} = \mathcal{E}_{ikj} \in L^\infty(\Omega)\,, \; \mathcal{M}_{ij} = \mathcal{M}_{ji} \in L^\infty(\Omega)\,, \; \mathcal{P}_i \in L^\infty(\Omega)\,, \; \mathcal{N}_i \in L^\infty(\Omega).$$

Under the previous assumptions, the following constants are well-defined:

$$M_{\mathcal{E}} = \sup_{ijk} \|\mathcal{E}_{ijk}\|_{L^\infty(\Omega)} \;, \quad M_{\mathcal{M}} = \sup_{ij} \|\mathcal{M}_{ij}\|_{L^\infty(\Omega)},$$
$$M_{\mathcal{P}} = \sup_{i} \|\mathcal{P}_i\|_{L^\infty(\Omega)} \;, \quad M_{\mathcal{N}} = \sup_{i} \|\mathcal{N}_i\|_{L^\infty(\Omega)}.$$

$\underline{A_3}$: The friction coefficient $\mu : \Gamma_C \times \mathbb{R} \to \mathbb{R}^+$ satisfies

(a) $\exists\, \mu^* > 0$ such that $|\mu(x,u)| \leq \mu^*, \; \forall\, u \in \mathbb{R}, \;$ a.e. $x \in \Gamma_C$,

(b) $\exists\, L_\mu > 0$ such that, for all $u, v \in \mathbb{R}$, one has

$$|\mu(x,u) - \mu(x,v)| \leq L_\mu |u - v|, \;\text{ a.e. } x \in \Gamma_C,$$

(c) $x \mapsto \mu(x,u)$ is measurable on Γ_C for all $u \in \mathbb{R}$.

$\underline{A_4}$: The function $k_c : \Gamma_C \times \mathbb{R} \to \mathbb{R}_+$ satisfies

(a) $\exists\, M_{k_c} > 0$ such that $|k_c(x,u)| \leq M_{k_c}, \; \forall\, u \in \mathbb{R}, \;$ a.e. $x \in \Gamma_C$,

(b) $\exists\, L_{k_c} > 0$ such that, for all u, $v \in \mathbb{R}$, one has

$$|k_c(x,u) - k_c(x,v)| \leq L_{k_c} |u - v|, \;\text{ a.e. } x \in \Gamma_C,$$

(c) $\mapsto k_c(x,u)$ is measurable on Γ_C for all $u \in \mathbb{R}$.

$\underline{A_5}$: The truncation function $\varphi_L : \Gamma_C \times \mathbb{R} \to \mathbb{R}$ satisfies

(a) $(\varphi_L(s_1) - \varphi_L(s_2))\,(s_1 - s_2) \geq 0 \quad$ for all $s_1, s_2 \in \mathbb{R}$ a.e. $x \in \Gamma_C$,

(b) $\exists\, L_\varphi > 0$ such that, for all $s_1, s_2 \in \mathbb{R}$, one has

$$|\varphi_L(x,s_1) - \varphi_L(x,s_2)| \leq L_\varphi\,|s_1 - s_2| \text{ a.e. } x \in \Gamma_C,$$

(c) $\exists\, M_\varphi > 0$ such that $|\varphi_L(x,s)| \leq M_\varphi,\ \forall\, s \in \mathbb{R}$ a.e. $x \in \Gamma_C$

(d) $x \mapsto \varphi_L(x,s)$ is measurable on Γ_C for all $s \in \mathbb{R}$.

$\underline{A_6}$: The function $R : H^{-\frac{1}{2}}(\Gamma_C) \to L^\infty(\Gamma_C)$ is bounded and Lipschitz continuous, i.e.,

(a) $\exists\, L_R > 0$ such that for all $s_1,\ s_2 \in H^{-\frac{1}{2}}(\Gamma_C)$, we have

$$\|Rs_1 - Rs_2\|_{L^\infty(\Gamma_C)} \leq L_R\,\|s_1 - s_2\|_{H^{\frac{1}{2}}(\Gamma_C)},$$

(b) $\exists\, M_R > 0$ such that for all $s \in H^{-\frac{1}{2}}(\Gamma_C)$, we have $\|Rs\|_{L^\infty(\Gamma_C)} \leq M_R$.

$\underline{A_7}$: The given forces, charge densities and heat sources satisfy the below regularity

(a) $f_0 \in L^2([0,T], L^2(\Omega)^d)\ ,\quad f_2 \in L^2([0,T]; L^2(\Gamma_N)^d)\ ,\quad g \in L^2(\Gamma_C)$,

(b) $\phi_0 \in L^2([0,T]; L^2(\Omega))\ ,\quad q_2 \in L^2([0,T]; L^2(\Gamma_b))\ ,\quad \varphi_f \in L^2([0,T], L^2(\Gamma_C))$.

$\underline{A_8}$: The initial data satisfy $u_0 \in V$, $v_0 \in V$, $\varphi_0 \in W$ and $\theta_0 \in Q$.

We move now to deriving the weak formulation of the problem (3)-(18). To this end, we assume that $(u, \sigma, \varphi, \theta)$ are smooth functions which solve (3)-(18), by invoking standard Green's formula, we obtain the following weak formulation of Problem (P).

Problem (PV) : *Find a displacement* $u \in \mathbb{V}$*, an electric potential* $\varphi \in \mathbb{W}$*, and a temperature* $\theta \in \mathbb{Q}$ *such that*

$$\begin{aligned}
&\langle \ddot{u}, v - \dot{u}\rangle_H + \langle \mathcal{A}\varepsilon(\dot{u}), \varepsilon(v - \dot{u})\rangle_{\mathcal{H}} + \langle \mathcal{F}\varepsilon(u), \varepsilon(v - \dot{u})\rangle_{\mathcal{H}} \\
&- \langle \mathcal{E}^* E(\varphi), \varepsilon(v - \dot{u})\rangle_H - \langle \mathcal{M}\theta, \varepsilon(v - \dot{u})\rangle_H + \int_{\Gamma_C} \mu(\theta)|R\sigma_\nu| \cdot (|v_\tau| - |\dot{u}_\tau|)\, da \\
&\geq \langle f_0, v - \dot{u}\rangle_H + \langle f_N, v - \dot{u}\rangle_{L^2(\Gamma_N)}\,, \quad \forall\, v \in \mathbb{V},
\end{aligned} \tag{28}$$

$$\begin{aligned}
&\langle \beta\nabla\varphi, \nabla\varphi - \nabla\psi\rangle_H - \langle \mathcal{E}\varepsilon(u), \nabla\varphi - \nabla\psi\rangle_H - \langle \mathcal{P}\theta, \nabla\varphi - \nabla\psi\rangle_H \\
&+ \int_{\Gamma_C} k \cdot (|\varphi| - |\psi|)\, da \geq \langle \phi_0, \varphi - \psi\rangle_H + \langle q_2, \varphi - \psi\rangle_{L^2(\Gamma_N)}\,, \quad \forall \psi \in \mathbb{W},
\end{aligned} \tag{29}$$

$$\begin{aligned}
&\langle \dot{\theta}, \theta - \eta\rangle_H + \langle \mathcal{K}\nabla\theta, \nabla\theta - \nabla\eta\rangle_H + \langle \mathcal{N}\varphi, \nabla\theta - \nabla\eta\rangle_H - \langle \mathcal{M}^*\varepsilon(\dot{u}), \nabla\theta - \nabla\eta\rangle_H \\
&+ \int_{\Gamma_C} k_c(u_\nu - g)\varphi_L(\theta - \theta_f) \cdot (\eta - \theta)\, da = \langle q_0, \eta - \theta\rangle_H\,, \quad \forall \eta \in \mathbb{Q}.
\end{aligned} \tag{30}$$

4 Existence and Uniqueness Result

In this section, we will prove the existence and uniqueness of the solution of the previous contact problem, by using the penalty method and intermediate problem. We first rewrite the variational formulation in abstract form. For this purpose, we consider the operators $A,\ F \in \mathcal{L}(\mathbb{V},\mathbb{V}^*)$, $B \in \mathcal{L}(\mathbb{W},\mathbb{W}^*)$, $K \in \mathcal{L}(\mathbb{Q},\mathbb{Q}^*)$, $E_1 \in \mathcal{L}(\mathbb{W},\mathbb{V}^*)$, $M_1 \in \mathcal{L}(\mathbb{Q},\mathbb{V}^*)$, $M_2 \in \mathcal{L}(\mathbb{V},\mathbb{Q}^*)$, $N \in \mathcal{L}(\mathbb{W},\mathbb{Q}^*)$, $E_2 \in \mathcal{L}(\mathbb{V},\mathbb{W}^*)$ and $P \in \mathcal{L}(\mathbb{Q},\mathbb{W}^*)$ defined as follows:

$$\begin{aligned}
&\langle Au,v\rangle = (\mathcal{A}\,\varepsilon(u),\varepsilon(v))_{\mathcal{H}}\,,\ \langle Fu,v\rangle = (\mathfrak{F}\varepsilon(u),\varepsilon(v))_{\mathcal{H}}\,,\ \langle E_1\varphi,v\rangle = (\mathcal{E}^*\nabla\varphi,\varepsilon(v))_H,\\
&\langle M_1\theta,v\rangle = (\mathcal{M}\theta,\varepsilon(v))_H\,,\ \langle B\varphi,\xi\rangle = (\beta\nabla\varphi,\nabla\xi)_H\,,\ \langle E_2u,\xi\rangle = (\mathcal{E}\varepsilon(u),\nabla\xi)_H,\\
&\langle P\theta,\xi\rangle = (\mathcal{P}\theta,\nabla\xi)_H\,,\ \langle K\theta,\eta\rangle = (\mathcal{K}\nabla\theta,\nabla\eta)_H\,,\ \langle M_2u,\eta\rangle = (\mathcal{M}^*\varepsilon(u),\eta)_H,\\
&\langle N\varphi,\eta\rangle = (\mathcal{N}\nabla\varphi,\eta)_H.
\end{aligned} \tag{31}$$

We next introduce the friction functional $j_1 : Q \times H^{-1/2} \times V \to \mathbb{R}$, the thermal and electrical transfer functional $j_2 : W \to \mathbb{R}$ and $h_c : V \times Q \to Q'$, respectively defined by

$$j_1(\theta,s,v) = \int_0^T \int_{\Gamma_C} \mu(\theta)\,|Rs|\cdot|v_\tau|\,da\,dt,\quad \forall v \in V, \tag{32}$$

$$j_2(\varphi) = \int_0^T \int_{\Gamma_C} k\ |\varphi|\,da\,dt,\quad \forall\eta \in W, \tag{33}$$

$$\langle h_c(u,\theta),\eta\rangle = \int_{\Gamma_C} k_c(u_\nu - g)\,\varphi_L(\theta-\theta_f)\cdot\eta\ da,\quad \forall\eta \in Q. \tag{34}$$

By Riesz's representation theorem, there exist $f \in \mathbb{V}'$, $q_e \in \mathbb{W}'$ and $\Theta \in \mathbb{Q}'$ such that

$$\langle f,v\rangle_{\mathbb{V}'\times V} = \int_0^T \int_\Omega f_0(t)\cdot v\,dx\,dt + \int_0^T \int_{\Gamma_N} f_2(t)\cdot v\,da\,dt,\quad \forall\, v \in V, \tag{35}$$

$$\langle q_e,\xi\rangle_{\mathbb{W}'\times W} = \int_0^T \int_\Omega \phi_0(t)\cdot\xi\,dx\,dt - \int_0^T \int_{\Gamma_b} q_2(t)\cdot\xi\,da\,dt,\quad \forall\,\xi \in W, \tag{36}$$

$$\langle\Theta,\eta\rangle_{\mathbb{Q}'\times Q} = \int_0^T \int_\Omega q_0(t)\cdot\eta\,dx\,dt,\quad \forall\,\eta \in Q. \tag{37}$$

Then, Problem $(\mathcal{PV})$ can be formulated in the following abstract form.

Problem $(\mathcal{PV}^1)$**:** Find $u \in \mathbb{V}$, $\varphi \in \mathbb{W}$, and $\theta \in \mathbb{Q}$ such that

$$\ddot u \in \mathbb{V}^*,\ \dot u \in \mathbb{V}\,,\ \dot\theta \in \mathbb{Q}, \tag{38}$$

$$f \in \ddot u + A\dot u + Fu + E_1\varphi - M_1\theta + \partial_3 j_1(\theta,\sigma_\nu,\dot u)\ \text{ in }\ \mathbb{V}^*, \tag{39}$$

$$q_e \in B\varphi - E_2u - P\theta + \partial j_2(\varphi)\ \text{ in }\ \mathbb{W}^*, \tag{40}$$

$$\dot\theta + K\theta + N\varphi - M_2\dot u + h_c(u,\theta) = \Theta \qquad \text{in } \mathbb{Q}^*, \tag{41}$$

$$u(\cdot,0) = u_0\,,\ \dot u(\cdot,0) = v_0\,,\ \theta(\cdot,0) = \theta_0\ \text{ in }\ \Omega, \tag{42}$$

where $\partial_3 j_1(\theta,s,v)$ denotes the partial sub-differential with respect to v of $j_1(\theta,s,v)$, and $\partial j_2(\varphi)$ for the partial sub-differential with respect to φ of $j_2(\varphi)$.

We are now able to state the following existence and uniqueness result.

Theorem 4.1 *Suppose assumptions* (A_1)-(A_8) *hold. Then Problem* $(\mathcal{PV}^1)$ *has at least one solution* $(u,\varphi,\theta) \in \mathbb{V} \times \mathbb{W} \times \mathbb{Q}$.

Proof. The proof is based on the arguments from the theory of multi-valued pseudo-monotone operators and the fixed point theorem. It consists of two steps. First, let $\zeta = (\zeta_1, \zeta_2) \in L^2(0,T; L^2(\Gamma_3) \times V)$ be a given function, we consider the mappings

$$j^1_\zeta(v) := \int_0^T \int_{\Gamma_C} \zeta_1(t)\, |v_\tau|\, da\, dt, \quad \forall v \in V, \tag{43}$$

$$\langle h_c(\theta_\zeta), \eta\rangle := \int_{\Gamma_C} \zeta_2(s)\varphi_L(\theta_\zeta - \theta_f)\cdot \eta\, da, \quad \forall \eta \in Q. \tag{44}$$

Then, for any given $\xi \in L^2(0,T; L^2(\Gamma_C) \times V)$, we consider the intermediate problem.

Problem $(\mathcal{PV}^1_\zeta)$**:** Find $u_\zeta \in \mathbb{V}$, $\varphi_\zeta \in \mathbb{W}$ and $\theta_\zeta \in \mathbb{Q}$ such that

$$\ddot{u}_\zeta \in \mathbb{V}',\ \dot{u}_\zeta \in \mathbb{V},\ \dot{\theta}_\zeta \in \mathbb{Q}', \tag{45}$$

$$f \in \ddot{u}_\zeta + A\,\dot{u}_\zeta + Fu_\zeta + E_1\varphi_\zeta - M_1\theta_\zeta + \partial j^1_\zeta(\dot{u}_\zeta) \quad \text{in} \quad \mathbb{V}', \tag{46}$$

$$q_e \in B\varphi_\zeta - E_2 u_\zeta - P\theta_\zeta + \partial j_2(\varphi_\zeta) \quad \text{in} \quad \mathbb{W}', \tag{47}$$

$$\dot{\theta}_\zeta + K\theta_\zeta + N\varphi_\zeta - M_2\dot{u}_\zeta + h_c(\theta_\zeta) = \Theta \quad \text{in} \quad \mathbb{Q}', \tag{48}$$

$$u_\zeta(\cdot,0) = u_0,\ \dot{u}_\zeta(\cdot,0) = v_0,\ \theta_\zeta(\cdot,0) = \theta_0 \quad \text{in} \quad \Omega. \tag{49}$$

Lemma 4.1 *Assume* (A_1)-(A_2), (A_4)-(A_5) *and* (A_7)-(A_8) *hold. Then, for any given* $\zeta = (\zeta_1, \zeta_2) \in L^2(0,T; L^2(\Gamma_C) \times V)$, *Problem* $(\mathcal{PV}^1_\zeta)$ *admits a unique solution* $(u_\zeta, \varphi_\zeta, \theta_\zeta)$.

Proof. The proof of Lemma 4.1 will be carried out by considering a sequence of regularized approximations to problem (PV_ζ), and the solution of this problem is the limit of the regularized problem. To this end, let $\{\psi^h\}_{h>0}$ be a sequence of positive convex functions of $C^1(\mathbb{R}^d)$ which approximate the inner product $|\cdot|_{\mathbb{R}^d}$, and satisfy for any $h > 0$, the following conditions:

$$|\nabla\psi^h(s)| \le 2\ ,\ 0 \le \langle \nabla\psi^h(s), s\rangle\ ,\ \big|\nabla\psi^h(s) - |s|\,\big| \le s, \quad \forall\, s \in \mathbb{R}^d. \tag{50}$$

We next consider the operator $J^h_\zeta : V \to V$ defined as follows:

$$(J^h_{\zeta(t)} v, w)_V := \int_{\Gamma_C} \zeta_1(t)\nabla\psi^h(v_\tau)\cdot w_\tau\, da, \quad \forall\, v \in V. \tag{51}$$

We then approximate the functional j_2 by a family of regularized functions $J^h_2 : V \to R$, depending on $h > 0$, given for all $v \in V$, by

$$J^h_2(\varphi) = \int_{\Gamma_C} \sqrt{|\varphi|^2 + h}\, da. \tag{52}$$

The functional J^h_2 is Gateaux-differentiable and its derivative J^h_2 is defined as follows:

$$\langle J^h_2\varphi, \xi\rangle = \int_{\Gamma_3} \frac{\varphi\,\xi}{\sqrt{|\varphi|^2 + h}}\, da, \quad \forall\, v \in V. \tag{53}$$

Let $R_e : V \to V'$ be a Riesz isomorphism, i.e., $u \mapsto \ell_u$, where $\ell_u(v) = \langle u, x\rangle$. Next, for each $h > 0$, we consider the following regularized problem.

Problem $(\mathcal{PV}_\zeta^{1h})$**:** Find $\omega_\zeta^h \in \mathbb{V}$, $u_\zeta^h \in \mathbb{V}$, $\varphi_\zeta^h \in \mathbb{W}$ and $\theta_\zeta^h \in \mathbb{Q}$ such that

$$\dot\omega_\zeta^h \in \mathbb{V}', \quad \dot\theta_\zeta^h \in \mathbb{Q}', \tag{54}$$

$$\dot\omega_\zeta^h + A\,\omega_\zeta^h + Fu_\zeta^h + E_1\varphi_\zeta^h - M_1\theta_\zeta^h + J_\zeta^h\omega_\zeta^h = f \ \text{ in } \ \mathbb{V}', \tag{55}$$

$$B\varphi_\zeta^h - E_2u_\zeta^h - P\theta_\zeta^h + J_2^h\varphi_\zeta^h = q_e \ \text{ in } \ \mathbb{W}', \tag{56}$$

$$\dot\theta_\zeta^h + K\theta_\zeta^h + N\varphi_\zeta^h - M_2\omega_\zeta^h + h_c(\theta_\zeta^h) = \Theta \ \text{ in } \ \mathbb{Q}', \tag{57}$$

$$R_e\dot u_\zeta^h - R_e\omega_\zeta^h = 0 \ \text{ in } \ \mathbb{V}', \tag{58}$$

$$u_\zeta^h(\cdot,0) = u_0\,, \quad \omega_\zeta^h(\cdot,0) = v_0\,, \quad \theta_\zeta^h(\cdot,0) = \theta_0 \ \text{ in } \ \Omega. \tag{59}$$

Lemma 4.2 *For every $h > 0$ and $\zeta \in L^2(0,T;L^2(\Gamma_C)\times V)$, Problem $(\mathcal{PV}_\zeta^{1h})$ has a unique solution $(w_\zeta^h, u_\zeta^h, \varphi_\zeta^h, \theta_\zeta^h)$. Moreover, under the assumptions of Theorem 4.1, the solution $(w_\zeta^h, u_\zeta^h, \varphi_\zeta^h, \theta_\zeta^h)$ of Problem $(\mathcal{PV}_\zeta^{1h})$ has the following estimation:*

$$\begin{aligned}&\|\omega_\zeta^h(t)\|^2_{L^2(\Omega)^d} + \|u_\zeta^h(t)\|^2_V + \int_0^t \|\omega_\zeta^h(s)\|^2_V ds + \|\theta_\zeta^h(t)\|^2_{L^2(\Omega)}\\ &+ \int_0^t \|\theta_\zeta^h(s)\|^2_Q ds + \|\varphi_\zeta^h(t)\|^2_W ds \le c, \quad \forall\, t\in(0,T),\end{aligned} \tag{60}$$

for a positive constant c which is independent of h.

Proof. To prove Lemma 4.2, we use Theorem 2.1 with $F := V\times W\times V\times Q$ and $G = L^2(\Omega)^d\times W\times V\times L^2(\Omega)$. We also define the following two operators defined by

$$\mathcal{B}: G\to G', \quad \mathcal{B}X = \mathcal{B}\begin{pmatrix}\omega\\ \varphi\\ u\\ \theta\end{pmatrix} = \begin{pmatrix}\omega\\ 0\\ \theta\\ R_e\,u\end{pmatrix}, \tag{61}$$

$$\mathcal{A}(t,\cdot): F\to F', \quad \mathcal{A}(t,X) = \begin{pmatrix} A\,\omega + F\,u + E_1\,\varphi - M_1\theta + J^h\omega\\ B\varphi + J_2^h\varphi - E_2\,u - P\,\theta\\ K\theta + h_\zeta(\theta) - M\omega + N\varphi\\ -R_e\,\omega\end{pmatrix}. \tag{62}$$

We also choose the two elements $X_0\in G$ and $L\in F$ given by

$$X_0 = \begin{pmatrix}v_0\\ \varphi_0\\ u_0\\ \theta_0\end{pmatrix} \quad \text{and} \quad L = \begin{pmatrix}f\\ q_e\\ \Theta\\ 0\end{pmatrix}. \tag{63}$$

By using (61) and (63), we get that Problem ($\mathcal{PV}_\zeta^{1h}$) is equivalent to the problem below.

$$\begin{aligned}&\text{Find } X_\zeta^h = (\omega_\zeta^h, \varphi_\zeta^h, \theta_\zeta^h, u_\zeta^h)' \text{ such that}\\ &\qquad \mathcal{B}\dot{X}_\zeta^h(t) + \mathcal{A}X_\zeta^h(t) = L \ \text{ in } \ \mathbb{F}',\\ &\qquad \mathcal{B}X_\zeta^h(0) = BX_0 \ \text{ in } \ G'.\end{aligned} \tag{64}$$

We will show that the operators $\mathcal{A}$ and $\mathcal{B}$ satisfy conditions of Theorem 2.1. Indeed, from the assumptions stated in Theorem 4.1, we can easily prove that the operator $\mathcal{B} : G \to G'$ is linear, bounded, positive and symmetric and the operator $\mathcal{A}(t, \cdot)$ verifies the conditions (1) and (2). Thus, due to Theorem 2.1, we get that problem (64) has a unique solution $X_\zeta^h = (w_\zeta^h, \varphi_\zeta^h, u_\zeta^h, \theta_\zeta^h)$. Consequently, Problem ($\mathcal{PV}_\zeta^{1h}$) admits a unique solution $(w_\zeta^h, \varphi_\zeta^h, u_\zeta^h, \theta_\zeta^h) \in \mathbb{V} \times \mathbb{W} \times \mathbb{V} \times \mathbb{Q}$. Next, in order to verify the estimate (60), we multiply (55) by w_ζ^h to obtain

$$\langle \dot{w}_\zeta^h, w_\zeta^h\rangle + \langle Aw_\zeta^h + Fu_\zeta^h + J_\zeta^h w_\zeta^h, w_\zeta^h\rangle - \langle M_1\theta_\zeta^h, w_\zeta^h\rangle = \langle f, w_\zeta^h\rangle. \tag{65}$$

Recalling (A_1), (A_2) and (65), by employing several times Cauchy's inequality

$$ab \le \epsilon a^2 + \frac{1}{4\epsilon}b^2, \quad \forall a, b \in \mathbb{R}, \epsilon > 0, \tag{66}$$

we find

$$\begin{aligned}&\|w_\zeta^h(t)\|^2_{L^2(\Omega)^d} + \|u_\zeta^h(t)\|_V^2 + \int_0^t \|w_\zeta^h(s)\|_V^2 ds\\ &\le c\Big(\|u_0\|_V^2 + \|v_0\|^2 + \int_0^t (\|\zeta_1(s)\|^2_{L^2(\Gamma_C)} + \|f(s)\|^2_{L^2(\Omega)})\, ds\\ &\qquad + \int_0^t \|\theta_\zeta^h(s)\|_Q^2 ds + \int_0^t \|\varphi_\zeta^h(s)\|_W^2 ds\Big), \quad \forall t \in (0, T).\end{aligned} \tag{67}$$

Next, we let the potential equation (56) act on φ_ζ^h to get

$$\langle B\varphi_\zeta^h, w_\zeta^h\rangle - \langle E_2 u_\zeta^h, \varphi_\zeta^h\rangle - \langle M_2 w_\zeta^h, \varphi_\zeta^h\rangle + \langle J_2^h \varphi_\zeta^h, \varphi_\zeta^h\rangle = \langle \Theta, \varphi_\zeta^h\rangle.$$

Then, from assumptions (A_1), (A_2) and (53), we deduce after some manipulations that

$$\|\varphi_\zeta^h(t)\|_W^2 \le c\left(\|u_\zeta^h(t)\|_V^2 + \|\theta_\zeta^h(t)\|^2_{L^2(\Omega)} + \|q_e(t)\|_W^2\right), \quad \forall\, t \in (0, T). \tag{68}$$

Next, let the energy equation (57) act on θ_ζ^h. Then we have

$$\langle \dot{\theta}_\zeta^h, \theta_\zeta^h\rangle + \langle K\theta_\zeta^h + N\varphi_\zeta^h - M_2 w_\zeta^h, \theta_\zeta^h\rangle + \langle h_c(\theta_\zeta^h), \theta_\zeta^h\rangle = \langle \Theta, \theta_\zeta^h\rangle.$$

It follows from hypotheses (A_1), (A_2), (A_4), (A_5) and the monotonicity of φ_L that

$$\begin{aligned}&\|\theta_\zeta^h(t)\|^2_{L^2(\Omega)} + \int_0^t \|\theta_\zeta^h(s)\|_Q^2 ds\\ &\le c\Big(\|\theta_0\|_Q^2 + \int_0^t \|\varphi_\zeta^h(s)\|_W^2 ds + \int_0^t \|w_\zeta^h(s)\|_V^2 ds\\ &\qquad + \int_0^t \|\Theta(s)\|^2_{Q'} ds + \int_0^t \|\zeta_2(s)\|_V^2 ds\Big).\end{aligned} \tag{69}$$

Inserting the estimation (67) and (68) into (69), we find

$$\begin{aligned}
&\|\theta_\zeta^h(t)\|^2_{L^2(\Omega)} + \int_0^t \|\theta_\zeta^h(s)\|^2_Q ds \\
&\le c\Big(\|\theta_0\|^2_{L^2(\Omega)} + \|u_0\|^2_V + \|v_0\|^2_{L^2(\Omega)^d} + \int_0^t \|\Theta(s)\|^2_{Q'} ds + \int_0^t \|q_e(s)\|^2_W ds \\
&\quad + \int_0^t \|f(s)\|^2_{L^2(\Omega)} ds + \int_0^t \|\zeta_1(s)\|^2_{L^2(\Gamma_C)} ds + \int_0^t \|\zeta_2(s)\|^2_V ds\Big).
\end{aligned}$$

Thus,

$$\|\theta_\zeta^h(t)\|^2_{L^2(\Omega)} + \int_0^t \|\theta_\zeta^h(s)\|^2_Q ds \le c. \tag{70}$$

Integrate (68) over $(0,t)$, where $t \in [0,T]$, and combine the result with (67) to obtain

$$\begin{aligned}
&\|w_\zeta^h(t)\|^2_{L^2(\Omega)^2} + \|u_\zeta^h(t)\|^2_V + \int_0^t \|w_\zeta^h(s)\|^2_V ds + \int_0^t \|\varphi_\zeta^h(s)\|^2_W ds \\
&\le c\Big(\|u_0\|^2_V + \|v_0\|^2_{L^2(\Omega)^d} + \int_0^t \|q_e(s)\|^2_W ds + \int_0^t \|\zeta_1(s)\|^2_{L^2(\Gamma_C)} ds \\
&\quad + \int_0^t \|f(s)\|^2_{L^2(\Omega)} ds + \int_0^t \|u_\zeta^h(s)\|^2_V ds\Big).
\end{aligned}$$

Then, Grönwall's inequality and the inequality (70) lead to

$$\|u_\zeta^h(t)\|_V \le c. \tag{71}$$

Hence,

$$\|\varphi_\zeta^h(t)\|_W \le c. \tag{72}$$

Moreover, the estimations (70), (71) and (72) complete the proof of Lemma 4.2. □

Now, we have the ingredient which allows us to prove the existence and uniqueness of the solution of Problem (PV_ζ). Indeed, due to Lemma 4.2, we have for a given set of initial conditions $\{w_0^h, u_0^h, \theta_0^h\}_{h>0}$ that the family of solutions $\{w_\zeta^h, \varphi_\zeta^h, u_\zeta^h, \theta_\zeta^h\}_{h>0}$ is bounded in $\mathbb{V} \times \mathbb{W} \times \mathbb{V} \times \mathbb{Q}$. Then, from the latter result and Problem (PV_ζ^h), we get that $\{\dot{w}_\zeta^h, R_e\dot{u}_\zeta^h, \dot{\theta}_\zeta^h\}$ is also bounded in $\mathbb{V}' \times \mathbb{V}' \times \mathbb{Q}'$. Then there exists a subsequence of parameters $\{h_k\}$ such that $h_k \to 0$ as $k \to \infty$ so that

$$u_\zeta^{h_k} \rightharpoonup u_\zeta^* \quad \text{weakly in } \mathbb{V}, \tag{73}$$

$$\omega_\zeta^{h_k} \rightharpoonup \omega_\zeta^* \quad \text{weakly in } \mathbb{V}, \tag{74}$$

$$\varphi_\zeta^{h_k} \rightharpoonup \varphi_\zeta^* \quad \text{weakly in } \mathbb{W}, \tag{75}$$

$$\theta_\zeta^{h_k} \rightharpoonup \theta_\zeta^* \quad \text{weakly in } \mathbb{Q}, \tag{76}$$

$$\dot{\theta}_\zeta^{h_k} \rightharpoonup \dot{\theta}_\zeta^* \quad \text{weakly in } \mathbb{Q}', \tag{77}$$

$$\dot{\omega}_\zeta^{h_k} \rightharpoonup \dot{\omega}_\zeta^* \quad \text{weakly in } \mathbb{V}', \tag{78}$$

$$R_e\dot{u}_\zeta^{h_k} \rightharpoonup R_e\dot{u}_\zeta^* \quad \text{weakly in } \mathbb{V}'. \tag{79}$$

We are going now to prove that the triplet $(w^*_\zeta, \varphi^*_\zeta, u^*_\zeta, \theta^*_\zeta)$ verifies (53)-(59). Indeed, by passing to the limit as $k \to \infty$ in the relations $J^{h_k}_\zeta, J^{h_k}_2$ and $h_c(\theta^{h_k}_\zeta)$, we obtain

$$\begin{aligned}
\lim_{k\to\infty} \langle J^{h_k}_\zeta \omega^{h_k}_\zeta, v-\omega^*_\zeta\rangle &= \lim_{k\to\infty} \int_0^T \int_{\Gamma_C} \zeta_1(t)\nabla\psi^h(\omega^{h_k}_{\zeta\tau}).(v_\tau - \omega^*_{\zeta\tau})\, da\, dt \\
&\le \lim_{k\to\infty} \int_0^T \int_{\Gamma_C} |\zeta_1(t)|(\psi^h(v_\tau - \omega^*_{\zeta\tau} + \omega^{h_k}_{\zeta\tau}) - \psi^h(\omega^*_{\zeta\tau}))\, da\, dt \\
&\le \lim_{k\to\infty} \int_0^T \int_{\Gamma_C} |\zeta_1(t)|(|v_\tau|) - |\omega^*_{\zeta\tau}|))\, da\, dt,
\end{aligned} \tag{80}$$

$$\begin{aligned}
\lim_{k\to\infty} \langle J^{h_k}_2 \varphi^{h_k}_\zeta, \xi-\varphi^*_\zeta\rangle &= \lim_{k\to\infty} \int_{\Gamma_C} \frac{\varphi^{h_k}_\zeta\,(\xi-\varphi^*_\zeta)}{\sqrt{|\varphi^{h_k}_\zeta|^2+h}}\, da \\
&\le \lim_{k\to\infty} \int_{\Gamma_C} \frac{|\varphi^{h_k}_\zeta|\,|\xi|}{\sqrt{|\varphi^{h_k}_\zeta|^2+h}} - \frac{\varphi^{h_k}_\zeta \varphi^*_\zeta}{\sqrt{|\varphi^{h_k}_\zeta|^2+h}}\, da \le j_2(\xi) - j_2(\varphi^*_\zeta),
\end{aligned} \tag{81}$$

and

$$\lim_{k\to\infty} \langle h_c(\theta^{h_k}_\zeta), \eta\rangle = \langle h_c(u^*_\zeta, \theta^*_\zeta), \eta\rangle. \tag{82}$$

Next, due to assumptions (A_1)-(A_6), and conditions (73)-(79) and (80)-(82), the weak limit $(u^*_\zeta, \varphi^*_\zeta, \theta^*_\zeta)$ of the subsequence $(u^{h_k}_\zeta, \varphi^{h_k}_\zeta, \theta^{h_k}_\zeta)$ is a solution to Problem $(\mathcal{PV}^1_\zeta)$. Furthermore, to prove the uniqueness of the solution of Problem (PV_ζ), let $(u_1, \varphi_1, \theta_1) \in \mathbb{V}\times\mathbb{W}\times\mathbb{Q}$ and $(u_2, \varphi_2, \theta_2) \in \mathbb{V}\times\mathbb{W}\times\mathbb{Q}$ be two solutions corresponding to the same data ζ. We substitute u_ζ in (46) by u_1 and u_2, respectively, we obtain

$$\begin{aligned}
f &\in \ddot u_1 + A\,\dot u_1 + Fu_1 + E_1\varphi_1 - M_1\theta_1 + \partial j^1_\zeta(\dot u_1) \;\text{ in }\; \mathbb{V}', \\
f &\in \ddot u_2 + A\,\dot u_2 + Fu_2 + E_1\varphi_2 - M_1\theta_2 + \partial j^1_\zeta(\dot u_2) \;\text{ in }\; \mathbb{V}'.
\end{aligned}$$

Let the resulting expressions act on $\dot u_2 - \dot u_1$ and $\dot u_1 - \dot u_2$, respectively, we find

$$\begin{aligned}
\langle f, \dot u_2 - \dot u_1\rangle = &\langle \ddot u_1, \dot u_2 - \dot u_1\rangle + \langle A\dot u_1, \dot u_2 - \dot u_1\rangle + \langle Fu_1, \dot u_2 - \dot u_1\rangle \\
&+ \langle E_1\varphi_1, \dot u_2 - \dot u_1\rangle - \langle M_1\theta_1, \dot u_2 - \dot u_1\rangle \\
&+ \langle \mathcal{Z}(\dot u_1), \dot u_2 - \dot u_1\rangle \;\text{ with }\; \mathcal{Z}(\dot u_1) \in \partial j^1_\zeta(\dot u_1) \;\text{ in }\; \mathbb{V}',
\end{aligned} \tag{83}$$

$$\begin{aligned}
\langle f, \dot u_1 - \dot u_2\rangle = &\langle \ddot u_2, \dot u_1 - \dot u_2\rangle + \langle A\dot u_2, \dot u_1 - \dot u_2\rangle + \langle Fu_2, \dot u_1 - \dot u_2\rangle \\
&+ \langle E_1\varphi_2, \dot u_1 - \dot u_2\rangle - \langle M_1\theta_2, \dot u_1 - \dot u_2\rangle \\
&+ \langle \mathcal{Z}(\dot u_2), \dot u_1 - \dot u_2\rangle \;\text{ with }\; \mathcal{Z}(\dot u_2) \in \partial j^1_\zeta(\dot u_2) \;\text{ in }\; \mathbb{V}'.
\end{aligned} \tag{84}$$

On the other hand, it comes from the convexity of the functional j^1_ζ that

$$\langle \mathcal{Z}(\dot u_1), \dot u_2 - \dot u_1\rangle \le j^1_\zeta(\dot u_2) - j^1_\zeta(\dot u_1) \;\text{ and }\; \langle \mathcal{Z}(\dot u_2), \dot u_1 - \dot u_2\rangle \le j^1_\zeta(\dot u_1) - j^1_\zeta(\dot u_2).$$

Therefore, the sum of two previous relations leads to

$$\langle \mathcal{Z}(\dot u_2) - \mathcal{Z}(\dot u_1), \dot u_1 - \dot u_2\rangle \le 0. \tag{85}$$

Keeping in mind (85), by adding two inequalities (83) and (84), we find

$$\begin{aligned}&\langle \ddot{u}_2-\ddot{u}_1,\dot{u}_1-\dot{u}_2\rangle+\langle A\dot{u}_2-A\dot{u}_1,\dot{u}_1-\dot{u}_2\rangle\\&+\langle Fu_2-Fu_1,\dot{u}_1-\dot{u}_2\rangle+\langle E_1\varphi_2-E_1\varphi_1,\dot{u}_1-\dot{u}_2\rangle\\&-\langle M_1\theta_2-M_1\theta_1,\dot{u}_1-\dot{u}_2\rangle=-\langle \mathcal{Z}(\dot{u}_2)-\mathcal{Z}(\dot{u}_1),\dot{u}_1-\dot{u}_2\rangle\geq 0.\end{aligned}$$

Now, by integrating the previous inequality over $(0,t)$, we deduce

$$\begin{aligned}&\int_0^t\langle \ddot{u}_1-\ddot{u}_2,\dot{u}_1-\dot{u}_2\rangle ds+\int_0^t\langle A\dot{u}_1-A\dot{u}_2,\dot{u}_1-\dot{u}_2\rangle ds+\int_0^t\langle Fu_1-Fu_2,\dot{u}_1-\dot{u}_2\rangle ds\\&\leq\int_0^t\langle E_1\varphi_1-E_2\varphi_2,\dot{u}_1-\dot{u}_2\rangle ds+\int_0^t\langle M_1\theta_1-M_1\theta_2,\dot{u}_1-\dot{u}_2\rangle ds.\end{aligned}$$

Then, using the same assumptions as in Lemma 4.1, we can obtain

$$\begin{aligned}&\|\dot{u}_1(t)-\dot{u}_2(t)\|^2_{L^2(\Omega)^d}+\int_0^t\|\dot{u}_1(s)-\dot{u}_2(s)\|^2_V\,ds+\|u_1(t)-u_2(t)\|^2_V\\&\leq c\Big(\int_0^t\|\varphi_1(s)-\varphi_2(s)\|^2_W\,ds+\int_0^t\|\theta_1(s)-\theta_2(s)\|^2_{L^2(\Omega)}\,ds\Big).\end{aligned}\tag{86}$$

Following the same tricks as above, we can also find

$$\int_0^t\|\varphi_1(s)-\varphi_2(s)\|^2_W ds\leq c\Big(\int_0^t\|u_1(s)-u_2(s)\|^2_V ds+\int_0^t\|\theta_1(s)-\theta_2(s)\|^2_{L^2(\Omega)}ds\Big).\tag{87}$$

Next, we replace θ_1 and θ_2 in the relation (48), respectively, we get

$$\begin{aligned}&\dot{\theta}_1+K\theta_1+N\varphi_1-M_2\dot{u}_1+h_c(\theta_1)=\Theta\quad\text{in}\quad\mathbb{Q}',\\&\dot{\theta}_2+K\theta_2+N\varphi_2-M_2\dot{u}_2+h_c(\theta_2)=\Theta\quad\text{in}\quad\mathbb{Q}'.\end{aligned}$$

Then, by acting the obtained results on $\theta_1-\theta_2$ and by subtracting them, we find

$$\begin{aligned}&\langle\dot{\theta}_1-\dot{\theta}_2,\theta_1-\theta_2\rangle+\langle K(\theta_1-\theta_2),\theta_1-\theta_2\rangle\\&+\langle N(\varphi_1-\varphi_2),\theta_1-\theta_2\rangle-\langle M_2(\dot{u}_1-\dot{u}_2),\theta_1-\theta_2\rangle=h_c(\theta_2)-h_c(\theta_1).\end{aligned}$$

We next integrate over $(0,t)$, then the first integrals are estimated by

$$\begin{aligned}\int_0^t\langle\dot{\theta}_1(s)-\dot{\theta}_2(s),\theta_1(s)-\theta_2(s)\rangle\,ds&=\frac{1}{2}\int_0^t\frac{d\,\|\theta_1(s)-\theta_2(s)\|^2_{L^2(\Omega)}}{ds}ds\\&=\frac{1}{2}\|\theta_1(s)-\theta_2(s)\|^2_{L^2(\Omega)}-\frac{1}{2}\|\theta_1(0)-\theta_2(0)\|^2_{L^2(\Omega)}.\end{aligned}$$

For more details on the relation above, see [18, Theorem 1(2)]. The other integrals are estimated by using Cauchy's inequality, the properties of the operators K, N and M_2, and of the functional h_c. Combining all estimates of these integrals, we obtain

$$\begin{aligned}&\|\theta_1(t)-\theta_2(t)\|^2_{L^2(\Omega)}+\int_0^t\|\theta_1(s)-\theta_2(s)\|^2_Q\,ds\\&\leq c\Big(\int_0^t\|\varphi_1(s)-\varphi_2(s)\|^2_W\,ds+\int_0^t\|\dot{u}_1(s)-\dot{u}_2(s)\|^2_V\,ds\Big).\end{aligned}\tag{88}$$

Then, we deduce that

$$\begin{aligned} &\|\theta_1(t)-\theta_2(t)\|^2_{L^2(\Omega)}+\|u_1(t)-u_2(t)\|^2_V \\ &\le c\,\Big(\int_0^t \|\theta_1(s)-\theta_2(s)\|^2_{L^2(\Omega)}\,ds+\int_0^t \|u_1(s)-u_2(s)\|^2_V\,ds\Big). \end{aligned} \tag{89}$$

Now Gronwall's inequality implies that $\theta_1=\theta_2$ and $u_1=u_2$, and consequently, $\varphi_1=\varphi_2$, which completes the proof of Lemma 4.1. $\square$

Now, we introduce $\mathcal{L}: L^2(0,T;L^2(\Gamma_C)\times V)\to L^2(0,T;L^2(\Gamma_C)\times V)$ defined by

$$\mathcal{L}(\zeta):=(\mu(\theta_\zeta)\,|R\sigma_\nu(u_\zeta,\varphi_\zeta,\theta_\zeta)|\,;\,k_c(u_{\zeta,\nu}-g)) \tag{90}$$

for all $\zeta=(\zeta_1,\zeta_2)\in L^2(0,T;L^2(\Gamma_C)\times V)$ and where $(u_\zeta,\varphi_\zeta,\theta_\zeta)$ is the unique solution of Problem (PV_ζ) corresponding to ζ. We also consider the space Y defined as follows:

$$Y=\big\{\zeta\in L^2([0,T],L^2(\Gamma_C)\times V):\ \|\zeta\|_{L^2([0,T],L^2(\Gamma_C)\times V)}\le T\sqrt{meas(\Gamma_C)}(M_\mu M_R+M_{k_c})\big\}.$$

Then, we provide the following result that states that $\mathcal{L}$ has a fixed point on Y.

Lemma 4.3 *The operator $\mathcal{L}$ has a unique fixed point $\zeta^*\in Y$.*

Proof. Let $\zeta=(\zeta_1,\zeta_2)$, $\lambda=(\lambda_1,\lambda_2)\in L^2(0,T;L^2(\Gamma_C)\times V)$, and let us denote by $(u_\zeta,\varphi_\zeta,\theta_\zeta)$ and $(u_\lambda,\varphi_\lambda,\theta_\lambda)$ the solution of Problem $(\mathcal{PV}^1_\zeta)$ corresponding to ζ and λ, respectively. By using the definition (90) of the operator $\mathcal{L}$, we obtain

$$\begin{aligned} &\|\mathcal{L}(\zeta)(t)-\mathcal{L}(\lambda)(t)\|^2_{L^2(\Gamma_C)\times V} \\ &=\|\mu(\theta_\zeta)\,|R\sigma_\nu(u_\zeta,\varphi_\zeta,\theta_\zeta)|-\mu(\theta_\lambda)\,|R\sigma_\nu(u_\lambda,\varphi_\lambda,\theta_\lambda)|\|^2_{L^2(\Gamma_C)} \\ &\quad+\|k_c(u_{\zeta\nu}-g)-k_c(u_{\lambda\nu}-g)\|^2_V. \end{aligned} \tag{91}$$

First, it comes from hypothesis $(A_4)(b)$ that

$$\|k_c(u_{\zeta\nu}-g)-k_c(u_{\lambda\nu}-g)\|^2_V\le L^2_{k_c}c_1^2\|u_\zeta-u_\lambda\|^2_V. \tag{92}$$

Also, by using the hypotheses (A_3) and (A_6), we deduce

$$\begin{aligned} &\big\|\mu(\theta_\zeta)\,|R\sigma_\nu(u_\zeta,\varphi_\zeta,\theta_\zeta)|-\mu(\theta_\lambda)\,|R\sigma_\nu(u_\lambda,\varphi_\lambda,\theta_\lambda)|\big\|^2_{L^2(\Gamma_3)} \\ &\le M_\mu^2L_R^2\|\sigma_\nu(u_\zeta,\varphi_\zeta,\theta_\zeta)-\sigma_\nu(u_\lambda,\varphi_\lambda,\theta_\lambda)\|^2_{H^{-\frac12}(\Gamma_C)}+M_R^2L_\mu^2\|\theta_\zeta-\theta_\lambda\|^2_{L^2(\Gamma_3)}. \end{aligned}$$

Moreover, we know that there exists a constant $c_F>0$ such that

$$\begin{aligned} &\big\|\sigma_\nu(u_\zeta,\varphi_\zeta,\theta_\zeta)-\sigma_\nu(u_\lambda,\varphi_\lambda,\theta_\lambda)\big\|_{H^{-\frac12}(\Gamma_C)} \\ &=\sup_{v\in H^{1/2}(\Gamma_C)}\frac{\langle\sigma_\nu(u_\zeta,\varphi_\zeta,\theta_\zeta)-\sigma_\nu(u_\lambda,\varphi_\lambda,\theta_\lambda),v_\nu\rangle_{L^2(0,T;\mathcal{H})\times H^{-1/2}(\Gamma_C)}}{\|v_\nu\|_{H^{-1/2}(\Gamma_C)}} \\ &\le c_F\,\left(\|\dot u_\zeta-\dot u_\lambda\|_V+\|u_\zeta-u_\lambda\|_V+\|\varphi_\zeta-\varphi_\lambda\|_W+\|\theta_\zeta-\theta_\lambda\|_Q\right). \end{aligned}$$

Thus, by combining two previous inequalities, we get

$$\begin{aligned} &\big\|\mu(\theta_\zeta)\,|R\sigma_\nu(u_\zeta,\varphi_\zeta,\theta_\zeta)|-\mu(\theta_\lambda)\,|R\sigma_\nu(u_\lambda,\varphi_\lambda,\theta_\lambda)|\big\|^2_{L^2(\Gamma_3)} \\ &\le c^*\,(M_\mu^2L_R^2+M_R^2L_\mu^2)\big(\|\dot u_\zeta-\dot u_\lambda\|^2_V+\|\theta_\zeta-\theta_\lambda\|^2_Q\big) \\ &\quad+c\,\big(\|\varphi_\zeta-\varphi_\lambda\|^2_W+\|\theta_\zeta-\theta_\lambda\|^2_{L^2(\Omega)}+\|u_\zeta-u_\lambda\|^2_V\big). \end{aligned} \tag{93}$$

Using relations (39), (51), and assumptions (A_1), (A_7), we get

$$\begin{aligned}&\|\dot u_\zeta(t)-\dot u_\lambda(t)\|^2_{L^2(\Omega)^d}+\|u_\zeta(t)-u_\lambda(t)\|^2_V+\int_0^t\|\dot u_\zeta(s)-\dot u_\lambda(s)\|^2_V\,ds\\&\le c\Big(\int_0^t\|\zeta_1(s)-\lambda_1(s)\|^2_{L^2(\Gamma_C)}ds+\int_0^t\|u_\zeta(s)-u_\lambda(s)\|_V\,ds\\&\quad+\int_0^t\|\varphi_\zeta(s)-\varphi_\lambda(s)\|^2_W\,ds\Big),\quad\forall\,t\in[0,T].\end{aligned}\tag{94}$$

From relation (40) and assumptions (A_1)-(A_2), we find

$$\|\varphi_\zeta(t)-\varphi_\lambda(t)\|^2_W\le c\left(\|u_\zeta-u_\lambda\|^2_V+\|\theta_\zeta(t)-\theta_\lambda(t)\|^2_Q\right),\quad\forall\,t\in[0,T].\tag{95}$$

Moreover, keeping in mind (41), assumptions (A_1), (A_2), (A_4) and (A_5), we obtain

$$\begin{aligned}&\|\theta_\zeta(t)-\theta_\lambda(t)\|^2_Q+\int_0^t\|\theta_\zeta(s)-\theta_\lambda(s)\|^2_Q\,ds\\&\le c\Big(\int_0^t\|\varphi_\zeta(s)-\varphi_\lambda(s)\|^2_W\,ds+\int_0^t\|\zeta_2(s)-\lambda_2(s)\|^2_V\,ds\\&\quad+\int_0^t\|\dot u_\zeta(s)-\dot u_\lambda(s)\|^2_V\,ds\Big),\quad\forall\,t\in(0,T).\end{aligned}\tag{96}$$

We combine now the inequalities (94)-(96) to get that for all $t\in[0,T]$, we have

$$\begin{aligned}&\|\theta_\zeta(t)-\theta_\lambda(t)\|^2_Q+\|\dot u_\zeta(t)-\dot u_\lambda(t)\|^2_V+\|u_\zeta(t)-u_\lambda(t)\|^2_V+\|\varphi_\zeta(t)-\varphi_\lambda(t)\|^2_W\\&+\int_0^t\|\dot u_\zeta(s)-\dot u_\lambda(s)\|^2_V\,ds+\int_0^t\|\theta_\zeta(s)-\lambda(s)\|^2_Q\,ds\\&\le c\Big(\int_0^t\|\zeta_2(s)-\lambda_2(s)\|^2_V\,ds+\int_0^t\|\zeta_1(s)-\lambda_1(s)\|^2_{L^2(\Gamma_C)}\,ds\\&+\int_0^t\|\dot u_\zeta(s)-\dot u_\lambda(s)\|^2_V\,ds+\int_0^t\|\theta_\zeta(s)-\theta_\lambda(s)\|^2_Q\,ds+\int_0^t\|\varphi_\zeta(s)-\varphi_\lambda(s)\|^2_W\,ds\Big).\end{aligned}$$

Moreover, by employing Gronwall's inequality, we deduce

$$\begin{aligned}&\|\theta_\zeta(t)-\theta_\lambda(t)\|^2_Q+\|\dot u_\zeta(t)-\dot u_\lambda(t)\|^2_V+\|u_\zeta(t)-u_\lambda(t)\|^2_V+\|\varphi_\zeta(t)-\varphi_\lambda(t)\|^2_W\\&+\int_0^t\|\dot u_\lambda(s)-\dot u_\lambda(s)\|^2_V+\int_0^t\|\theta_\zeta(s)-\theta_\lambda(s)\|^2_Q\,ds\\&\le c\Big(\int_0^t\|\zeta_2(s)-\lambda_2(s)\|^2_V\,ds+\int_0^t\|\zeta_1(s)-\lambda_1(s)\|^2_{L^2(\Gamma_C)}\,ds\Big),\ \forall\,t\in(0,T).\end{aligned}\tag{97}$$

We integrate (91) over $[0,t]$ for a given $t\in(0,T)$, then use (92), (93) and (97) to get

$$\|\mathcal{L}(\zeta)(t)-\mathcal{L}(\lambda)(t)\|^2_{L^2([0,T],L^2(\Gamma_C)\times V)}\le c\,\|\zeta(s)-\lambda(s)\|^2_{L^2([0,T],L^2(\Gamma_C)\times V)}.\tag{98}$$

Thus, the operator $\mathcal{L}$ is Lipschitz continuous on $L^2([0,T],L^2(\Gamma_C)\times V)$. In addition, we can easily show that the operator Λ is continuous from Y into itself. We have also Y is a nonempty, convex closed subset of the reflexive space $L^2([0,T],L^2(\Gamma_C)\times V)$, then

Y is weakly compact. Finally, it follows from Schauder's fixed point theorem that the operator $\mathcal{L}$ admits a unique fixed point $\zeta^* \in Y$, and finally, Lemma 4.3 holds. □

To finish the proof of Theorem 4.1, let $\zeta^* \in Y$ be a fixed point $\mathcal{L}$ defined by (90), it is straightforward to show that the solution $(u_{\zeta^*}, \varphi_{\zeta^*}, \theta_{\zeta^*})$ of Problem (PV_ζ) corresponding to ζ^*, is also a solution of Problem (PV) (called the weak solution of Problem (P)). Moreover, the uniqueness of the solution to Problem (PV) can be provided from the uniqueness of the solution to Problem (PV_ζ), which ends the proof of Theorem 4.1. □

5 Conclusion

The main contribution here is the proof of the unique solvability of a new dynamic thermo-electro-elastic contact model, i.e., Problem (P), which takes into account the effects of thermal softening and frictional heating at the contact surface. So, an existence and uniqueness result has been obtained when $M_\mu L_R + M_R L_\mu$ is sufficiently small. We note here that estimating the allowed size of coefficient $M_\mu L_R + M_R L_\mu$ remains an open and very interesting question.

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