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Regional Weak and Strong Stabilization of Time Delay Infinite Dimensional Bilinear Systems

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Abstract: The current study focuses on the regional stabilization of time delay infinite dimensional bilinear systems evolving in a spatial domain Ω. It consists in studying the asymptotic behavior of such a system in a subregion ω of Ω. Then we demonstrate regional weak stabilization under weak observability conditions, while regional strong stabilization can be achieved under the exact observability condition. Illustrative examples and simulations are included to affirm the accuracy of the theoretical results.

Keywords: *infinite dimensional systems; delay bilinear systems; regional stabilization; weak and strong observability.*

Mathematics Subject Classification (2010): 93D15.

1 Introduction

There has been a growing interest in the study of infinite dimensional bilinear systems, which are a type of nonlinear systems that exhibit nonlinearity as a result of the interaction between the state and control. These systems are widely used in various industrial and natural processes, including heat transfer through conduction-convection, neutron kinetics in nuclear reactors, and dynamic heat exchanger with a controlled flow [3], among others. Bilinear systems are also often used as simple approximations for nonlinear systems [6]. In some cases, time delay may also be present in the system variables, either due to intrinsic delays or due to delays in the reaction of the control. This makes it important to consider time delay when designing these systems to accurately reflect real processes. Bilinear systems with time delay can be found in the fields such as viscoelasticity, mechanics, nuclear reactions, heat flow, and neural networks, etc [9].

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The stabilization of an infinite dimensional bilinear system with time delay has been a topic of discussion in numerous research studies: in [7], the researchers delved into the stabilization of a category of time-delayed bilinear systems within a Hilbert space. Their approach was rooted in the decreasing energy property of the system under examination. By employing a set of continuous controls, the researchers analyzed both weak and strong stabilization, and established a polynomial decay rate estimate for the stabilized state. They also tackled exponential stabilization through bounded feedback control and provided a clear decay rate estimate for the stabilized state.

The regional stabilization of an infinite dimensional bilinear system evolving over a spatial domain $\Omega \subset R^n$ $(n \geq 1)$ and ω being a subregion of Ω refers to the ability to stabilize the system only in the region ω or where the focus is solely on the state's behavior within ω. This concept is crucial because stabilizing the system in a subregion is more cost-effective than stabilizing it over the entire domain. Additionally, there are systems that cannot be stabilized over the entire domain of evolution, but stabilization can be achieved in specific subdomains within it (see [11]). This notion has been developed in many works: in [13], the researchers studied the regional stabilization of an infinite dimensional bilinear system that evolves in a spatial domain Ω with an unbounded control operator. Likewise, in [12], the researchers introduced controls that guarantee weak, strong, and exponential regional stabilization for a particular class of bilinear systems, as well as a control that regionally weakly stabilizes these systems with a minimal performance cost. In [14], the authors investigated the regional exponential stabilization of an infinite dimensional bilinear systems using bounded controls.
In [4], the authors presented preliminary results on the well-posedness of time-delayed bilinear systems in a real Hilbert space. They used a decomposition technique to demonstrate strong stabilization.

In this study, we delve into the subject of regional stabilization for infinite dimensional bilinear systems with time delay. Then we present sufficient conditions ensuring the regional weak and strong stabilization of such systems. The approach is a combination of energy decay, weak and exact observability conditions, and properties of semigroups. Furthermore, we provide application examples and simulations to support the obtained theoretical results.

More precisely, we consider a system defined by

$$\begin{cases} \dot{y}(t) = Ay(t) + u(t)By(t-r), \quad t \geq 0, \\ y_0 = \varphi \in \mathcal{C}, \end{cases} \tag{1}$$

where A is the infinitesimal generator of a linear $\mathcal{C}_0$-semigroup $S(t)$ on $L^2(\Omega)$, $B : L^2(\Omega) \longrightarrow L^2(\Omega)$ is a linear bounded control operator and $u \in L^2([0, +\infty[: R)$ is the scalar-valued control. The delay is indicated by the positive constant r, and $\mathcal{C}$ is a Banach space of continuous functions $\psi : [-r, 0] \rightarrow L^2(\Omega)$ with the norm $||\psi||_c = \sup_{-r \leq \theta \leq 0} ||\psi(\theta)||$.
The history function $y_t : [-r; \infty) \longrightarrow L^2(\Omega)$ is given by $y_t(\theta) = y(t+\theta)$ for all $\theta \in [-r, 0]$.
Let us consider a non-empty open subregion ω of Ω, with a positive Lebesgue measure. Define the restriction operator to ω by

$$\begin{aligned} \chi_\omega : \ L^2(\Omega) &\longrightarrow L^2(\omega) \\ y &\longmapsto y|_\omega, \end{aligned}$$

and $\chi_\omega^* : L^2(\omega) \longrightarrow L^2(\Omega)$ is the adjoint operator of χ_ω given by

$$\chi_\omega^* y(x) = \begin{cases} y(x) & \text{if} \quad x \in \omega, \\ 0 & \text{else} \;\; x \in \Omega \backslash \omega. \end{cases}$$

Let us recall that system (1) is said to be regionally weakly stabilizable if there exists a feedback control u such that for any initial condition $\varphi \in \mathcal{C}$, the corresponding solution $y(t)$ of system (1) is global and verifies $\forall \phi \in L^2(\omega),\ \ \big\langle \chi_\omega y(t), \phi \big\rangle_{L^2(\omega)} \longrightarrow 0 \; as \; t \longrightarrow \infty$, and regionally strongly stabilizable if there exists a feedback control u such that for any initial condition $\varphi \in \mathcal{C}$, the corresponding solution $y(t)$ of system (1) is global and verifies $||\chi_\omega y(t)||_{L^2(\omega)} \longrightarrow 0 \; as \; t \longrightarrow \infty$.

The paper is structured as follows. Section 2 focuses on the regional weak stabilization of (1). Section 3 is dedicated to the regional strong stabilization of (1). Multiple examples are given in Section 4 as applications. The final Sections 5 and 6 present illustrative simulations and conclusions.

2 Regional Weak Stabilization

This section gives sufficient conditions for the regional weak stabilization of (1).

Theorem 2.1 *Assume that A generates a semigroup $(S(t))_{t\geq 0}$ of contractions on $L^2(\Omega)$, and B is a compact operator. If the conditions*

1. $\big\langle \chi_\omega^* \chi_\omega A\phi, \phi \big\rangle \leq 0, \forall \phi \in \mathcal{D}(A)$,
2. $\big\langle \chi_\omega^* \chi_\omega B y(t-r), y(t) \big\rangle \big\langle B y(t-r), y(t) \big\rangle \geq 0,\ \forall y \in L^2(\Omega)$,
3. $\big\langle \chi_\omega^* \chi_\omega B S(t-r) y(t), S(t) y(t) \big\rangle = 0,\ \forall t \geq r \Longrightarrow \chi_\omega y(t) = 0$

hold, then the control

$$u(t) = -\big\langle \chi_\omega^* \chi_\omega B y(t-r), y(t) \big\rangle \tag{2}$$

regionally weakly stabilizes the system (1).

Proof. Since $\varphi \in \mathcal{C}$, the function $t \longmapsto \parallel \chi_\omega y(t) \parallel^2_{L^2(\omega)}$ is continuously differentiable (see [10]). Then

$$\frac{1}{2}\frac{d}{dt}||\chi_\omega y(t)||^2_{L^2(\omega)} = \big\langle \chi_\omega^* \chi_\omega A y(t), y(t) \big\rangle + u(t) \big\langle \chi_\omega^* \chi_\omega B y(t-r), y(t) \big\rangle.$$

Thanks to condition (1) of Theorem 2.1, it follows that

$$\frac{1}{2}\frac{d}{dt}||\chi_\omega y(t)||^2_{L^2(\omega)} \leq u(t) \big\langle \chi_\omega^* \chi_\omega B y(t-r), y(t) \big\rangle. \tag{3}$$

In order to make the function $\frac{1}{2}\frac{d}{dt}||\chi_\omega y(t)||^2_{L^2(\omega)}$ nonincreasing, we take the control

$$u(t) = -\big\langle \chi_\omega^* \chi_\omega B y(t-r), y(t) \big\rangle. \tag{4}$$

The resulting closed-loop system becomes

$$\begin{cases} \dot{y}(t) = A y(t) + f(y_t), \quad t \geq 0, \\ y_0 = \varphi \in \mathcal{C}, \end{cases} \tag{5}$$

where $f(\psi) = -\langle \chi_\omega^* \chi_\omega B\psi(-r), \psi(0)\rangle B\psi(-r), \ \ \forall \psi \in \mathcal{C}$. The function f is locally Lipschitz, indeed, for all $\psi_1, \psi_2 \in \mathcal{C}$, with $||\psi_1||_\mathcal{C} \leq R$ and $||\psi_2||_\mathcal{C} \leq R$, we have

$$||f(\psi_1) - f(\psi_2)|| \leq M||\psi_1 - \psi_2||_\mathcal{C},$$

where $M = ||\chi_\omega||^2_{L^2(\omega)}||B||^2(||\psi_1||^2_\mathcal{C} + ||\psi_1||_\mathcal{C}||\psi_2||_\mathcal{C} + ||\psi_2||^2_\mathcal{C})$. Then the system (5) possesses a unique mild solution $y \in \mathcal{C}([-r, t_{max}[; \mathcal{H})$ expressed as

$$\begin{cases} y(t) = S(t)\varphi(0) + \displaystyle\int_0^t S(t-s)f(y_s)ds, \quad t \in [0, t_{max}[, \\ y_0 = \varphi \in \mathcal{C}. \end{cases} \tag{6}$$

Since A generates a semigroup of contractions, we have

$$\frac{d}{dt}||y(t)||^2 \leq -2\langle \chi_\omega^*\chi_\omega By(t-r), y(t)\rangle \langle By(t-r), y(t)\rangle. \tag{7}$$

By integrating the inequality (7), for all $t \geq 0$, we get

$$||y(t)||^2 - ||y(0)||^2 \leq -2\int_0^t \langle \chi_\omega^*\chi_\omega By(s-r), y(s)\rangle \langle By(s-r), y(s)\rangle \, ds.$$

Using condition (2) of Theorem 2.1, we obtain

$$||y(t)|| \leq ||\varphi||_\mathcal{C}, \ t \geq -r. \tag{8}$$

Therefore, the system (1) possesses a unique global solution $y \in \mathcal{C}([-r, \infty); \mathcal{H})$ (see Theorem 2.6 in [10]).

By using (3) together with the control (2), we get

$$\frac{d}{dt}||\chi_\omega y(t)||^2_{L^2(\omega)} \leq -2|\langle \chi_\omega^*\chi_\omega By(t-r), y(t)\rangle|^2. \tag{9}$$

By integrating the inequality (9) over the interval $[0, t]$, we get

$$||\chi_\omega y(t)||^2_{L^2(\omega)} - ||\chi_\omega y(0)||^2_{L^2(\omega)} \leq -2\int_0^t |\langle \chi_\omega^*\chi_\omega By(s-r), y(s)\rangle|^2 \, ds. \tag{10}$$

From (6), we have

$$y(t) - S(t)\varphi(0) = -\int_0^t S(t-s)\langle \chi_\omega^*\chi_\omega By(s-r), y(s)\rangle By(s-r)ds. \tag{11}$$

For $T > r$ fixed, using (8), (11), Schwartz's inequality and the fact $||S(t)|| \leq 1, \ \forall t \geq 0$, we get

$$||y(t) - S(t)\varphi(0)|| \leq ||B||||\varphi||_\mathcal{C}\left((T-r)\int_r^T |\langle \chi_\omega^*\chi_\omega By(s-r), y(s)\rangle|^2 ds\right)^{\frac{1}{2}}, \ \ \forall t \in [r, T]. \tag{12}$$

For all $\varphi \in \mathcal{C}$ and $t \geq 0$, we have

$$\begin{aligned} |\langle \chi_\omega^*\chi_\omega BS(t-r)\varphi(0), S(t)\varphi(0)\rangle| \ &\leq |\langle \chi_\omega^*\chi_\omega BS(t-r)\varphi(0) - \chi_\omega^*\chi_\omega By(t-r), S(t)\varphi(0)\rangle| \\ &+|\langle \chi_\omega^*\chi_\omega By(t-r), y(t) - S(t)\varphi(0)\rangle| \\ &+|\langle \chi_\omega^*\chi_\omega By(t-r), y(t)\rangle|. \end{aligned} \tag{13}$$

By using (13), and taking into account that both B and χ_ω are bounded operators, we obtain

$$\begin{aligned} |\langle \chi_\omega^* \chi_\omega BS(t-r)\varphi(0), S(t)\varphi(0)\rangle| \quad & \leq ||\chi_\omega||^2_{L^2(\omega)}||B||\,||y(t-r)-S(t-r)\varphi(0)||\,||\varphi||_{\mathcal{C}} \\ & +||\chi_\omega||^2_{L^2(\omega)}||B||\,||y(t)-S(t)\varphi(0)||\,||\varphi||_{\mathcal{C}} \\ & +|\langle \chi_\omega^* \chi_\omega By(t-r), y(t)\rangle|. \end{aligned} \tag{14}$$

From (11), (12), and Schwartz's inequality, we have

$$\begin{aligned} & ||y(t-r)-S(t-r)\varphi(0)|| \leq \\ & ||B||||\varphi||_c(T-r)^{\frac{1}{2}}\left(\int_r^T |\langle \chi_\omega^* \chi_\omega By(s-r), y(s)\rangle|^2 ds\right)^{\frac{1}{2}}, \quad \forall t \in [r,T]. \end{aligned} \tag{15}$$

Replacing $\varphi(0)$ by $y(t)$ in (12) and (14), we get

$$\begin{aligned} |\langle \chi_\omega^* \chi_\omega BS(t-r)y(t), S(t)y(t)\rangle| \quad & \leq N(T-r)^{\frac{1}{2}}\left(\int_r^T |\langle \chi_\omega^* \chi_\omega By(s-r), y(s)\rangle|^2\, ds\right)^{\frac{1}{2}} \\ & +|\langle \chi_\omega^* \chi_\omega By(t-r), y(t)\rangle|, \end{aligned} \tag{16}$$

where $N = 2||\chi_\omega||^2_{L^2(\omega)}||B||^2||\varphi||^2_{\mathcal{C}}$.

By integrating this relation over the interval $[r,T]$ and applying Schwartz's inequality, we get

$$\begin{aligned} & \int_r^T |\langle \chi_\omega^* \chi_\omega BS(s-r)y(s), S(s)y(s)\rangle| ds \leq \\ & (N+1)(T-r)^{\frac{1}{2}}\left(\int_{t+r}^{t+T} |\langle \chi_\omega^* \chi_\omega By(s-r), y(s)\rangle|^2 ds\right)^{\frac{1}{2}}. \end{aligned} \tag{17}$$

Thus, as $t \longrightarrow +\infty$,

$$\left(\int_r^T |\langle \chi_\omega^* \chi_\omega BS(s-r)y(s), S(s)y(s)\rangle| ds\right)^2 = O\left(\int_{t+r}^{t+T} |\langle \chi_\omega^* \chi_\omega By(s-r), y(s)\rangle|^2 ds\right). \tag{18}$$

Let us consider the nonlinear semi-group $\Gamma(t)\varphi(0) := y(t)$ and let $(t_n)_{n\geq 0}$ be a sequence of real numbers such that $t_n \longrightarrow +\infty \; as \; n \longrightarrow +\infty$.

From (8), $\Gamma(t)\varphi(0)$ is bounded in $L^2(\Omega)$, and since $L^2(\Omega)$ is reflexive, there exists a subsequence $(t_{\phi(n)})$ of (t_n) such that $\Gamma(t_{\phi(n)})\varphi(0) \rightharpoonup \psi \; as \; n \longrightarrow +\infty$.

Since B is compact and χ_ω continuous, we have

$$\lim_{n \longmapsto +\infty} \langle \chi_\omega^* \chi_\omega BS(t-r)\Gamma(t_{\phi(n)})\varphi(0), \mathrm{S(t)}\Gamma(t_{\phi(n)})\varphi(0)\rangle = \langle \chi_\omega^* \chi_\omega BS(t-r)\psi, S(t)\psi\rangle.$$

For all $n \geq 0$, we set

$$\Lambda_n(t_{\phi(n)}) := \int_{t_{\phi(n)}+r}^{t_{\phi(n)}+T} |\langle \chi_\omega^* \chi_\omega B\Gamma(s-r)\varphi(0), \Gamma(s)\varphi(0)\rangle|^2 ds.$$

Using (10), we obtain

$$\int_0^t |\langle \chi_\omega^* \chi_\omega By(s-r), y(s)\rangle|^2\, ds \leq \frac{1}{2}||\chi_\omega y(0)||^2_{Y},$$

which gives $\displaystyle\int_0^t |\langle \chi_\omega^* \chi_\omega By(s-r), y(s)\rangle|^2\, ds < +\infty$. Then $\forall t \geq 0$, $\Lambda_n(t) \longrightarrow 0$ as $n \longrightarrow +\infty$.

By replacing $y(t)$ by $\Gamma(t)\varphi(0)$ and $y(t-r)$ by $\Gamma(t-r)\varphi(0)$ in (17), we have

$$\int_r^T |\langle \chi_\omega^* \chi_\omega BS(s-r)\Gamma(s)\varphi(0), S(s)\Gamma(s)\varphi(0)\rangle| ds \leq (N+1)(T-r)^{\frac{1}{2}}\sqrt{\Lambda_n(t)}.$$

Since $\Lambda_n(t_{\phi(n)}) \longrightarrow 0$ as $n \longrightarrow +\infty$, we get

$$\lim_{n \longmapsto +\infty} \int_r^T |\langle \chi_\omega^* \chi_\omega BS(s-r)\Gamma(t_{\phi(n)})\varphi(0), S(s)\Gamma(t_{\phi(n)})\varphi(0)\rangle| ds = 0.$$

Then, using the dominated convergence theorem, we have

$$\int_r^T |\langle \chi_\omega^* \chi_\omega BS(s-r)\psi, \mathrm{S(s)}\psi\rangle| ds = 0.$$

It follows that

$$\langle \chi_\omega^* \chi_\omega BS(s-r)\psi, S(s)\psi\rangle = 0, \ \forall s \in [0, t].$$

Using condition (3) of Theorem 2.1, we deduce that

$$\chi_\omega \Gamma(t_{\phi(n)})\varphi(0) \rightharpoonup 0 \ as \ n \longrightarrow +\infty. \tag{19}$$

On the other hand, it is clear that (19) holds for each subsequence $(t_{\phi(n)})$ of (t_n), and $\chi_\omega\Gamma(t_{\phi(n)})\varphi(0)$ weakly converges in $L^2(\Omega)$. This implies that $\forall \psi \in L^2(\Omega)$, we have $\langle \chi_\omega \Gamma(t_n)\varphi(0), \psi\rangle \longrightarrow 0 \ as \ n \longrightarrow +\infty$ and hence $\chi_\omega\Gamma(t)\varphi(0) \rightharpoonup 0 \ as \ t \longrightarrow +\infty$.

3 Regional Strong Stabilization

The following results provide conditions for the regional strong stabilization of the system (1).

Theorem 3.1 *Assume that A generates a semiproup $(S(t))_{t\geq 0}$ of contractions on $L^2(\Omega)$ and B is a bounded linear operator. If the conditions (1) and (2) of Theorem 2.1 hold, and there exist T, $\delta > 0$ such that the inequality*

$$\int_r^T |\langle \chi_\omega^* \chi_\omega BS(s-r)\psi, S(s)\psi\rangle| ds \geq \delta(r)||\chi_\omega \psi||^2_{L^2(\omega)}, \quad \forall \psi \in L^2(\Omega), \tag{20}$$

holds, then the control defined in (2) regionally strongly stabilizes the system (1).

Proof. Let $T \geq r$ be such that (20) is satisfied. Using (10) allows

$$||\chi_\omega y(kT)||^2_{L^2(\omega)} - ||\chi_\omega y((k+1)T)||^2_{L^2(\omega)} \geq 2\int_{kT}^{T(k+1)} |\langle \chi_\omega^* \chi_\omega By(s-r), y(s)\rangle|^2 ds, \ \ k \geq 0.$$

From (18) and (20), we have

$$||\chi_\omega y(kT)||^2_{L^2(\omega)} - ||\chi_\omega y((k+1)T)||^2_{L^2(\omega)} \geq \alpha ||\chi_\omega y(kT)||^4_{L^2(\omega)}, \tag{21}$$

where $\alpha = \dfrac{2\delta(r)^2}{(N+1)^2(T-r)}$. Taking $p_k = ||\chi_\omega y(kT)||^2_{L^2(\omega)}$, the inequality (21) can be expressed as follows:

$$\beta p_k^2 + p_{k+1} \leq p_k, \ \ \forall k \geq 0.$$

As $p_{k+1} \leq p_k$, we get $\beta p_{k+1}^2 + p_{k+1} \leq p_k, \ \ \forall k \geq 0.$

Using Lemma 5.2 from [1], we have $p_k \leq \dfrac{M}{k+1} \quad for \ all \ k \geq 0$. As $||\chi_\omega y(t)||_{L^2(\omega)}$ decreases, we conclude the estimate

$$||\chi_\omega y(t)||_{L^2(\omega)} = O\left(\frac{1}{\sqrt{t}}\right) \ as \ t \longrightarrow +\infty,$$

which proves the regional strong stabilization of the system (1).

4 Applications

This section presents illustrative examples with respect to regional stabilization.

Example 4.1 Let us consider the system defined on $\Omega =]0, +\infty)$ by

$$\begin{cases} \dfrac{\partial y(t,x)}{\partial t} = -\dfrac{\partial y(t,x)}{\partial x} + u(t)By(t-1,x), & x \in \Omega, \ t \geq 0, \\ y(t,x) = t\sin(\pi x), & x \in \Omega, \ t \in [-1,0], \end{cases} \tag{22}$$

where $u \in L^2([0,+\infty[: \mathbb{R})$, $By(.) = \displaystyle\int_\omega y(x)dx \mathbb{1}_\omega(.)$ for $\omega \subset \Omega$.

Let $Ay = -\dfrac{\partial y}{\partial x}$, with the domain $\mathcal{D}(A) = \{y \in H^1(\Omega) | \ y(0) = 0, \ y(x) \longrightarrow 0 \ as \ x \longrightarrow +\infty\}$, the operator A generates a semigroup of contractions

$$S(t)y = \begin{cases} y(x-t) & if \ x \geq t \\ 0 & if \ x < t. \end{cases}$$

Let $\omega =]0,1[$ be a subregion of Ω. For all $y \in \mathcal{D}(A)$, we have

$$\left\langle \chi_\omega^* \chi_\omega Ay, y \right\rangle = -\int_0^1 y^{'}(x)y(x)dx = -\frac{y^2(1)}{2} \leq 0.$$

Then condition (1) of Theorem 2.1 is verified. The condition (2) of Theorem 2.1 is satisfied since

$$\begin{aligned} \left\langle \chi_\omega^* \chi_\omega By(t-1), y(t) \right\rangle \left\langle By(t-1), y(t) \right\rangle &= \int_0^1 By(t-1)y(t)dx \int_\Omega By(t-1)y(t)dx \\ &= \int_0^1 \big(By(t-1)y(t)\big)^2 dx \geq 0. \end{aligned}$$

The operator B is compact and verifies

$$\left\langle \chi_\omega^* \chi_\omega BS(t-1)y, S(t)y \right\rangle = \left(\int_0^{1-t} y(x)dx\right)^2.$$

Therefore

$$\langle \chi_\omega^* \chi_\omega BS(t-1)y, S(t)y \rangle = 0, \ \forall t \geq 0 \Longrightarrow \chi_\omega y = 0.$$

Then the condition (3) of Theorem 2.1 holds.

We deduce that the control

$$u(t) = -\int_0^1 y(t-1,x)y(t,x)dx$$

regionally weakly stabilizes the system (22).

Example 4.2 On $\Omega =]0,1[$, let us consider the following system:

$$\begin{cases} \dfrac{\partial y(t,x)}{\partial t} = -i\Delta y(t,x) + u(t)y(t-2,x), & x \in \Omega, \ t \geq 0, \\ y(t,0) = y(t,1) = 0, & x \in \partial\Omega, \ t \geq 0, \\ y(t,x) = tx(1-x), & x \in \Omega, \ t \in [-2;0], \end{cases} \tag{23}$$

where $u \in L^2([0,+\infty[: \mathbb{R})$ and $B = I$.

Let ω be a subregion of Ω that verifies the geometric control condition (GCC) (see [2]). The operator $A = -i\Delta$ ($i \in \mathbb{C}$) with the domain $\mathcal{D}(A) = H^2(\Omega) \cap H_0^1(\Omega)$ generates a semigroup of isometry on $L^2(\Omega)$.

We have $Re(\langle \chi_\omega^* \chi_\omega Ay, y \rangle) = 0, \ \forall y \in \mathcal{D}((A))$. Then the condition (1) of Theorem 2.1 is satisfied. Since the operator B is the identity, the condition (2) of Theorem 2.1 is satisfied. Indeed,

$$\begin{aligned} \langle \chi_\omega^* \chi_\omega By(t-2), y(t) \rangle \langle By(t-2), y(t) \rangle &= \int_\omega y(t-2)y(t)dx \int_\Omega y(t-2)y(t)dx \\ &= \int_\omega \int_\Omega \big(y(t-2)y(t)\big)^2 dx \geq 0. \end{aligned}$$

For all $\psi \in L^2(\Omega)$, we obtain $\langle \chi_\omega^* \chi_\omega BS(t-2)\psi, S(t)\psi \rangle = \langle S(t-2)\psi, S(t)\psi \rangle_{L^2(\omega)}$.

Integrating this inequality, we get

$$\int_2^T \langle \chi_\omega^* \chi_\omega BS(t-2)\psi, S(t)\psi \rangle dt \geq \int_2^T \langle S(t-2)\psi, S(t)\psi \rangle_{L^2(\omega)} dt.$$

Since the subregion ω verifies GCC, there exist $\alpha, T > 0$ such that the inequality

$$\int_r^T \langle \chi_\omega^* \chi_\omega BS(t-2)\psi, S(t)\psi \rangle dt \geq \alpha ||\psi||^2 \geq \alpha ||\chi_\omega \psi||^2_{L^2(\omega)}$$

holds (see [2]). We deduce that the control

$$u(t) = -\int_\omega y(t-2,x)y(t,x)dx$$

regionally strongly stabilizes the system (23).

5 Algorithm and Simulations

The algorithm for the computation of the stabilizing control is as follows.

Algorithm 5.1 Step 1: Initial data.
The information required includes the evolution domain Ω, a time discretization (t_i), initial state y_0, delay r, initial control u_0, and desired precision ε.
Step 2 : Calculating the Control.
Using formula (2), calculate the control to get $u(t_i)$.
Step 3 : Finding the State.
Solve system (24) using the explicit finite difference method to obtain the state $y(t_{i+1})$.
Step 4 : Checking the Accuracy.
Proceed to Step 2 again by incrementing i if $\| y(t_i) \| > \varepsilon$.

For simulation purposes, we examine the transport equation, which is defined on $\Omega =]0, +\infty)$ by

$$\begin{cases} \dfrac{\partial y(x,t)}{\partial t} = -0.01 \dfrac{\partial y(x,t)}{\partial x} + u(t) By(x, t-r), & x \in \Omega,\ t \geq 0, \\ y(x,t) = t \sin(\pi x), & x \in \Omega,\ t \in [-r, 0], \end{cases} \tag{24}$$

where $u \in L^2([0, +\infty[: \mathbb{R})$, $By(.) = \displaystyle\int_\omega y(x) dx \mathbb{1}_\omega(.)$ for $\omega \subset \Omega$.
Consider the subregion $\omega =]0, 4[$. The control

$$u(t) = -\int_\omega y(x, t-r) y(x,t) dx \tag{25}$$

regionally weakly stabilizes (24) on ω.
We take the delay $r = 1$, and applying the above algorithm with $\varepsilon = 10^{-4}$, we have the following figures.

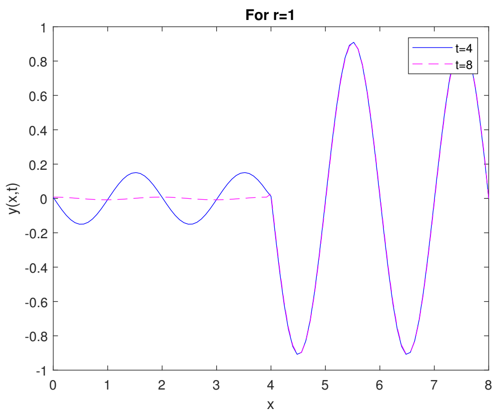

Figure 1: Stabilization on ω.

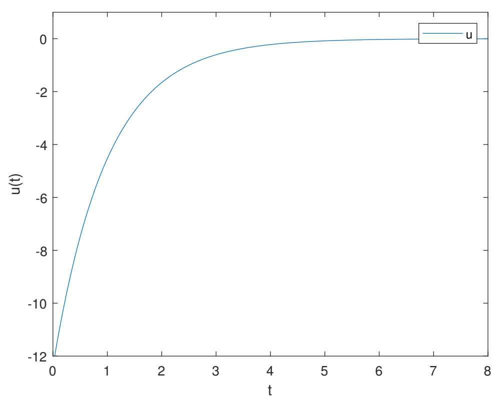

Figure 2: The control function.

Figure 1 shows the stabilization of the state on ω with the error equal to $0.1725*10^{-4}$. Figure 2 shows the decay of the system energy.

For $r = 2$, we get Figure 4 which shows that system (24) is still stabilized on the subregion ω.

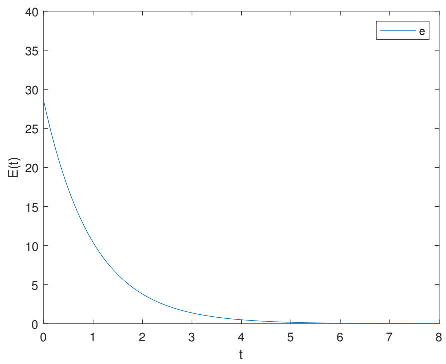

Figure 3: The energy decay.

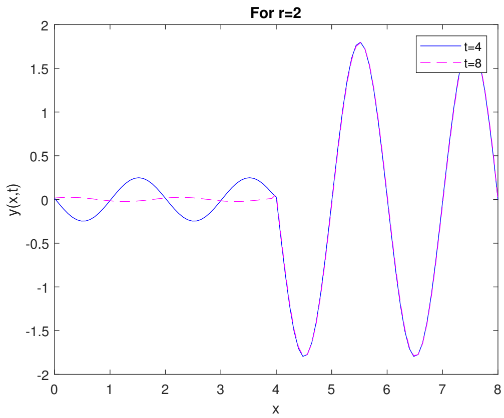

Figure 4: Stabilization on ω.

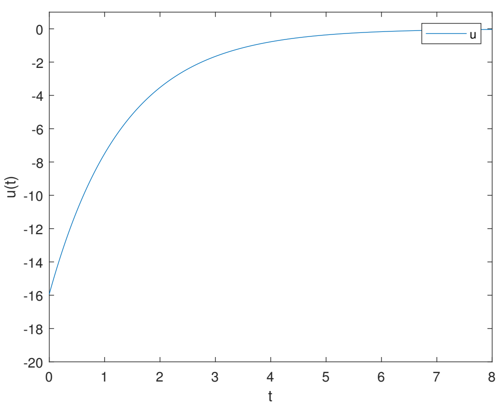

Figure 5: The control function.

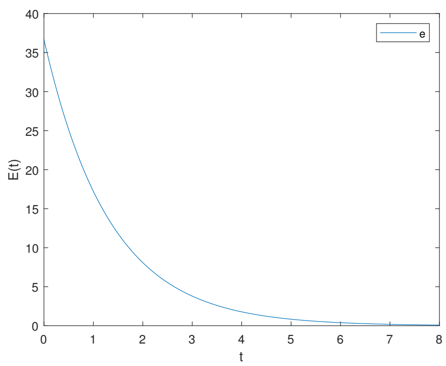

Figure 6: The energy decay.

Remark 5.1 The control (25) only stabilizes the state within the subregion ω. Yet, if condition (2) outlined in Theorem 2.1 is satisfied, the state remains bounded even in the residual region $\Omega\backslash\omega$.

Now we consider system (24) with $r = 10$. Applying the previous algorithm with control (25), we obtain Figures 7, 9 which show the instability of system (24) over the

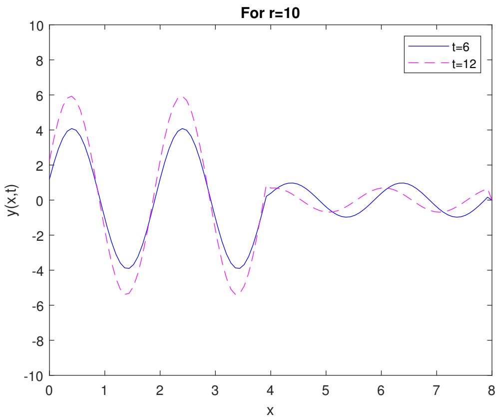

Figure 7: Instability on ω.

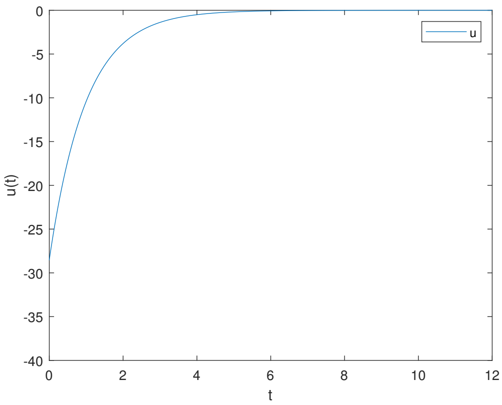

Figure 8: The control function.

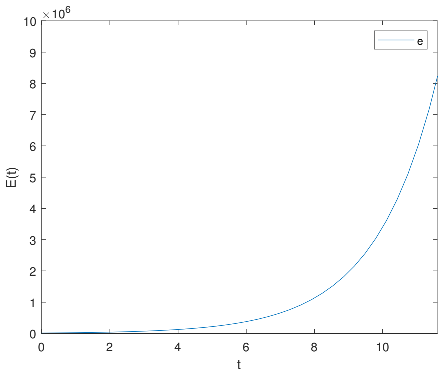

Figure 9: The energy growth.

subregion ω.

Remark 5.2 The stabilization on ω is obtained for low delay, and when the delay increases, the system (24) becomes unstable on ω.

6 Conclusion

This study examines the regional stabilization of an infinite dimensional delayed bilinear system. It offers sufficient conditions for weak and strong stabilization. The approach is a combination of energy decay, weak and exact observability conditions, and properties of semigroups. Furthermore, we provide examples and illustrations to support the obtained theoretical results. At the same time, there are still some unanswered questions such as the application of these results to delayed semilinear systems, these issues are currently under investigation and will be dealt with in a future paper.

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