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Boundary Value Problem for Fractional q-Difference Equations

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Abstract: In this paper, we study the existence of solutions for a class of boundary value problems for fractional q-difference equations involving the Caputo fractional q-difference derivative. Our results are given by applying some standard fixed point theorems. Furthermore, an example is presented to illustrate one of the main results.

Keywords: *fractional q-difference equations; Caputo fractional q-derivative; existence; fixed point; Leray-Schauder nonlinear alternative.*

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1 Introduction

Fractional calculus is an important branch in mathematical analysis, currently being addressed by many researchers in various fields of science and engineering such as physics, chemistry, biology, economics, control theory, and biophysics, etc. For more details, see [15, 19, 22, 23, 27]. Recently, considerable attention has been given to the existence of solutions to the boundary value problems for fractional differential equations. See for example, the papers of Benchohra *et al.* [4, 10, 11] and references therein.

In 1910, Jackson [16, 17], the first researcher to develop q-difference calculus or quantum calculus in a systematic way, introduced the notions of the q-integral and some classical concepts.

Combining fractional calculus and q-calculus, we obtain fractional q-difference calculus. This generalizes q-calculus by defining the q-derivatives and q-integrals in an arbitrary order. The fractional q-difference calculus had its origin in the end of the

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sixties with the works of Al-Salam [7] and Agarwal [3]. For an extended book on the subject, we suggest to the reader the recent book [9]. Since then, there has appeared much work on the theory of fractional q-difference calculus and fractional q-difference equations, see [8, 24, 25] for example.

Moreover, fractional q-difference equations have wide applications in several fields such as engineering, economics, chemistry, physics, and so on. So, the boundary value problems for fractional q-difference equations involving the Caputo fractional q-derivative have become of importance and the existence of their solutions has been studied by a great number of researchers, see the references [1, 2, 5, 6, 26].

In this paper, motivated by the works of Benchohra *et al.* [11] and Benhamida *et al.* [12], we wish to discuss the existence of solutions of the boundary value problem for fractional q-difference equations of the form

$$({}^{C}D_q^{\alpha}y)(t) = f(t, y(t)), \text{ for a.e. } t \in J = [0, T], \quad 0 < \alpha \leq 1, \tag{1}$$

$$ay(0) + by(T) = c, \tag{2}$$

where $T > 0$, $q \in (0, 1)$, ${}^{C}D_q^{\alpha}$ is the Caputo fractional q-difference derivative of order $0 < \alpha \leq 1$, $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is a given function and a, b and c are real constants such that $a + b \neq 0$.

We give three existence results, one based on Banach's fixed point theorem, another based on Schaefer's fixed point theorem and the third based on Leray-Schauder nonlinear alternative theorem. Finally, we present an example.

2 Preliminaries

In this section, we introduce notations, definitions, and preliminary facts that will be used in the remainder of this paper.

Let $T > 0$ and define $J := [0, T]$. Consider the Banach space $C(J, \mathbb{R})$ of continuous functions from J into $\mathbb{R}$, with the norm

$$\|y\|_{\infty} = \sup\{|y(t)| : 0 \leq t \leq T\}.$$

Let $L^1(J, \mathbb{R})$ denote the Banach space of measurable functions $y : J \to \mathbb{R}$ which are Lebesgue integrable, with the norm

$$\|y\|_{L^1} = \int_J |y(t)| dt.$$

Now, we recall some definitions and properties of the fractional q-calculus [13, 18]. For $a \in \mathbb{R}$ and $0 < q < 1$, we set

$$[a]_q = \frac{1 - q^a}{1 - q}.$$

The q-analogue of the power $(a - b)^{(n)}$ is expressed by

$$(a - b)^{(0)} = 1, \ (a - b)^{(n)} = \prod_{k=0}^{n-1} (a - bq^k), \ a, b \in \mathbb{R}, \ n \in \mathbb{N}.$$

In general,

$$(a - b)^{(\alpha)} = a^{\alpha} \prod_{k=0}^{\infty} \left(\frac{a - bq^k}{a - bq^{k+\alpha}} \right), \ a, b, \alpha \in \mathbb{R}.$$

Note that if $b = 0$, then $a^{(\alpha)} = a^{\alpha}$.

Definition 2.1 [18] The q-gamma function is defined by

$$\Gamma_q(\alpha) = \frac{(1-q)^{(\alpha-1)}}{(1-q)^{\alpha-1}},\ \alpha \in \mathbb{R} - \{0,-1,-2,\ldots\}.$$

Notice that the q-gamma function satisfies $\Gamma_q(\alpha+1) = [\alpha]_q\Gamma_q(\alpha)$.

Definition 2.2 [18] The q-derivative of order $n \in \mathbb{N}$ of a function $f : J \to \mathbb{R}$, is defined by $(D_q^0 f)(t) = f(t)$,

$$(D_q f)(t) = (D_q^1 f)(t) = \frac{f(t) - f(qt)}{(1-q)t},\ t \neq 0,\ (D_q f)(0) = \lim_{t\to 0}(D_q f)(t),$$

and

$$(D_q^n f)(t) = (D_q^1 D_q^{n-1} f)(t),\ t \in J,\ n \in \{1,2,\ldots\}.$$

Set $J_t := \{tq^n : n \in \mathbb{N}\} \cup \{0\}$.

Definition 2.3 [18] The q-integral of a function $f : J_t \to \mathbb{R}$, is defined by

$$(I_q f)(t) = \int_0^t f(s)d_q s = \sum_{n=0}^{\infty} t(1-q)q^n f(tq^n),$$

provided that the series converges.

We note that $(D_q I_q f)(t) = f(t)$, while if f is continuous at 0, then

$$(I_q D_q f)(t) = f(t) - f(0).$$

Definition 2.4 [3] The Riemann-Liouville fractional q-integral of order $\alpha \geq 0$ of a function $f : J \to \mathbb{R}$ is defined by $(I_q^0 f)(t) = f(t)$, and

$$(I_q^\alpha f)(t) = \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s)d_q s,\ t \in J.$$

Note that for $\alpha = 1$, we have $(I_q^1 f)(t) = (I_q f)(t)$.

Lemma 2.1 *[25] For $\alpha \geq 0$ and $\beta \in (-1,+\infty)$, we have*

$$(I_q^\alpha (t-a)^{(\beta)})(t) = \frac{\Gamma_q(\beta+1)}{\Gamma_q(\alpha+\beta+1)}(t-a)^{(\alpha+\beta)},\ 0 < a < t < T.$$

In particular,

$$(I_q^\alpha 1)(t) = \frac{1}{\Gamma_q(\alpha+1)} t^{(\alpha)}.$$

Definition 2.5 [24] The Riemann-Liouville fractional q-derivative of order $\alpha \geq 0$ of a function $f : J \to \mathbb{R}$ is defined by $(D_q^0 f)(t) = f(t)$, and

$$(D_q^\alpha f)(t) = (D_q^{[\alpha]} I_q^{[\alpha]-\alpha} f)(t),\ t \in J,$$

where $[\alpha]$ is the integer part of α.

Definition 2.6 [24] The Caputo fractional q-derivative of order $\alpha \geq 0$ of a function $f : J \to \mathbb{R}$ is defined by $(D_q^0 f)(t) = f(t)$, and

$$({}^C D_q^\alpha f)(t) = (I_q^{[\alpha]-\alpha} D_q^{[\alpha]} f)(t),\ t \in J,$$

where $[\alpha]$ is the integer part of α.

Lemma 2.2 *[24] Let $\alpha, \beta \geq 0$ and let f be a function defined on J. Then the following identities hold:*

(i) $(I_q^\alpha I_q^\beta f)(t) = (I_q^{\alpha+\beta} f)(t)$,

(ii) $(D_q^\alpha I_q^\alpha f)(t) = f(t)$.

Lemma 2.3 *[24] Let $\alpha \geq 0$ and let f be a function defined on J. Then the following equality holds:*

$$(I_q^\alpha\ {}^C D_q^\alpha f)(t) = f(t) - \sum_{k=0}^{[\alpha]-1} \frac{t^k}{\Gamma_q(k+1)} (D_q^k f)(0).$$

In particular, if $\alpha \in (0,1)$, then

$$(I_q^\alpha\ {}^C D_q^\alpha f)(t) = f(t) - f(0).$$

Next, we offer a variety of fixed point theorems.

Theorem 2.1 (Banach contraction principle) [14]
Let C be a non-empty closed subset of a Banach space X, then any contraction mapping H of C into itself has a unique fixed point.

Theorem 2.2 *(Schaefer) [28] Let X be a Banach space and $H : X \to X$ be a completely continuous operator. If the set*

$$E(H) := \{y \in X : y = \lambda H(y),\ for\ \lambda \in (0,1)\}$$

is bounded, then H has a fixed point.

Theorem 2.3 *(Nonlinear alternative of Leray-Schauder) [14] Let X be a Banach space and C be a closed, convex subset of X. Let U be an open subset of C with $0 \in U$ and $H : \overline{U} \to C$ be a continuous and compact operator. Then either*

(a) H has fixed points, or

(b) There exist $y \in \partial U$ and $\lambda \in (0,1)$ with $y = \lambda H(y)$.

3 Main Results

Let us start by defining what we mean by a solution of the problem (1)-(2).

Definition 3.1 A function $y \in C(J, \mathbb{R})$ is said to be a solution of the problem (1)-(2) if y satisfies the equation $({}^C D_q^\alpha y)(t) = f(t, y(t))$ on J, and satisfies the condition $ay(0) + by(T) = c$.

For the existence of solutions to the problem (1)-(2), we need the following auxiliary lemma.

Lemma 3.1 *Let $h : J \to \mathbb{R}$ be continuous, the solution of the boundary value problem*

$$({}^C D_q^\alpha y)(t) = h(t), \, t \in J = [0, T], \quad 0 < \alpha \leq 1, \tag{3}$$

$$ay(0) + by(T) = c, \tag{4}$$

is given by

$$y(t) = \int_0^t \frac{(t - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} h(s) d_q s - \frac{b}{a+b} \int_0^T \frac{(T - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} h(s) d_q s + \frac{c}{a+b}. \tag{5}$$

Proof. Applying the Riemann-Liouville fractional q-integral of order α to both sides of equation (3), and by using Lemma 2.3, we have

$$y(t) = \int_0^t \frac{(t - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} h(s) d_q s + c_0. \tag{6}$$

Using the boundary condition of the problem in (4), we obtain

$$ay(0) + by(T) = ac_0 + b(I_q^\alpha h(T) + c_0) = c.$$

So

$$c_0 = \frac{c}{a+b} - \frac{b}{a+b} I_q^\alpha h(T).$$

Finally, by substitution of c_0 into (6), we give

$$y(t) = \int_0^t \frac{(t - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} h(s) d_q s - \frac{b}{a+b} \int_0^T \frac{(T - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} h(s) d_q s + \frac{c}{a+b}.$$

The proof is completed. □

In the following subsections we prove the existence and uniqueness results of the problem (1)-(2) by using a variety of fixed point theorems.

We consider the following hypotheses:

$(H1)$ The function $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is continuous.

$(H2)$ There exists a constant $L > 0$ such that

$$|f(t, x) - f(t, y)| \leq L|x - y|, \ for\ each\ t \in J\ and\ each\ x, y \in \mathbb{R}.$$

$(H3)$ There exists a constant $M > 0$ such that

$$|f(t, x)| \leq M, \ for\ each\ t \in J\ and\ each\ x \in \mathbb{R}.$$

The first result is based on the Banach contraction principle theorem (Theorem 2.1).

Theorem 3.1 *Assume that the hypothesis (H2) is satisfied. If*

$$\left(1 + \frac{|b|}{|a+b|}\right) \frac{LT^{(\alpha)}}{\Gamma_q(\alpha+1)} < 1, \tag{7}$$

then the problem (1)-(2) has a unique solution on $[0, T]$.

Proof. Transform the problem (1)-(2) into a fixed point problem. Consider the operator

$$H : C(J, \mathbb{R}) \longrightarrow C(J, \mathbb{R})$$

defined by

$$\begin{aligned}
(Hy)(t) &= \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s, y(s)) d_q s \\
&\quad - \frac{b}{a+b} \int_0^T \frac{(T-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s, y(s)) d_q s + \frac{c}{a+b}. \qquad (8)
\end{aligned}$$

By Lemma 3.1, the fixed points of H are the solutions of the problem (1)-(2). We shall prove that H is a contraction mapping on $C(J, \mathbb{R})$.

For $x, y \in C(J, \mathbb{R})$ and for each $t \in J = [0, T]$, we have

$$\begin{aligned}
|(Hx)(t) - (Hy)(t)| &= \left| \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \left(f(s, x(s)) - f(s, y(s))\right) d_q s \right. \\
&\quad \left. - \frac{b}{a+b} \int_0^T \frac{(T-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \left(f(s, x(s)) - f(s, y(s))\right) d_q s \right|.
\end{aligned}$$

Therefore, by $(H2)$, we obtain

$$\begin{aligned}
|(Hx)(t) - (Hy)(t)| &\leq \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \left|f(s, x(s)) - f(s, y(s))\right| d_q s \\
&\quad + \frac{|b|}{|a+b|} \int_0^T \frac{(T-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \left|f(s, x(s)) - f(s, y(s))\right| d_q s, \\
&\leq L \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} |x(s) - y(s)| d_q s \\
&\quad + \frac{L|b|}{|a+b|} \int_0^T \frac{(T-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} |x(s) - y(s)| d_q s.
\end{aligned}$$

Hence

$$\|H(x) - H(y)\|_\infty \leq \left(1 + \frac{|b|}{|a+b|}\right) \frac{LT^{(\alpha)}}{\Gamma_q(\alpha+1)} \|x - y\|_\infty.$$

By (7), H is a contraction, and by the Banach contraction principle theorem, we deduce that H has a unique fixed point, which is the unique solution of the problem (1)-(2). □

The second result is based on Schaefer's fixed point theorem (Theorem 2.2).

Theorem 3.2 *Assume that the hypotheses* $(H1)$ *and* $(H3)$ *hold. Then the problem (1)-(2) has at least one solution on* $[0,T]$.

Proof. We shall use Schaefer's fixed point theorem to prove that H defined by (8) has a fixed point.
The proof will be given in several steps.

Step 1: H *is continuous.*

Let $\{y_n\}_{n\in\mathbb{N}}$ be a sequence such that $y_n \to y$ in $C(J,\mathbb{R})$. Then, for each $t \in J$, we have

$$\begin{aligned} |(Hy_n)(t) - (Hy)(t)| &\leq \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} |f(s,y_n(s)) - f(s,y(s))|\, d_q s \\ &+ \frac{|b|}{|a+b|} \int_0^T \frac{(T-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} |f(s,y_n(s)) - f(s,y(s))|\, d_q s. \end{aligned}$$

Hence, for each $t \in J$, we find

$$\|H(y_n) - H(y)\|_\infty \leq \left(1 + \frac{|b|}{|a+b|}\right) \frac{T^{(\alpha)}}{\Gamma_q(\alpha+1)} \|f(.,y_n(.)) - f(.,y(.))\|_\infty.$$

Since f is continuous, we have

$$\|H(y_n) - H(y)\|_\infty \to 0 \quad as \quad n \to \infty.$$

Consequently, H is continuous on $C(J,\mathbb{R})$.

Step 2: H *maps bounded sets into bounded sets in* $C(J,\mathbb{R})$.

Indeed, it is enough to show that for any $r > 0$, there exists a positive constant R such that for each $y \in B_r = \{y \in C(J,\mathbb{R}) : \|y\|_\infty \leq r\}$, we have $\|H(y)\|_\infty \leq R$.
Let $y \in B_r$. Then, for each $t \in J$, we have

$$|(Hy)(t)| = \left| \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s,y(s)) d_q s - \frac{b}{a+b} \int_0^T \frac{(T-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s,y(s)) d_q s + \frac{c}{a+b} \right|.$$

By $(H3)$, we obtain

$$\begin{aligned} |(Hy)(t)| &\leq \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} |f(s,y(s))| d_q s \\ &+ \frac{|b|}{|a+b|} \int_0^T \frac{(T-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} |f(s,y(s))| d_q s + \frac{|c|}{|a+b|}, \\ &\leq \left(1 + \frac{|b|}{|a+b|}\right) \frac{MT^{(\alpha)}}{\Gamma_q(\alpha+1)} + \frac{|c|}{|a+b|}. \end{aligned}$$

Hence

$$\|H(y)\|_\infty \leq \left(1 + \frac{|b|}{|a+b|}\right) \frac{MT^{(\alpha)}}{\Gamma_q(\alpha+1)} + \frac{|c|}{|a+b|} := R.$$

Consequently, H is uniformly bounded on B_r.

Step 3: *H maps bounded sets into equicontinuous sets of $C(J,\mathbb{R})$.*

Let $t_1, t_2 \in J$, $t_1 < t_2$ and let B_r be a bounded set of $C(J,\mathbb{R})$ as in Step 2. Let $y \in B_r$, then

$$\begin{aligned}
|(Hy)(t_2)-(Hy)(t_1)| &= \Big| \int_0^{t_2} \frac{(t_2-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s,y(s))d_qs \\
&\quad - \int_0^{t_1} \frac{(t_1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s,y(s))d_qs \Big|, \\
&\leq \int_0^{t_1} \frac{\left((t_2-qs)^{(\alpha-1)}-(t_1-qs)^{(\alpha-1)}\right)}{\Gamma_q(\alpha)} |f(s,y(s))|d_qs \\
&\quad + \int_{t_1}^{t_2} \frac{(t_2-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} |f(s,y(s))|d_qs.
\end{aligned}$$

By $(H3)$, we get

$$\begin{aligned}
|(Hy)(t_2)-(Hy)(t_1)| &\leq \frac{M}{\Gamma_q(\alpha)} \int_0^{t_1} \left((t_2-qs)^{(\alpha-1)}-(t_1-qs)^{(\alpha-1)}\right) d_qs \\
&\quad + \frac{M}{\Gamma_q(\alpha)} \int_{t_1}^{t_2} (t_2-qs)^{(\alpha-1)} d_qs.
\end{aligned}$$

Thus

$$\|(Hy)(t_2)-(Hy)(t_1)\|_\infty \leq \frac{M}{\Gamma_q(\alpha+1)} \left(t_2^{(\alpha)} - t_1^{(\alpha)}\right).$$

As $t_1 \to t_2$, the right-hand side of the above inequality tends to zero.

As a consequence of Steps 1, 2 and 3, together with the Arzela-Ascoli theorem, we can conclude that H is completely continuous.

Step 4: *A priori bound.*

Now, we prove that the set $\Omega = \{y \in C(J,\mathbb{R}) : y = \lambda H(y), 0 < \lambda < 1\}$ is bounded. Let $y \in \Omega$. Thus, for each $t \in J$, we have

$$\begin{aligned}
y(t) &= \lambda (Hy)(t), \\
&= \lambda \left(\int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s,y(s))d_qs \right. \\
&\quad \left. - \frac{b}{a+b} \int_0^T \frac{(T-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s,y(s))d_qs + \frac{c}{a+b} \right).
\end{aligned}$$

Then, by $(H3)$ (as in Step 2), it follows that, for each $t \in J$, we get

$$\|y(t)\|_\infty \leq \left(1 + \frac{|b|}{|a+b|}\right) \frac{MT^{(\alpha)}}{\Gamma_q(\alpha+1)} + \frac{|c|}{|a+b|}.$$

Consequently, $\|y(t)\|_\infty \leq R < \infty$, the set Ω is bounded.

As a consequence of Schaefer's fixed point theorem, we deduce that H has a fixed point which is a solution of the problem (1)-(2). □

The third result is based on the Leray-Schauder nonlinear alternative theorem (Theorem 2.3).

Theorem 3.3 *Assume that (H1) holds and the following hypotheses are satisfied:*

(H4) There exist $\phi_f \in L^1(J,\mathbb{R}_+)$ and $\psi : [0,\infty) \to (0,\infty)$ are continuous and nondecreasing such that

$$|f(t,y)| \le \phi_f(t)\psi(|y|) \text{ for each } t \in J \text{ and each } y \in \mathbb{R}.$$

(H5) There exists a number $\nu > 0$ such that

$$\frac{\nu}{\left(1+\frac{|b|}{|a+b|}\right)\psi(\nu)(I_q^{\alpha}\phi_f)(T)+\frac{|c|}{|a+b|}} \quad > \quad 1.$$

Then the problem (1)-(2) has at least one solution on $[0,T]$.

Proof. We shall use the Leray-Schauder theorem to prove that H defined by (8) has a fixed point. As shown in Theorem 3.2, we see that the operator H is continuous and completely continuous.

Let $y \in C(J,\mathbb{R})$ such that for each $t \in [0,T]$, we have $y(t) = \lambda(Hy)(t)$ for $\lambda \in (0,1)$. Then, from $(H4)$ and for every $t \in J$, we give

$$\begin{aligned} |y(t)| \quad &\le \quad \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}|f(s,y(s))|d_qs \\ &\quad + \frac{|b|}{|a+b|}\int_0^T \frac{(T-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}|f(s,y(s))|d_qs + \frac{|c|}{|a+b|}, \\ &\le \quad \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}\phi_f(s)\psi(|y|)d_qs \\ &\quad + \frac{|b|}{|a+b|}\int_0^T \frac{(T-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}\phi_f(s)\psi(|y|)d_qs + \frac{|c|}{|a+b|}. \end{aligned}$$

Thus

$$\|y\|_\infty \quad \le \quad \left(1+\frac{|b|}{|a+b|}\right)\psi(\|y\|_\infty)(I_q^{\alpha}\phi_f)(T)+\frac{|c|}{|a+b|}.$$

Hence

$$\frac{\|y\|_\infty}{\left(1+\frac{|b|}{|a+b|}\right)\psi(\|y\|_\infty)(I_q^{\alpha}\phi_f)(T)+\frac{|c|}{|a+b|}} \quad \le \quad 1.$$

Then, by condition $(H5)$, there exists ν such that $\|y\|_\infty \neq \nu$. Let us set

$$U = \{y \in C(J,\mathbb{R}) : \|y\|_\infty < \nu\}.$$

The operator $H : \overline{U} \to C(J,\mathbb{R})$ is completely continuous. From the choice of U, there is no $y \in \partial U$ such that $y = \lambda H(y)$, for some $\lambda \in (0,1)$. As a result, by the nonlinear alternative of Leray-Schauder type, H has a fixed point $y \in \overline{U}$, which is a solution of the problem (1)-(2). The proof is completed. □

4 Example

Consider the boundary value problem for fractional $\frac{1}{3}$-difference equations

$$({}^C D_{\frac{1}{3}}^{\frac{1}{2}} y)(t) = \frac{e^{-t^2} y(t)}{(6+t)(1+y(t))}, \ t \in J = [0,1], \ 0 < \alpha \le 1, \tag{9}$$

$$y(0) + y(1) = 0, \tag{10}$$

where $\alpha = \frac{1}{2}, \ q = \frac{1}{3}, \ a = 1, \ b = 1, \ c = 0, \ T = 1$, and

$$f(t,y) = \frac{e^{-t^2} y}{(6+t)(1+y)}, \ (t,y) \in J \times [0,\infty).$$

Let $x, y \in [0,\infty)$ and $t \in J$. Then we have

$$\begin{aligned} |f(t,x) - f(t,y)| &= \left| \frac{e^{-t^2}}{(6+t)} \left(\frac{x}{1+x} - \frac{y}{1+y} \right) \right|, \\ &\le \frac{e^{-t^2}}{(6+t)} |x-y|, \\ &\le \frac{1}{6} |x-y|. \end{aligned}$$

Hence, the condition $(H2)$ holds with $L = \frac{1}{6}$. We shall check that condition (7) is satisfied with $T = 1$. Indeed,

$$\begin{aligned} \left(1 + \frac{|b|}{|a+b|}\right) \frac{LT^{(\alpha)}}{\Gamma_q(\alpha+1)} &= \left(1 + \frac{1}{2}\right) \frac{1}{6\Gamma_q(\frac{3}{2})}, \\ &= 0.2666 < 1, \end{aligned}$$

where $\Gamma_{\frac{1}{3}}(\frac{3}{2}) \approx 0.9376$. Then, by Theorem 3.1, the problem (9)-(10) has a unique solution on $[0,1]$.

5 Conclusion

In this paper, we have presented the existence and uniqueness of solutions of the boundary value problem for fractional q-difference equations. The uniqueness result is obtained by applying the Banach contraction principle theorem, while the existence results are proved via Schaefer's fixed point theorem and the Leray-Schauder nonlinear alternative theorem. Finally, we illustrated our main results by providing an example.

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