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Numerical Approach for Solving Incommensurate Higher-Order Fractional Differential Equations

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Abstract: In this research, we present a novel numerical approach to tackle an incommensurate system of fractional differential equations of 2α-order, where $\alpha = (\alpha_1, \alpha_2, \alpha_3, \cdots, \alpha_n)$ with $0 < \alpha_i \leq 1$, $\forall i = 1, 2, 3, \cdots, n$. Our proposed method involves reducing the system to α-fractional differential equations using a newly derived result, followed by the implementation of the Modified Fractional Euler Method (MFEM), a recent numerical technique. We demonstrate the efficacy of our approach through an illustrative example, providing validation for our proposed methodology.

Keywords: *incommensurate system; fractional differential equations; modified fractional Euler method.*

Mathematics Subject Classification (2010): 34A08, 26C10.

1 Introduction

In recent years, Fractional Differential Equations (FDEs) have been extensively studied and applied due to their ability to capture the dynamics of systems with long-range interactions, anomalous diffusion, and viscoelasticity. The fractional derivatives allow for the inclusion of memory and hereditary properties, making them suitable for modeling phenomena that exhibit memory retention and relaxation effects. While significant progress has been made in solving FDEs, there remains a challenging class of problems known as incommensurate higher-order FDEs. These equations involve fractional derivatives of different orders that are not rational multiples of each other. As a result, they

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lack a common denominator, leading to difficulties in analytical solutions and numerical treatments.

In this paper, we focus on addressing incommensurate higher-order FDEs, which are characterized by having fractional derivatives of different orders. Such systems present unique challenges in numerical solutions due to their complex nature [1–4]. Our objective is to develop an efficient and accurate numerical method to handle this class of FDEs. The proposed numerical approach is based on two main steps. First, we derive a result that enables us to transform the incommensurate system into a set of α-FDEs, where $\alpha = (\alpha_1, \alpha_2, \alpha_3, \cdots, \alpha_n)$. This transformation simplifies the problem and prepares it for numerical treatment. The parameters α_i are limited to the range $0 < \alpha_i \leq 1$, for $i = 1, 2, 3, \cdots, n$, encompassing various degrees of fractional order, allowing for a comprehensive analysis of the system's behavior. Next, we employ the Modified Fractional Euler Method (MFEM), a recently developed numerical technique tailored to solve FDEs efficiently and accurately. The MFEM incorporates adaptive step-size control and higher-order approximation schemes, making it well-suited for addressing the complexities of incommensurate higher-order fractional systems.

To demonstrate the effectiveness of our numerical approach, we present two illustrative examples. The results obtained by our proposed method are compared with existing analytical solutions, showcasing the accuracy and reliability of our approach. The rest of the paper is organized as follows. Section 2 provides a brief overview of the relevant background and related works. Section 3 outlines the theoretical framework and the proposed numerical approach in detail. In Section 4, we present two numerical examples with some comparisons, and in Section 5, we conclude the whole paper.

2 Preliminaries

In this section, we recall some preliminaries and basic results related to fractional calculus. For more about FDEs and fractional calculus, see [5].

Definition 2.1 Let α be a real nonnegative number. Then the Riemann-Liouville fractional-order integrator J_a^α is defined by

$$J_a^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \ a \leq x \leq b. \tag{1}$$

Definition 2.2 Let $\alpha \in \mathbb{R}^+$ and $m = \lceil \alpha \rceil$ such that $m - 1 < \alpha \leq m$. Then the Caputo fractional-order differentiator of order α is given by

$$D_a^\alpha f(x) = \frac{1}{\Gamma(m-\alpha)} \int_a^x (x-t)^{m-\alpha-1} f^{(m)}(t)dt, \ x > a. \tag{2}$$

Theorem 2.1 *[6] (Generalized Taylor's Theorem). Suppose that $D_*^{k\alpha} f(x) \in C(0,b]$ for $k = 0, 1, \cdots, n+1$, where $0 < \alpha \leq 1$. Then the function f can be expanded about $x = x_0$ as*

$$f(x) = \sum_{i=0}^{n} \frac{x^{i\alpha}}{\Gamma(i\alpha+1)} D_*^{i\alpha} f(x_0) + \frac{x^{(n+1)\alpha}}{\Gamma((n+1)\alpha+1)} D_*^{(n+1)\alpha} f(\xi), \tag{3}$$

with $0 < \xi < x$, $\forall x \in (0,b]$.

Now, by using the first three terms of the generalized Taylor theorem and for $\xi \in (a,b)$, $t_i \in [a,b]$, in which the interval is divided as $a = t_0 < t_1 = t_0 + h < t_2 = t_0 + 2h < \cdots < t_n = t_0 + nh = b$ with $h = \frac{b-a}{n}$ for $i = 1, 2, \cdots, n$, we can expand $y(t)$ about $t = t_i$ to develop a new further modification for the Fractional Euler Method (FEM), called the MFEM. This formula has the form [7,8]

$$\begin{aligned} y(t_{i+1}) = y(t_i) + \frac{h^\alpha}{\Gamma(\alpha+1)} f\left(t_i + \frac{h^\alpha}{2\Gamma(\alpha+1)}, y(t_i) + \frac{h^\alpha}{2\Gamma(\alpha+1)} f(t_i, y(t_i))\right) \\ + \frac{h^{2\alpha}}{\Gamma(2\alpha+1)} D^{2\alpha}(\xi), \end{aligned} \tag{4}$$

where $\xi \in (a,b)$.

3 Numerical Approach

In this section, we propose a novel result that can reduce the higher incommensurate fractional system of 2α-order into an α-fractional system, where $\alpha = (\alpha_1, \alpha_2, \alpha_3, \cdots, \alpha_n)$ with $0 < \alpha_i \le 1$, $\forall i = 1, 2, 3, \cdots, n$. Then, we describe how one can deal with the produced system.

Lemma 3.1 *Any FDE of order $n\alpha, n \in \mathbb{Z}^+$ and $\alpha \in (0,1]$, with functions possessing values in $\mathbb{R}$, can be converted into a system of FDEs of order α with values in $\mathbb{R}^{nd}$.*

Proof. To prove this result, we should first take the scalar case that takes place whenever $d = 1$ and then we will consider the remaining case that occurs when $\alpha > 1$. For this reason, we should note that the general form of the FDE of order $n\alpha$ in its scalar case can be given by

$$D^{n\alpha} y(t) = G(t, y(t), D^\alpha y(t), D^{2\alpha} y(t), \cdots, D^{(n-1)\alpha} y(t)), \tag{5}$$

where G is a continuous function defined on the subset $I \times \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}$ so that it takes values in $\mathbb{R}$ for a given interval I. Now, define the function

$$\mathbf{\Psi}(t, v_0, v_1, \cdots, v_{n-1}) = (v_1, v_2, \cdots, G(t, v_0, v_1, \cdots, v_{n-1})) \tag{6}$$

as a continuous function defined on $I \times \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}$ as G, but it takes the values in $\mathbb{R}^n$. In this regard, we consider the following equation:

$$D^\alpha \mathbf{Y}(t) = \mathbf{\Psi}(t, \mathbf{Y}(t)), \text{ for } t \in I. \tag{7}$$

Now, we want to show that $x : I \to \mathbb{R}$ is a solution of equation (6) if and only if the function

$$\begin{aligned} \mathbf{X} :& I \to \mathbb{R}^n, \\ & t \to (x(t), D^\alpha x(t), D^{2\alpha} x(t), \cdots, D^{(n-1)\alpha} x(t)), \end{aligned} \tag{8}$$

is a solution of equation (7). To this end, we assume that x is a solution to equation (6) such that $\mathbf{X}$ is as defined above. Then we have

$$D^\alpha \mathbf{X}(t) = \begin{pmatrix} D^\alpha x(t) \\ D^{2\alpha} x(t) \\ \vdots \\ D^{(n-1)\alpha} x(t) \\ D^{n\alpha} x(t) \end{pmatrix} = \begin{pmatrix} D^\alpha x(t) \\ D^{2\alpha} x(t) \\ \vdots \\ D^{(n-1)\alpha} x(t) \\ G(t, x(t), D^\alpha x(t), D^{2\alpha} x(t), \cdots, D^{(n-1)\alpha} x(t)) \end{pmatrix}, \tag{9}$$

i.e.,

$$D^{\alpha}\mathbf{X}(t) = \mathbf{\Psi}(t, \mathbf{X}(t)). \tag{10}$$

Herein, the converse of the above discussion is similar. Now, for the case of $\alpha > 1$, one can reread the above proof again, and substitute each occurrence of $\mathbb{R}$ by $\mathbb{R}^d$ to get the result.

Corollary 3.1 *Lemma 3.1 can hold for $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$, where $0 < \alpha_i \leq 1$, for all $i = 1, 2, \cdots, n$.*

For the purpose of addressing FDEs of 2α-order, where $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ such that $0 < \alpha_i \leq 1$, for all $i = 1, 2, \cdots, n$, we consider this system has the following form:

$$\begin{aligned} &D^{2\alpha_1} y_1(t) = g_1(t, \mathbf{Y}(t), D^{2\alpha}\mathbf{Y}(t)),\\ &D^{2\alpha_2} y_2(t) = g_2(t, \mathbf{Y}(t), D^{2\alpha}\mathbf{Y}(t)),\\ &D^{2\alpha_3} y_3(t) = g_3(t, \mathbf{Y}(t), D^{2\alpha}\mathbf{Y}(t)),\\ &\vdots\\ &D^{2\alpha_n} y_n(t) = g_n(t, \mathbf{Y}(t), D^{2\alpha}\mathbf{Y}(t)) \end{aligned} \tag{11}$$

with the initial conditions

$$y_i(0) = a_i, \quad D^{\alpha} y_i(0) = b_i \tag{12}$$

such that a_i and b_i are constants for all $i = 1, 2, \cdots, n$, where

$$\mathbf{Y}(t) = (y_1(t), y_2(t), y_3(t), \cdots, y_n(t))$$

and

$$D^{2\alpha}\mathbf{Y}(t) = \left(D^{2\alpha_1} y_1(t), D^{2\alpha_2} y_2(t), D^{2\alpha_3} y_3(t), \cdots, D^{2\alpha_n} y_n(t)\right).$$

In order to obtain an approximate solution to system (11), we reduce it with the use of Lemma 3.1 into α-FDEs. In particular, we suppose that

$$\begin{aligned} &v_1(t) = D^{\alpha_1} y_1(t),\\ &v_2(t) = D^{\alpha_2} y_2(t),\\ &v_3(t) = D^{\alpha_3} y_3(t),\\ &\vdots\\ &v_n(n) = D^{\alpha_n} y_n(t). \end{aligned} \tag{13}$$

Actually, the above assumption would convert system (11) to the following form:

$$\begin{aligned} &D^{\alpha_1} y_1(t) = v_1(t) = h_1(t, \mathbf{X}(t)),\\ &D^{\alpha_1} v_1(t) = g_1(t, \mathbf{X}(t)),\\ &D^{\alpha_2} y_2(t) = v_2(t) = h_2(t, \mathbf{X}(t)),\\ &D^{\alpha_2} v_2(t) = g_2(t, \mathbf{X}(t)),\\ &D^{\alpha_3} y_3(t) = v_3(t) = h_3(t, \mathbf{X}(t)),\\ &D^{\alpha_3} v_3(t) = g_3(t, \mathbf{X}(t)),\\ &\vdots\\ &D^{\alpha_n} y_n(t) = v_n(t) = h_n(t, \mathbf{X}(t)).\\ &D^{\alpha_n} v_n(t) = g_n(t, \mathbf{X}(t)) \end{aligned} \tag{14}$$

with the initial conditions

$$y_i(0) = a_i, \quad v_i(0) = b_i, \ \forall i = 1, 2, \cdots, n, \tag{15}$$

where $\mathbf{X}(t) = (y_1(t), v_1(t), y_2(t), v_2(t), y_3(t), v_3(t), \cdots, y_n(t), v_n(t))$. Now, to solve system (14), we use the MFEM. This method divides the solutions interval $[a, b]$ as $a = t_0 < t_1 = t_0 + h < t_2 = t_0 + 2h < \cdots < t_n = t_0 + nh = b$, in which $t_i = a + ih$ are called the mesh points for $i = 1, 2, \cdots, n$, and $h = \frac{b-a}{n}$ is the step size of the algorithm. Accordingly, based on the MFEM, we obtain the following states:

$$\begin{aligned}
y_1(t_{i+1}) &= y_1(t_i) + \frac{h^{\alpha_1}}{\Gamma(\alpha_1+1)} h_1\left(t_i + \frac{h^{\alpha_1}}{2\Gamma(\alpha_1+1)}, \mathbf{X}(t_i) + \frac{h^{\alpha_1}}{\Gamma(\alpha_1+1)} h_1(t_i, \mathbf{X}(t_i))\right),\\
v_1(t_{i+1}) &= v_1(t_i) + \frac{h^{\alpha_1}}{\Gamma(\alpha_1+1)} g_1\left(t_i + \frac{h^{\alpha_1}}{2\Gamma(\alpha_1+1)}, \mathbf{X}(t_i) + \frac{h^{\alpha_1}}{\Gamma(\alpha_1+1)} g_1(t_i, \mathbf{X}(t_i))\right),\\
y_2(t_{i+1}) &= y_2(t_i) + \frac{h^{\alpha_2}}{\Gamma(\alpha_2+1)} h_2\left(t_i + \frac{h^{\alpha_2}}{2\Gamma(\alpha_2+1)}, \mathbf{X}(t_i) + \frac{h^{\alpha_2}}{\Gamma(\alpha_2+1)} h_2(t_i, \mathbf{X}(t_i))\right),\\
v_2(t_{i+1}) &= v_2(t_i) + \frac{h^{\alpha_2}}{\Gamma(\alpha_2+1)} g_2\left(t_i + \frac{h^{\alpha_2}}{2\Gamma(\alpha_2+1)}, \mathbf{X}(t_i) + \frac{h^{\alpha_2}}{\Gamma(\alpha_2+1)} g_2(t_i, \mathbf{X}(t_i))\right),\\
y_3(t_{i+1}) &= y_3(t_i) + \frac{h^{\alpha_3}}{\Gamma(\alpha_3+1)} h_3\left(t_i + \frac{h^{\alpha_3}}{2\Gamma(\alpha_3+1)}, \mathbf{X}(t_i) + \frac{h^{\alpha_3}}{\Gamma(\alpha_3+1)} h_3(t_i, \mathbf{X}(t_i))\right),\\
v_3(t_{i+1}) &= v_3(t_i) + \frac{h^{\alpha_3}}{\Gamma(\alpha_3+1)} g_3\left(t_i + \frac{h^{\alpha_3}}{2\Gamma(\alpha_3+1)}, \mathbf{X}(t_i) + \frac{h^{\alpha_3}}{\Gamma(\alpha_3+1)} g_3(t_i, \mathbf{X}(t_i))\right),\\
&\vdots\\
y_n(t_{i+1}) &= y_n(t_i) + \frac{h^{\alpha_n}}{\Gamma(\alpha_n+1)} h_n\left(t_i + \frac{h^{\alpha_n}}{2\Gamma(\alpha_n+1)}, \mathbf{X}(t_i) + \frac{h^{\alpha_n}}{\Gamma(\alpha_n+1)} h_n(t_i, \mathbf{X}(t_i))\right),\\
v_n(t_{i+1}) &= v_n(t_i) + \frac{h^{\alpha_n}}{\Gamma(\alpha_n+1)} g_n\left(t_i + \frac{h^{\alpha_n}}{2\Gamma(\alpha_n+1)}, \mathbf{X}(t_i) + \frac{h^{\alpha_n}}{\Gamma(\alpha_n+1)} g_n(t_i, \mathbf{X}(t_i))\right)
\end{aligned} \tag{16}$$

for all $i = 1, 2, \cdots, n$. As a matter of fact, formulas (16) represent an approximate solution of system (14) and therefore $(y_1(t), y_2(t), y_3(t), \cdots, y_n(t)$ is then the desired solution of system (11).

4 Illustrative Examples

In this part, we illustrate our proposed approach by considering two incommensurate systems of FDEs, each of them is of 2α-order, where $\alpha = (\alpha, \beta)$ with $0 < \alpha, \beta \leq 1$.

Example 4.1 Consider the following system:

$$\begin{aligned}
D^{2\alpha} x_1(t) + \frac{1}{2}(x_1(t) - x_2(t)) &= 1,\\
D^{2\beta} x_2(t) + \frac{1}{2}(x_2(t) - x_1(t)) &= 2
\end{aligned} \tag{17}$$

with the initial conditions

$$\begin{aligned}
x_1(0) &= 1, D^{\alpha} x_1(0) = 0,\\
x_2(0) &= \frac{1}{2}, D^{\beta} x_2(0) = 0.
\end{aligned} \tag{18}$$

To solve system (17)-(18) with the use of Lemma 3.1, we assume $u_1(t) = D^\alpha x_1(t)$ and $u_2(t) = D^\beta x_2(t)$. This would convert system (17)-(18) to be as follows:

$$\begin{aligned} D^\alpha x_1(t) &= u_1(t), \\ D^\alpha u_1(t) &= 1 - \frac{1}{2}(x_1(t) - x_2(t)), \\ D^\beta x_2(t) &= u_2(t), \\ D^\beta u_2(t) &= 2 - \frac{1}{2}(x_2(t) - x_1(t)) \end{aligned} \tag{19}$$

with the initial conditions

$$\begin{aligned} x_1(0) &= 1, u_1(0) = 0, \\ x_2(0) &= \frac{1}{2}, u_2(0) = 0. \end{aligned} \tag{20}$$

For simplicity, one might suppose

$$\begin{aligned} f_1(t, \mathbf{X}(t)) &= u_1(t), \\ f_2(t, \mathbf{X}(t)) &= 1 - \frac{1}{2}(x_1(t) - x_2(t)), \\ f_3(t, \mathbf{X}(t)) &= u_2(t), \\ f_4(t, \mathbf{X}(t)) &= 2 - \frac{1}{2}(x_2(t) - x_1(t)), \end{aligned} \tag{21}$$

where $\mathbf{X}(t) = (x_1(t), u_1(t), x_2(t), u_2(t))$. This would make system (19)-(20) to be as

$$\begin{aligned} D^\alpha x_1(t) &= f_1(t, \mathbf{X}(t)), \\ D^\alpha u_1(t) &= f_2(t, \mathbf{X}(t)), \\ D^\beta x_2(t) &= f_3(t, \mathbf{X}(t)), \\ D^\beta u_2(t) &= f_4(t, \mathbf{X}(t)) \end{aligned} \tag{22}$$

with the initial conditions

$$\begin{aligned} x_1(0) &= 1, u_1(0) = 0, \\ x_2(0) &= \frac{1}{2}, u_2(0) = 0. \end{aligned} \tag{23}$$

To solve system (22)-(23) by the MFEM, we are applying the solution's formula (16) to obtain

$$\begin{aligned} x_1(t_{i+1}) &= x_1(t_i) + \frac{h^\alpha}{\Gamma(\alpha+1)} f_1\left(t_i + \frac{h^\alpha}{2\Gamma(\alpha+1)}, \mathbf{X}(t_i) + \frac{h^\alpha}{\Gamma(\alpha+1)} f_1(t_i, \mathbf{X}(t_i))\right), \\ u_1(t_{i+1}) &= u_1(t_i) + \frac{h^\alpha}{\Gamma(\alpha+1)} f_2\left(t_i + \frac{h^\alpha}{2\Gamma(\alpha+1)}, \mathbf{X}(t_i) + \frac{h^\alpha}{\Gamma(\alpha+1)} f_2(t_i, \mathbf{X}(t_i))\right), \\ x_2(t_{i+1}) &= x_2(t_i) + \frac{h^\beta}{\Gamma(\beta+1)} f_3\left(t_i + \frac{h^\beta}{2\Gamma(\beta+1)}, \mathbf{X}(t_i) + \frac{h^\beta}{\Gamma(\beta+1)} f_3(t_i, \mathbf{X}(t_i))\right), \\ u_2(t_{i+1}) &= u_2(t_i) + \frac{h^\beta}{\Gamma(\beta+1)} f_4\left(t_i + \frac{h^\beta}{2\Gamma(\beta+1)}, \mathbf{X}(t_i) + \frac{h^\beta}{\Gamma(\beta+1)} f_4(t_i, \mathbf{X}(t_i))\right) \end{aligned} \tag{24}$$

for all $i = 1, 2, \cdots, n$. In fact, the first and third equations of system (24) represent the numerical solution of system (17). In order to see the validity of our scheme, we make some comparisons between our approximate solutions

$$\begin{aligned} x_1(t_{i+1}) &= x_1(t_i) + \frac{h^\alpha}{\Gamma(\alpha+1)} f_1\left(t_i + \frac{h^\alpha}{2\Gamma(\alpha+1)}, \mathbf{X}(t_i) + \frac{h^\alpha}{\Gamma(\alpha+1)} f_1(t_i, \mathbf{X}(t_i))\right) \\ x_2(t_{i+1}) &= x_2(t_i) + \frac{h^\beta}{\Gamma(\beta+1)} f_3\left(t_i + \frac{h^\beta}{2\Gamma(\beta+1)}, \mathbf{X}(t_i) + \frac{h^\beta}{\Gamma(\beta+1)} f_3(t_i, \mathbf{X}(t_i))\right) \end{aligned} \tag{25}$$

and the following exact solution that could be obtained when $\alpha = \beta = 1$:

$$\begin{aligned} x_1(t) &= \frac{3}{4}t^2 + \frac{3}{4}cos(t) - \frac{5}{4}, \\ x_2(t) &= \frac{3}{4}t^2 - \frac{3}{4}cos(t) + \frac{5}{4} \end{aligned} \tag{26}$$

for system (17)-(18). In particular, Figure 1 depicts a graphical comparison between the numerical solution (25) and the exact solution (26). In addition, we plot in Figures 2 and 3 the numerical solution (25) of system (17)-(18) in accordance with commensurate and incommensurate fractional-order values, respectively.

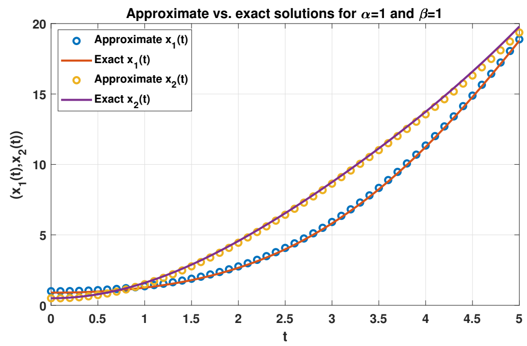

Figure 1: Approximate vs. exact solution of $(x_1(t), x_2(t))$ for $\alpha = 1$ and $\beta = 1$.

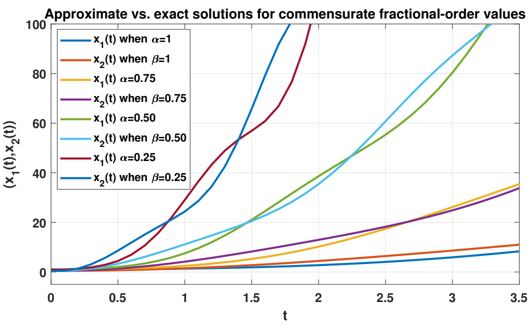

Figure 2: Approximate vs. exact solutions of $(x_1(t), x_2(t))$ for commensurate fractional-order values.

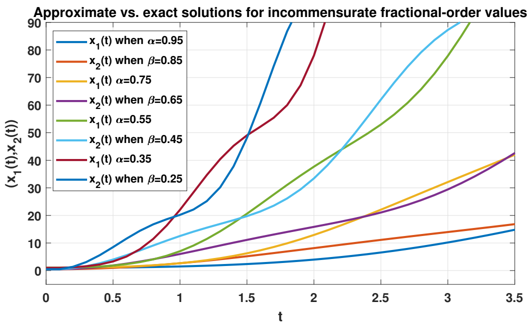

Figure 3: Approximate solutions of $(x_1(t), x_2(t))$ for incommensurate fractional-order values.

Example 4.2 Consider the following system:

$$\begin{aligned} D^{2\alpha}x_1(t) &= 3x_1(t) + 3x_2(t) - D^{\alpha}x_1(t),\\ D^{2\beta}x_2(t) &= 3x_1(t) + 3x_2(t) - D^{\beta}x_2(t) \end{aligned} \tag{27}$$

with the initial conditions

$$\begin{aligned} x_1(0) = 1, D^{\alpha}x_1(0) = 0,\\ x_2(0) = 1, D^{\beta}x_2(0) = 0. \end{aligned} \tag{28}$$

To solve system (27)-(28) with the use of Lemma 3.1, we assume $u_1(t) = D^{\alpha}x_1(t)$ and $u_2(t) = D^{\beta}x_2(t)$. This would convert system (27)-(28) to be as follows:

$$\begin{aligned} D^{\alpha}x_1(t) &= u_1(t),\\ D^{\alpha}u_1(t) &= 3x_1(t) + 3x_2(t) - u_1(t),\\ D^{\beta}x_2(t) &= u_2(t),\\ D^{\beta}u_2(t) &= 3x_1(t) + 3x_2(t) - u_2(t) \end{aligned} \tag{29}$$

with the initial conditions

$$\begin{aligned} x_1(0) = 1, u_1(0) = 0,\\ x_2(0) = 1, u_2(0) = 0. \end{aligned} \tag{30}$$

For simplicity, one might suppose

$$\begin{aligned} f_1(t, \mathbf{X}(t)) &= u_1(t),\\ f_2(t, \mathbf{X}(t)) &= 3x_1(t) + 3x_2(t) - u_1(t),\\ f_3(t, \mathbf{X}(t)) &= u_2(t),\\ f_4(t, \mathbf{X}(t)) &= 3x_1(t) + 3x_2(t) - u_2(t), \end{aligned} \tag{31}$$

where $\mathbf{X}(t) = (x_1(t), u_1(t), x_2(t), u_2(t))$. This would make system (29)-(30) to be as

$$\begin{aligned} D^{\alpha}x_1(t) &= f_1(t, \mathbf{X}(t)),\\ D^{\alpha}u_1(t) &= f_2(t, \mathbf{X}(t)),\\ D^{\beta}x_2(t) &= f_3(t, \mathbf{X}(t)),\\ D^{\beta}u_2(t) &= f_4(t, \mathbf{X}(t)) \end{aligned} \tag{32}$$

with the initial conditions

$$\begin{aligned} x_1(0) &= 1, u_1(0) = 0, \\ x_2(0) &= 1, u_2(0) = 0. \end{aligned} \tag{33}$$

To solve system (32)-(33) by the MFEM, we are applying the solution's formula (16) to obtain

$$\begin{aligned} x_1(t_{i+1}) &= x_1(t_i) + \frac{h^\alpha}{\Gamma(\alpha+1)} f_1\left(t_i + \frac{h^\alpha}{2\Gamma(\alpha+1)}, \mathbf{X}(t_i) + \frac{h^\alpha}{\Gamma(\alpha+1)} f_1(t_i, \mathbf{X}(t_i))\right), \\ u_1(t_{i+1}) &= u_1(t_i) + \frac{h^\alpha}{\Gamma(\alpha+1)} f_2\left(t_i + \frac{h^\alpha}{2\Gamma(\alpha+1)}, \mathbf{X}(t_i) + \frac{h^\alpha}{\Gamma(\alpha+1)} f_2(t_i, \mathbf{X}(t_i))\right), \\ x_2(t_{i+1}) &= x_2(t_i) + \frac{h^\beta}{\Gamma(\beta+1)} f_3\left(t_i + \frac{h^\beta}{2\Gamma(\beta+1)}, \mathbf{X}(t_i) + \frac{h^\beta}{\Gamma(\beta+1)} f_3(t_i, \mathbf{X}(t_i))\right), \\ u_2(t_{i+1}) &= u_2(t_i) + \frac{h^\beta}{\Gamma(\beta+1)} f_4\left(t_i + \frac{h^\beta}{2\Gamma(\beta+1)}, \mathbf{X}(t_i) + \frac{h^\beta}{\Gamma(\beta+1)} f_4(t_i, \mathbf{X}(t_i))\right) \end{aligned} \tag{34}$$

for all $i = 1, 2, \cdots, n$. In fact, the first and third equations of system (34) represent the numerical solution of system (27). In order to see the validity of our scheme, we make some comparisons between our approximate solutions

$$\begin{aligned} x_1(t_{i+1}) &= x_1(t_i) + \frac{h^\alpha}{\Gamma(\alpha+1)} f_1\left(t_i + \frac{h^\alpha}{2\Gamma(\alpha+1)}, \mathbf{X}(t_i) + \frac{h^\alpha}{\Gamma(\alpha+1)} f_1(t_i, \mathbf{X}(t_i))\right) \\ x_2(t_{i+1}) &= x_2(t_i) + \frac{h^\beta}{\Gamma(\beta+1)} f_3\left(t_i + \frac{h^\beta}{2\Gamma(\beta+1)}, \mathbf{X}(t_i) + \frac{h^\beta}{\Gamma(\beta+1)} f_3(t_i, \mathbf{X}(t_i))\right) \end{aligned} \tag{35}$$

and the following exact solution that could be obtained when $\alpha = \beta = 1$:

$$\begin{aligned} x_1(t) &= c_1 e^{2t} + c_2 e^{-3t} + c_3 + c_4 e^{-t} \\ x_2(t) &= c_1 e^{2t} + c_2 e^{-3t} - c_3 - c_4 e^{-t} \end{aligned} \tag{36}$$

for system (27)-(28), where $c_1 = \frac{1}{5}$, $c_2 = -\frac{1}{5}$ and $c_3 = c_4 = 0$. In particular, Figures 4 and 5 depict graphical comparisons between the numerical solutions given in (35) and the exact solution (36). In addition, we plot in Figures 6 and 7 the numerical solution (35) of system (27)-(28) in accordance with commensurate fractional-order values, and similarly, we plot in Figures 8 and 9 the numerical solution (35) of the same system according to some incommensurate fractional-order values.

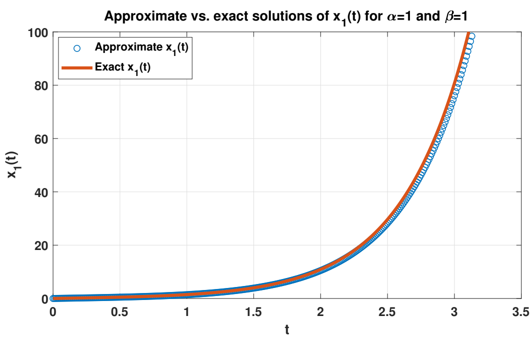

Figure 4: Approximate vs. exact solutions of $x_1(t)$ for $\alpha = 1$ and $\beta = 1$.

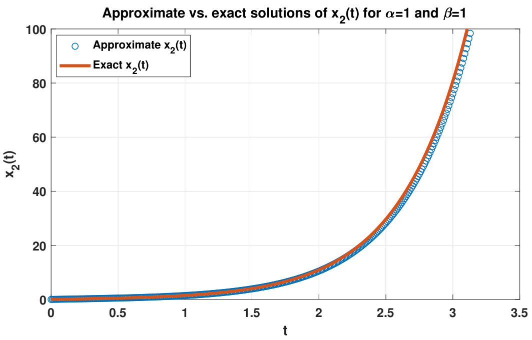

Figure 5: Approximate vs. exact solutions of $x_2(t)$ for $\alpha = 1$ and $\beta = 1$.

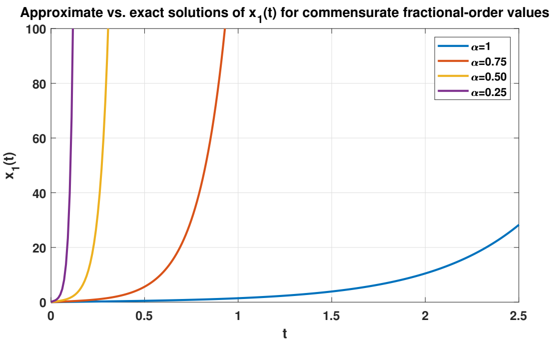

Figure 6: Approximate vs. exact solutions of $x_1(t)$ for commensurate fractional-order values.

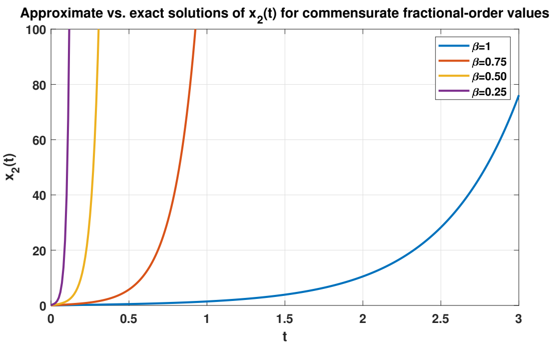

Figure 7: Approximate vs. exact solutions of $x_2(t)$ for commensurate fractional-order values.

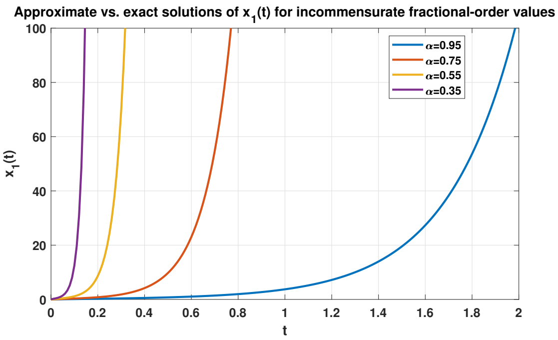

Figure 8: Approximate vs. exact solutions of $x_1(t)$ for incommensurate fractional-order values.

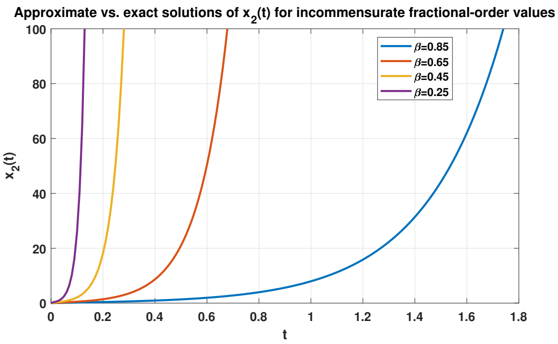

Figure 9: Approximate vs. exact solutions of $x_2(t)$ for incommensurate fractional-order values.

5 Conclusion

In conclusion, this research introduces a novel and effective numerical approach for addressing the challenges posed by incommensurate systems of fractional differential equations of 2α-order, where $\alpha = (\alpha_1, \alpha_2, \alpha_3, \cdots, \alpha_n)$ with $0 < \alpha_i \leq 1$, $\forall i = 1, 2, 3, \cdots, n$. Our proposed method offers a systematic solution by transforming the incommensurate system into a set of α-fractional differential equations using a newly derived result. Subsequently, we successfully apply the Modified Fractional Euler Method (MFEM), a recently developed numerical technique, to efficiently solve the transformed equations. Through an illustrative example, we demonstrate the practical efficacy and reliability of our numerical approach.

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