Nonlinear Dynamics and Systems Theory, **24** (2) (2024) 135–146

Derivation of Multi-Asset Black-Scholes Differential Equations

K. Fahim *, A. U. Alfajriyah and E. R. M. Putri

Department of Mathematics, Sepuluh Nopember Institute of Technology, Sukolilo, Surabaya, Indonesia

Received: September 3, 2023; Revised: February 18, 2024

Abstract: The Black-Scholes differential equations are extensively proposed in multi-asset option prices. Modelling of the Black-Scholes differential equation is generally completed by applying a Δ-hedging method, which could first-rate be accomplished on entire markets. Another technique, which is done in this work, is by first modelling multi-asset option prices in a backward stochastic differential equation. This study starts constructing a multi-asset portfolio which is written in BSDEs. The Feynman-Kac concept offers the relation between BSDEs and the Black-Scholes differential equations. Then we obtain a theorem which explains that the solution of BSDEs of multi-asset portfolios exists and is unique. It is also a solution to the Black-Scholes differential equations. Finally, in the last part of this work, we give some simulations of multi-asset option prices which are executed in a software.

Keywords: *backward stochastic differential equations (BSDEs); Black-Scholes differential equations; Feynman-Kac theorem; multi-asset option; partial differential equations (PDEs).*

Mathematics Subject Classification (2010): 22E10, 28B05, 35R60, 93-10, 70K75.

* Corresponding author: mailto:kfahim@matematika.its.ac.id

1 Introduction

Determining the price of derivative products in the capital market is an important problem in the field of financial mathematics. Derivative products are contracts whose profit value is determined by the underlying asset. One of the derivative products are options. An option is an agreement between two parties whereby the contract holder has the right to sell or buy an amount of the underlying asset at a certain price and at a stated time. The most widely used option pricing method is the Black-Scholes differential equation since it provides an option price solution for the options of one or more underlying assets. Options with several underlying assets are termed as multi-asset options. Derivation of the Black-Scholes differential equation is generally done using the Δ-hedging theory that can only be done on a complete market by setting the stock portfolio value equal to the option value. This is called a portfolio replication. The opposite of a complete market is an incomplete market. In the incomplete market, there are only a few securities or financial products. Portfolios or wealth processes on the incomplete market can only be built using primary securities, so there is an impossibility to replicate the payoff with a portfolio of underlying assets [2]. Pricing of financial products in the incomplete market is one of the BSDEs applications.

Several previous studies stated that the Black-Scholes differential equation can be obtained through BSDEs (Backward Stochastic Differential Equations). BSDEs are a useful method for studying options pricing problems because they have several advantages. BSDEs can be used on the incomplete market. Another advantage of BSDEs is that there is no need to change the measurement to a neutral risk condition. When determining the price of options, the return value is adjusted according to the investor's risk preferences, while the discount rate differs between the investors. There is an alternative method, namely by adjusting all investors' risk preferences. This risk preference is marked with a number called the risk premium and then the expected value is calculated in the new probability measure, which is a neutral risk measure. The BSDE solution corresponds to the PDE (partial differential equation) solution. This result is given by an analogy called the Feynman-Kac theorem.

BSDEs application in financial mathematics was first introduced by El Karoui et al. [2]. This study shows that the Black-Scholes differential equation for one asset can be obtained using BSDEs [2]. Actually, the process of changing portfolio values (wealth process) can be represented in the form of BSDEs. This research was conducted on one asset options. Pricing of single asset options, non-linear in the incomplete market can also be modeled using BSDEs [2]. In the *complete market*, option prices are a BSDEs solution with a linear generator function f. On the other hand, in the incomplete market, changes in portfolio wealth are given by BSDEs with the non-linear generator function f [2]. The generalization of the Feynman-Kac theorem has an important function in the solution of parabolic PDEs including the BSDEs in option pricing.

This research begins by formulating risky asset prices and a risk-free asset. Then we construct a multi-asset portfolio that contains risky and risk-free assets. The portfolio is structured in such a way that it satisfies the system of BSDEs (6) and (14). Then the existence and uniqueness of the solution of the System (6) and (14) are proven. Using the Feynman-Kac theorem, a multi-asset Black-Scholes differential equation can be obtained. Simulation and the analysis of multi-asset option price simulation results are also carried out.

2 The Feynman-Kac Theorem

In this section, we firstly present a theorem on the existence and uniqueness of the solution to BSDE (1). Then we give the Feynman-Kac theorem which provides the relation between BSDE (1) and semilinear PDE (3). These two theorems are essential for the main result of this study and are given without proof. The complete proofs can be found in [2, 9]. Let us first consider a BSDE

$$\begin{cases} -dY(t) = f\left(t, X(t), Y(t), Z(t)\right) dt - Z(t)\, dW(t),\ t \in [0, T), \\ Y_T = \xi, \end{cases} \tag{1}$$

or, equivalently,

$$Y_t = \xi + \int_t^T f(s, X_s, Y_s, Z_s) ds - \int_t^T Z_s dW_s,\ t \in [0, T), \tag{2}$$

where W is an m-dimensional Brownian motion on a probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$, the terminal condition $\xi : \Omega \mapsto \mathbb{R}$ is a random variable being $\mathcal{F}_T$-measurable, and f is a generator function from $\mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^{1\times n}$ into $\mathbb{R}$. Here, X_t is an $\mathbb{R}^n$-valued stochastic process satisfying the differential equation

$$dX(t) = \mu(X(t))dt + \sigma(X(t))dW(t).$$

Moreover, the terminal condition ξ and the generator function f are called a pair of standard parameters if they satisfy the following conditions:

1. $\xi \in L^2_T(\mathbb{R})$,
2. $f(\cdot, \cdot, 0, 0) \in H^2_T(\mathbb{R})$,
3. f is uniformly Lipschitz, i.e., there exists a number $M > 0$ such that

$$|f(\cdot, \cdot, y_1, z_1) - f(\cdot, \cdot, y_2, z_2)| \le M(|y_1 - y_2| + |z_1 - z_2|), \text{ a.s. for all } y_1, y_2, z_1, z_2,$$

where $L^2_T(\mathbb{R}^d)$ is a space of d-dimensional random variables which are $\mathcal{F}_T$-measurable and square integrable, and $H^2_T(\mathbb{R}^{m\times n})$ is a space of $\mathbb{R}^{m\times n}$-valued predictable processes Y such that $\int_0^T |Y_t|^2 dt$ is an integrable random variable.

Theorem 2.1 *If (f, ξ) is a pair of standard parameters, then (1) has a unique solution (Y, Z) in $H^2_T(\mathbb{R}) \times H^2_T(\mathbb{R}^{1\times m})$.*

In the following theorem, we give the Feynman-Kac theorem which explains the relation between BSDE (1) and the semilinear PDE

$$\begin{cases} -\frac{\partial V(t,x)}{\partial t} - \mathcal{L}V(t, x) - f(t, x, V(t, x), (\bigtriangledown_x V(t, x))^T \sigma(x)) = 0,\ t \in [0, T), x \in \mathbb{R}^n, \\ V(T, x) = \zeta(x),\ x \in \mathbb{R}^n, \end{cases} \tag{3}$$

where $\mathcal{L}$ is a generator function given by

$$\mathcal{L}V(t, x) = (\bigtriangledown_x V(t, x))^T \mu(x) + \frac{1}{2} Tr(\sigma^T(x) H_x V(t, x)\sigma(x)). \tag{4}$$

Theorem 2.2 *Let V be a solution of (3) and let*

$$Y_t = V\big(t, X_t\big),\ Z_t = \big(\bigtriangledown_x V(t, X_t)\big)^T \sigma(X_t), Y_T,$$

then (Y_t, Z_t) is a solution of BSDE (1) with the terminal condition $\xi = V(T, X_T)$.

3 The BSDE of Multi-Asset Options

In this section, we will derive the BSDE of multi-asset options. But first, let us introduce the assumptions we use to derive the BSDE.

Assumption 3.1 The following statements are some assumptions:

(1) we use basket options;

(2) the payoff of basket options is a geometric mean, i.e.,

$$Payoff = (\prod_{i=1}^{n} S_i^{\alpha_i} - K)^+, \tag{5}$$

where $\Sigma\alpha_i = 1$ and $\alpha_i \geq 0$;

(3) the wealth process can be replicated;

(4) there exists a risk premium.

The equation of multi-asset options has the form of multi-dimensional parabolic partial differential equations. There are several types of multi-asset options that have different payoffs. According to Assumption 3.1 (1), we use basket options that consist of more than two underlying assets. The basket option pricing formula can be obtained from the multi-asset Black-Scholes differential equation. The multivariable problem of pricing basket options can be reduced to one dimension if the payoff of basket options satisfies Assumption 3.1 (2) (see [5]).

Multi-asset options are formed from several underlying assets consisting of risky assets and risk-free assets. In this study, stocks and bonds are used as risky and risk-free assets, respectively. Suppose S_i denotes the price of the i-th stock which satisfies the geometric Brownian motion equation as follows:

$$\frac{dS_i}{S_i} = \mu_i\, dt + \sum_{j=1}^{m} \sigma_{ij}\, dW_j, \qquad i = 1, 2, ..., n\,, \tag{6}$$

where μ_i is the i-th stock return, σ_{ij} is the volatility of the i-th stock due to the j-th Brownian motion and $W = [W_1, W_2, \dots, W_m]^T$ is an m-dimensional Brownian motion of a probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$. The movement price of the bond P at time t satisfies the equation

$$P(t) = \upsilon\, e^{-\int_t^T r(u)\, du}$$

or

$$\begin{cases} dP(t) = P(t)\, r\, dt, & 0 \leq t < T \\ P(T) = \upsilon, \end{cases} \tag{7}$$

where r is a risk-free interest rate and T is a due time.

Here, we use a multi-asset portfolio equation to determine the option price. Once the price of options is obtained, we can construct a multi-asset portfolio of stocks that has the same value as basket options. This is called a replicating portfolio strategy. In particular, the multi-asset portfolio equation Y is made to have the equal value as the options at time $t \in [0, T]$. A portfolio with this strategy is also known as a wealth process.

A multi-asset portfolio Y consists of n stock assets and a bond asset. Suppose the stock asset S_i, $i = 1, 2, 3, \cdots, n$, has a proportion of Δ_i and a bond P has a proportion of Δ_P. The value of the portfolio at t is defined as

$$Y(t) = \sum_{i=1}^{n} \Delta_i S_i + \Delta_P P. \tag{8}$$

We assume that the value of (8) satisfies the self-financing strategy, therefore we obtain

$$dY(t) = \sum_{i=1}^{n} \Delta_i dS_i + \Delta_P dP. \tag{9}$$

Substituting (6) and (7) into (9), we get

$$dY(t) = \left(Y r + \sum_{i=1}^{n} \Delta_i S_i (\mu_i - r) \right)dt + \sum_{i=1}^{n}\sum_{j=1}^{m} \Delta_i S_i \sigma_{ij} dW_j . \tag{10}$$

From Assumption 3.1 (3), the portfolio Y can be replicated, hence there exists a risk-free portfolio Π that has the same value as the portfolio Y and satisfies

$$d\Pi = dV(S_1, S_2, \ldots, S_n, t) - \sum_{i=1}^{n} \Delta_i dS_i . \tag{11}$$

Using the multi-dimensional Itô formula, we obtain

$$\begin{aligned} dV(S_1, S_2, \ldots, S_n, t) = &\left(\frac{\partial V}{\partial t} + \frac{1}{2} Tr(\sigma(\mathbf{S})^T H_S V \sigma(\mathbf{S})) + \bigtriangledown_S V(t, \mathbf{S}(t))^T \mu(S)\right) dt \\ &+ (\bigtriangledown_S V(t, \mathbf{S}(t))^T \sigma(\mathbf{S})) dW(t) , \end{aligned} \tag{12}$$

where

$$\frac{1}{2} Tr\left(\sigma(\mathbf{S})^T H_S V \sigma(\mathbf{S})\right) = \sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij} S_i S_j ,$$

$$a_{ij} = \sum_{k=1}^{m} \sigma_{ik} \sigma_{jk} ,$$

and

$$\bigtriangledown_S V (t, \mathbf{S}(t))^T \sigma(\mathbf{S})) d\mathbf{W_t} = \sum_{i=1}^{n}\sum_{j=1}^{m} \sigma_{ij} S_i \frac{\partial V}{\partial S_i} dW_j .$$

Substituting (6) and (12) into (11), we have

$$\begin{aligned} d\Pi = &\left(\frac{\partial V}{\partial t} + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij} S_i S_j \frac{\partial^2 V}{\partial S_i \partial S_j} - \sum_{i=1}^{n} \mu_i (\Delta_i S_i - \frac{\partial V}{\partial S_i})\right) dt \\ &+ \sum_{i=1}^{n}\sum_{j=1}^{m} S_i \sigma_{ij} \left(\Delta_i - \frac{\partial V}{\partial S_i}\right) dW_j . \end{aligned}$$

Since Π is a risk-free portfolio, we obtain $\Delta_i = \dfrac{\partial V}{\partial S_i}$, $(i = 1, 2, \ldots, n)$. Substituting it to (10), we get

$$-dY(t) = -\left(Y\,r + \sum_{i=1}^{n} \frac{\partial V}{\partial S_i}\, S_i\,(\mu_i - r)\right) dt - \sum_{i=1}^{n}\sum_{j=1}^{m} \frac{\partial V}{\partial S_i}\, S_i\,\sigma_{ij}\,dW_j. \qquad (13)$$

According to Assumption 3.1 (4), there exists a risk premium

$$\boldsymbol{\lambda} = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_m \end{bmatrix}$$

such that

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} & \ldots & \sigma_{1m} \\ \sigma_{21} & \sigma_{22} & \ldots & \sigma_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \ldots & \sigma_{nm} \end{bmatrix} \boldsymbol{\lambda} = \begin{bmatrix} \mu_1 - r \\ \mu_2 - r \\ \vdots \\ \mu_n - r \end{bmatrix}.$$

Furthermore, (13) becomes

$$\begin{aligned} -dY(t) = -\Bigg(Y\,r &+ \begin{bmatrix} \frac{\partial V}{\partial S_1} S_1 & \frac{\partial V}{\partial S_2} S_2 & \ldots & \frac{\partial V}{\partial S_n} S_n \end{bmatrix} \\ &\times \begin{bmatrix} \sigma_{11} & \sigma_{12} & \ldots & \sigma_{1m} \\ \sigma_{21} & \sigma_{22} & \ldots & \sigma_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \ldots & \sigma_{nm} \end{bmatrix} \times \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_m \end{bmatrix}\Bigg) dt \\ &- \begin{bmatrix} \frac{\partial V}{\partial S_1} S_1 & \frac{\partial V}{\partial S_2} S_2 & \ldots & \frac{\partial V}{\partial S_n} S_n \end{bmatrix} \times \begin{bmatrix} \sigma_{11} & \sigma_{12} & \ldots & \sigma_{1m} \\ \sigma_{21} & \sigma_{22} & \ldots & \sigma_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \ldots & \sigma_{nm} \end{bmatrix} \\ &\times \begin{bmatrix} dW_1 \\ dW_2 \\ \vdots \\ dW_m \end{bmatrix} \end{aligned}$$

and can be written as

$$-dY(t) = f(t, \mathbf{S}(t), Y(t), \mathbf{Z}(t))\,dt - \mathbf{Z}(t)\,d\mathbf{W}(t), \qquad (14)$$

where

$$f(t, \mathbf{S}(t), Y(t), \mathbf{Z}(t)) = -\,Y\,r - \mathbf{Z}(t)\boldsymbol{\lambda} \qquad (15)$$

with

$$\mathbf{Z}(t) = \left[\frac{\partial V}{\partial S_1}S_1 \quad \frac{\partial V}{\partial S_2}S_2 \quad \dots \quad \frac{\partial V}{\partial S_n}S_n\right] \times \begin{bmatrix} \sigma_{11} & \sigma_{12} & \dots & \sigma_{1m} \\ \sigma_{21} & \sigma_{22} & \dots & \sigma_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \dots & \sigma_{nm} \end{bmatrix}.$$

Then, based on the fact that the portfolio Y is constructed such that it has the same value as the basket option, we have the terminal condition of BSDE (14) given by

$$Y_T = (\prod_{i=1}^{n} S_i(T)^{\alpha_i} - K)^+ \ , \tag{16}$$

where $K \in \mathbb{R}^+$.

4 Existence and Uniqueness of Solution to the System (6) and (14)

This section provides a theorem on the existence and uniqueness of solution of the system (6) and (14). This theorem explains the sufficient condition under which a solution of the system (6) and (14) exists and is unique.

Theorem 4.1 *Let* (3) *with* $f(t,x,y,z) = -yr - z\lambda$ *and*

$$V(T,(x_1,x_2,\cdots,x_n)^T) = (\prod_{i=1}^{n} x_i^{\alpha_i} - K)^+$$

have solutions in $H_T^2(\mathbb{R})$. *Then there exists a unique solution of the system* (6) *and* (14) *in* $H_T^2(\mathbb{R})$. *Furthermore,* (3) *also has a unique solution in* $H_T^2(\mathbb{R})$.

Proof. Firstly, we will prove that the BSDE

$$d\tilde{Y}(t) = f(t,\boldsymbol{S}(t),\tilde{Y}(t),\boldsymbol{U})dt - \boldsymbol{U}(t)d\boldsymbol{W}(t) \tag{17}$$

has a unique solution when the terminal condition is given by

$$U(T) = (\prod_{i=1}^{n} S_i(T)^{\alpha_i} - K)^+.$$

The proof is given as follows.

1. In this part, we will prove that $\tilde{Y}_T \in L_T^2(\mathbb{R})$.
   The terminal condition of BSDE (17) is given by $\tilde{Y}_T = (\prod_{i=1}^{n} S_i(T)^{\alpha_i} - K)^+$. Since $S_i(T)$ is $\mathcal{F}_T$-measurable, we can obtain that so is $\tilde{Y}_T$. Moreover, we also have

$$\mathbb{E}[S_i(T)^2] = S_i(0)^2 \exp\left\{2\mu T + \sum_{j=1}^{m} \sigma_{ij}^2\, T\right\} < \infty\,,$$

   hence

$$\mathbb{E}[|\tilde{Y}_T^2|] = \mathbb{E}\left[\left|\left(\left(\prod_{i=1}^{n} S_i(T)^{\alpha_i} - K\right)^+\right)^2\right|\right] < \infty\ .$$

Therefore, we get

$$\tilde{Y}_T = \left(\prod_{i=1}^{n} S_i(T)^{\alpha_i} - K\right)^+ \in L_T^2(\mathbb{R}) \ .$$

2. In this part, we will prove that $f(\cdot,\cdot,0,0) \in H_T^2(\mathbb{R})$.
   Based on (15), we obtain

   $$\mathbb{E}\left[\int_0^T |f(\cdot,\cdot,0,0)|^2 \, dt\right] = 0 < \infty \ .$$

   Thus, $f(\cdot,\cdot,0,0)$ is a predictable process and we also have

   $$f(\cdot,\cdot,0,0) \in H_T^2(\mathbb{R}).$$

3. In this last part, we prove that f is uniformly Lipschitz.
   We can observe that for $L = \max(r, |\boldsymbol{\lambda}|)$, we have

   $$f(t, \mathbf{S}(t), y_1, z_1) - f(t, \mathbf{S}(t), y_1, z_1)| \le L(|y_1 - y_2| + |z_1 - z_2|) \ .$$

   Therefore f is a uniform Lipschitz function.

Thus three conditions in Theorem 2.1 are satisfied, so (17) has a unique solution. Moreover, suppose that V is one of the solutions of (3), which is possible because we assume that (3) has a solution. According to the Feynman-Kac theorem and the uniqueness of the solution (17), we obtain that the pair $(\tilde{Y}, \boldsymbol{U})$ given by

$$\tilde{Y}(t) = V(t, \boldsymbol{S}(t)) \tag{18}$$

and

$$\boldsymbol{U}(t) = (\nabla_x V(t, \boldsymbol{S}(t)))^T \boldsymbol{\sigma}(\boldsymbol{S}(t)) \tag{19}$$

is the unique solution of (17). Consequently, by a contradiction argument, we have $Y(t) = V(t, \boldsymbol{S}(t))$ is a unique solution of the system (6) and (14). Furthermore, by a similar argument, we can also conclude that the solution of (3) is unique.

5 Simulations of Multi-Asset Option Prices

In this section, we present the simulation of pricing of the multi-asset option in the basket option. From Theorem 4.1, we obtain that the solution of the system (6) and (14) is equivalent to the solution of

$$\begin{cases} \frac{\partial V}{\partial t} + \dfrac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij}\, S_i\, S_j \dfrac{\partial^2 V}{\partial S_i \, \partial S_j} + \sum_{i=1}^{n} \dfrac{\partial V}{\partial S_i}\, r\, S_i - rV = 0 \ , \\ dS_i = \mu_i S_i \, dt + \sum_{j=1}^{m} \sigma_{ij} S_i \, dW_j, \qquad i = 1, 2, ..., n \, , \\ V(T, (S_1, S_2, \cdots, S_n)^T) = (\prod_{i=1}^{n} x_i^{\alpha_i} - K)^+ \end{cases} \tag{20}$$

if (3) has solutions in $H_T^2(\mathbb{R})$. We can see that the last equation above is the multi-asset Black-Scholes differential equation and the exact solution is given by (see [5])

$$V(S_1, \ldots, S_n, t) = e^{-\hat{q}(T-t)}\, S_1^{\alpha_1}\, S_2^{\alpha_2} \ldots S_n^{\alpha_n} N(\hat{d}_1) - Ke^{-r(T-t)} N(\hat{d}_2), \tag{21}$$

where

$$\begin{aligned}\hat{d}_1 &= \frac{\ln\left(\frac{S_1^{\alpha_1} S_2^{\alpha_2} \ldots S_n^{\alpha_n}}{K}\right) + (r - \hat{q} + \frac{\hat{\sigma}^2}{2})(T-t)}{\hat{\sigma}\sqrt{T-t}},\\ \hat{d}_2 &= \frac{\ln\left(\frac{S_1^{\alpha_1} S_2^{\alpha_2} \ldots S_n^{\alpha_n}}{K}\right) + (r - \hat{q} - \frac{\hat{\sigma}^2}{2})(T-t)}{\hat{\sigma}\sqrt{T-t}}\\ &= \hat{d}_1 - \hat{\sigma}\sqrt{T-t},\\ \hat{\sigma}^2 &= \sum_{i=1}^{n} a_{ij}\alpha_i\alpha_j,\\ q &= \sum_{i=1}^{n} \frac{a_{ii}}{2}\alpha_i + \frac{\hat{\sigma}^2}{2}.\end{aligned}$$

Here, we know that (21) is also the solution of the BSDE of multi-asset options (14). Simulations were carried out to determine the effect of several variables on the option price. These variables are the strike price, interest rate, maturity date, and stock volatility. The simulation is carried out on the assets of 2, 5, 10 and 30. In each option price simulation, the number of the Brownian motion used is equal to the number of stock assets, the price of each stock is 100 and the proportion of each stock in basket options has the same value.

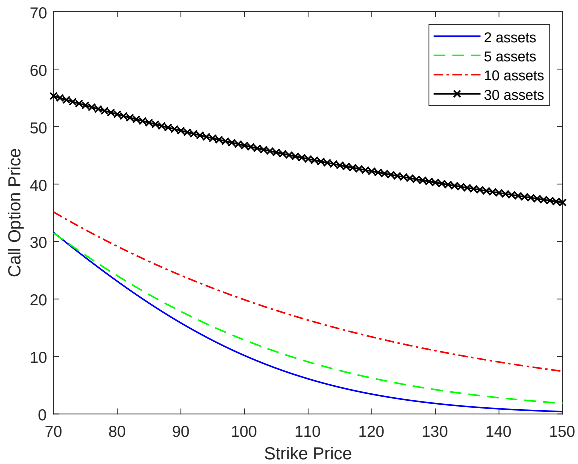

Figure 1: The effect of the strike price on the option price.

We plot the call option price influenced by different strike prices given by (21), see Figure 1. The call option price decreases when the value of strike price increases. If the call option is exercised at maturity, then the profit from the call option is the share price that exceeds the strike, $S_i^{\alpha_i} - K$. Therefore, the price of the call option is more expensive, the lower the strike price is. This also relates to the stock prices.

If the stock price is fixed, while the *strike* price is getting smaller, then the profit will be getting bigger. Therefore, the call option is more valuable as the stock price increases. Figure 1 also shows that the more assets in the call option, the higher the price of the call option.

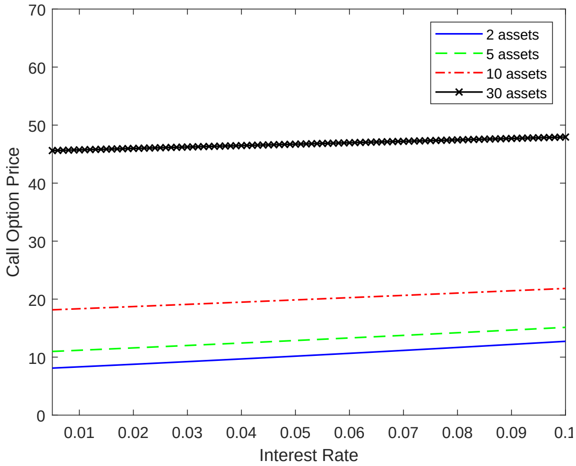

Figure 2: The effect of the interest rate on the option price.

Figure 2 shows that a higher interest rate causes a higher price of the call option. This is caused by the fact that if the interest rate in economy rises, the *return* expected by investors on stock assets tends to rise. On the other hand, the current value of funds received by the option holder is decreasing. The combination of these two causes an increase in the call option price.

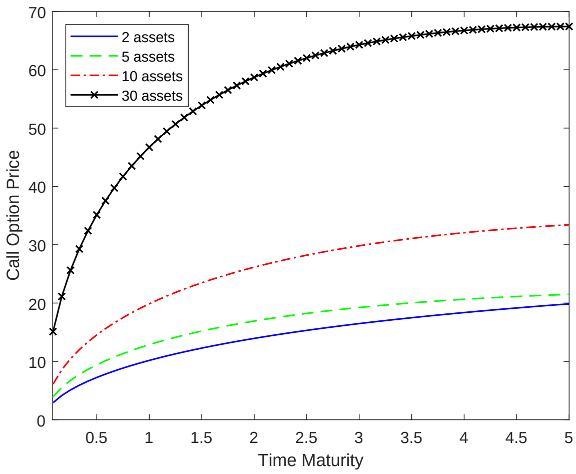

Figure 3: The effect of the time maturity on the option price.

Based on the simulation results, see Figure 3, it is found that a longer expiration time caused a higher call option price. The longer the maturity of a call option, the more opportunities the option holder gets to exercise his rights on that option. Based on Figure 4, it is found that the higher the volatility, the higher the price of the call option. Volatility shows a measure of the uncertainty of stock prices in the future. When volatility increases, stock prices also increase. As a result, the profit derived from the call option also increases. On the other hand, the loss from the call option is the price

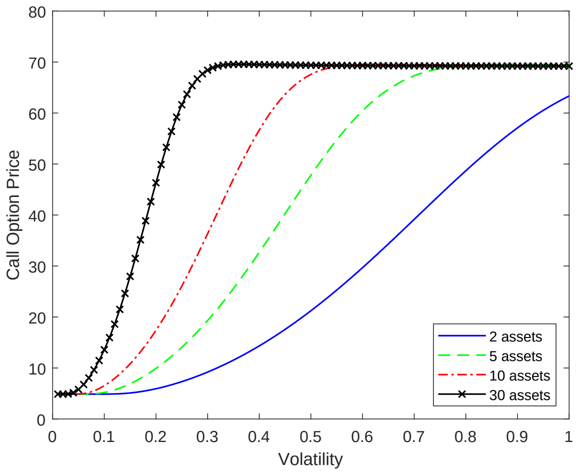

Figure 4: The effect of the volatility on the option price.

of the premium or the price of the call option paid at the beginning of the transaction. Therefore, it is found that a higher volatility resulted in a higher call option price.

6 Conclusion

In this work, we obtain a model of the multi-asset portfolio given by the system (6) and (14). The terminal condition of the BSDE is made equal to the payoff of the multi-asset option. From the theorems on the existence and uniqueness of the solution of the BSDE (see Theorem 1 and Theorem 3), we prove that there exists a unique solution of the system (6) and (14). Utilizing the Feynman-Kac theorem (Theorem 2), we can obtain the solution of the system (6) and (14), which is consistent with the multi-asset Black-Scholes differential equation's solution. In addition, there are also several factors that affect the price of multi-asset options.

Acknowledgment

The first author of this paper is fully supported by the Department of Mathematics, Institut Teknologi Sepuluh Nopember, with contract number 1367/PKS/ITS/2023.

References

[1] Z. Brzezniak and T. Zastawniak. *Basic Stochastic Processes: A Course Through Exercises*. Springer Science and Business Media, 2000.

[2] N. El Karoui, S. Peng and M. C. Quenez. Backward stochastic differential equations in finance. *Mathematical Finance* **7** (1997) 1–71.

[3] N. El Karoui , S. Hamadéne and A. Matoussi. Backward stochastic differential equations and applications. *Indifference Pricing: Theory and Applications* (2008) 267–320.

[4] J. C. Hull. *Options Futures and Other Derivatives*. Pearson Education India, 2008.

[5] L. Jiang. *Mathematical Modeling and Methods of Option Pricing*. World Scientific Publishing Company, 2005.

[6] Ning Ma, Zhen Wu and Huaizhong Zhao. Backward stochastic differential equations with Markov chains and associated PDEs. *Journal of Differential Equations* **302** (2021) 854–894.

[7] Y. Peng, B. Gong, H. Liu and B. Dai Option pricing on the GPU with backward stochastic differential equation. *2011 Fourth International Symposium on Parallel Architectures, Algorithms and Programming* (2011) 19–23.

[8] N. Perkowski. *Backward Stochastic Differential Equations: An Introduction.* https://api.semanticscholar.org/CorpusID:51816221.

[9] H. Pham. *Continuous-Time Stochastic Control and Optimization with Financial Applications.* Springer, 2009.

[10] P. Sahoo. Probability and mathematical statistics. University of Louisville, 2013.

[11] A. P. Selvadurai. *Partial Differential Equations in Mechanics 1: Fundamental, Laplace's Equation, Diffusion Equation, Wave Equation.* Springer Science and Bussiness Media, 2000.

[12] V. Shcherbakov and E. Larsson. Radial basis function partition of unity methods for pricing vanilla basket options. *Computers and Mathematics with Applications* **71** (2016) 185-200.

[13] K. Shiraya and A. Takahashi. A general control variate method for multi-dimensional SDEs: An application to multi-asset options under local stochastic volatility with jumps models in finance. *European Journal of Operational Research* **258** (2017) 358–371.

[14] F. Soleymani and A. Akgül. Improved numerical solution of multi-asset option pricing problem: A localized RBF-FD approach. *Chaos, Solitons and Fractals* **119** (2019) 298–309.

[15] L. Teng, A. Lapitckii and M. Günther. A multi-step scheme based on cubic spline for solving backward stochastic differential equations. *Applied Numerical Mathematics* **150** (2020) 117–138.

[16] B. Yu, H. Zhu and P. Wu. The closed-form approximation to price basket options under stochastic interest rate. *Finance Research Letters* **46** (2022) 102434.