Nonlinear Dynamics and Systems Theory, **24** (2) (2024) 147–158

Dynamic Analysis of a New Hyperchaotic System with Infinite Equilibria and Its Synchronization

Yasmina Ghettout [1], Lotfi Meddour [2*], Tayeb Hamaizia [2] and Rabiaa Ouahabi [2]

[1] *Laboratory of Differential Equations, Department of Mathematics, Faculty of Exact Sciences, University of Mentouri Brothers, Constantine, Algeria.*
[2] *Department of Mathematics, Faculty of Exact Sciences, University of Mentouri Brothers, Constantine, Algeria.*

Received: August 13, 2023; Revised: February 19, 2024

Abstract: In this paper, a new 4D autonomous hyperchaotic system with an infinite number of equilibrium points is introduced and analyzed. This hyperchaotic system is constructed by introducing an additional dimension with a linear state feedback controller to the third equation in the Lorenz system. The dynamical properties of the new hyperchaotic system are discussed by means of dissipation, symmetry, Lyapunov exponents, bifurcation diagrams, equilibrium points and coexisting attractors. Finally, the synchronization of the novel hyperchaotic system is discussed.

Keywords: *Lorenz system; hyperchaos; infinite equilibria; Lyapunov exponent; synchronization.*

Mathematics Subject Classification (2010): 93B52, 70K50, 34H10, 37G35, 34D06.

1 Introduction

A hyperchaotic system is a type of the dynamical system that exhibits chaotic behavior with at least two positive Lyapunov exponents, and the minimal dimension of the phase space that embeds the hyperchaotic attractor should be at least four. In 1963, Lorenz proposed a three-dimensional system with two scrolls, which is recognized as the first chaotic model reported in literature [10], it has been the subject of many studies (see, for example, [11]). Subsequently, in 1976, Rossler proposed another chaotic system like the Lorenz one, in 1979, Rossler put forward the concept of hyperchaos and proposed the

* Corresponding author: mailto:meddour_lotfi@umc.edu.dz

hyperchaotic Rossler system [13]. As a kind of behavior that is more complex than chaos, hyperchaos has greater application potential in some engineering and technological fields, which need strong complexity, including secure communications [23], nonlinear circuits, encryption, and other fields [20]. Constructing a hyperchaotic system is a real challenge because of the absence of a uniform method to generate this kind of complex systems. Until now, the main method to build a hyperchaotic system is to design it by changing an existing chaotic system. A hyperchaotic system can be generated by adding a simple state feedback controller to a regular chaotic system. The addition of feedback control introduces additional nonlinearities that can lead to more complex and richer dynamical behavior, e.g., the Lü system [14] and the Lorenz system [6].

Hyperchaotic systems with an infinite number of equilibrium points have been classified as a newly introduced class of dynamical systems with hidden attractors. This classification highlights the richness and complexity of these systems and their unique properties that were not fully understood before [21], [15], [8], [1]. The coexistence of multiple attractors in a system is indeed an exceedingly interesting phenomenon that has attracted increasing attention in the scientific community. It challenges our conventional understanding of systems having a single stable state and opens up new avenues for studying and understanding complex dynamics. The coexistence of attractors introduces the concept of multistability, where a system can exhibit different long-term behaviors depending on its initial conditions or parameters. This phenomenon has been observed in various fields of study, including physics, biology, chemistry, engineering, and social sciences. The Lorenz system is a well-known example that demonstrates the phenomenon of coexisting attractors, what makes the Lorenz system particularly interesting is that depending on the system's initial conditions, it can yield a symmetric pair of strange attractors, breaking the symmetry of the butterfly attractor. These two attractors exist simultaneously alongside each other in the system's phase space [9], [2], [22].

The synchronization of chaotic systems is a fascinating area of research that has attracted considerable attention due to its theoretical significance and practical applications, especially in the field of secure communication. Chaotic systems are highly sensitive to initial conditions, and even a tiny change in the initial state can lead to drastically different trajectories over time. Despite this sensitivity, it is possible to synchronize two or more chaotic systems in a way that they evolve with similar dynamics, exhibiting identical or correlated chaotic behaviors [16], [17], [5], [7], [18].

The rest of the paper is organized as follows. In Section 2, the novel hyperchaotic system is introduced with brief details. The dynamical system analysis and its properties are presented in Section 3. In Section 4, a synchronization control system is designed. Finally, conclusions are presented in Section 5.

2 The Novel 4D System with an Infinite Number of Equilibrium Points

The celebrated Lorenz system is described by the following equations [10]:

$$\begin{cases} \dot{x} = -a(y - x), \\ \dot{y} = -xz + cx - y, \\ \dot{z} = xy - bz, \end{cases} \tag{1}$$

where a, b, c are real parameters. When $a = 10$, $b = 8/3$, $c = 28$, it shows chaotic behavior. According to this system, a new 4D system is obtained by introducing an

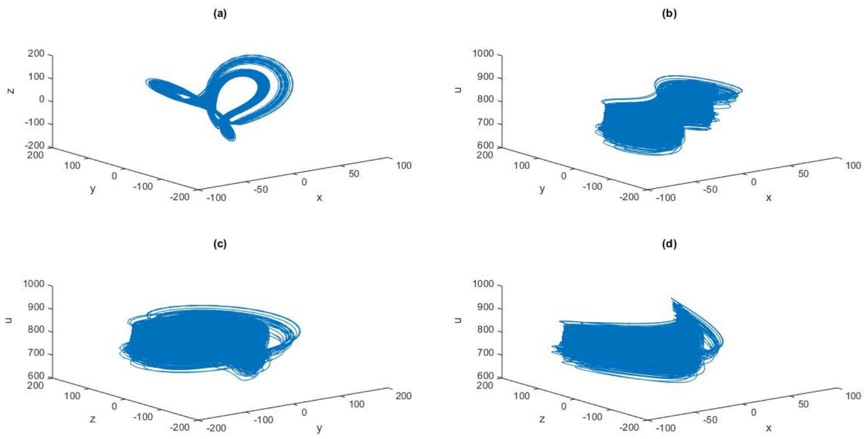

Figure 1: Three-dimensional projections of the hyperchaotic attractor of system (2): $(a, b, c, d) = (10, 5, 30, 20)$. (a) x-y-z phase space; (b) x-y-u phase space; (c) y-z-u phase space; (d) x-z-u phase space.

additional dimension with a linear state feedback controller to the third equation, and our new 4D system is given by the following dynamics:

$$\begin{cases} \dot{x} = -a(x - y), \\ \dot{y} = -xz + cx - y, \\ \dot{z} = xy - bz - u, \\ \dot{u} = dx. \end{cases} \tag{2}$$

We represent the state of the 4D system (2) by $X = (x, y, z, u)$, and a, b, c, d are positive parameters. When the parameters of system (2) are taken as $(a, b, c, d) = (10, 5, 30, 20)$ and the initial conditions $(x_0, y_0, z_0, u_0) = (0.1, 0.1, 0.1, 0.1)$, the system has a hyperchaotic attractor, and the corresponding four Lyapunov exponents can be calculated: $L_1 = 0.8289, L_2 = 0.5692, L_3 = -0.2836, L_4 = -16.9904$. The phase portraits of the new hyperchaotic system (2) are shown in Figure 1.

The Lyapunov dimension, commonly known as the Kaplan-Yorke dimension D_{KY} [3], of this system is

$$D_{KY} = j + \frac{1}{|L_{j+1}|} \sum_{i=1}^{j} L_i = 3 + \frac{L_1 + L_2 + L_3}{|L_4|},$$

$$D_{KY} = 3 + \frac{0.8289 + 0.5692 - 0.2836}{|-16.9904|} = 3.07.$$

3 Dynamical Analysis

3.1 Symmetry

The system (2) is symmetrical with respect to the z-axis for its invariance under the coordinate transformation $(x, y, z, u) \longrightarrow (-x, -y, z, -u)$.

3.2 Dissipativity and existence of attractor

For a dynamical system, the divergence of the system (2) is defined by

$$\nabla V = \frac{d\dot{x}}{dx} + \frac{d\dot{y}}{dy} + \frac{d\dot{z}}{dz} + \frac{d\dot{u}}{du} = -(a + b + 1) = -16 < 0. \tag{3}$$

Therefore, the above analysis proves that our system is dissipative. The exponential contraction rate is calculated as follows:

$$V(t) = V(0)e^{-(a+b+1)t}. \tag{4}$$

It shows that each volume containing the system trajectories shrinks to zero as $t \longrightarrow \infty$ at an exponential rate $-(a + b + 1)$. There exists an attractor in system (2).

3.3 Equilibrium points

The equilibria of the system (2) can be found by setting $\dot{x} = \dot{y} = \dot{z} = \dot{u} = 0$ and $a, b, c, d > 0$,

$$a(y - x) = 0, \tag{5}$$

$$-xz + cx - y = 0, \tag{6}$$

$$xy - bz - u = 0, \tag{7}$$

$$dx = 0. \tag{8}$$

Equation (8) reveals that $x = 0$. By substituting $x = 0$ into (5), we have $y = 0$, by substituting $x = 0$ and $y = 0$ into (7), we have

$$-bz - u = 0.$$

In other words, system (2) has an infinite number of equilibrium points

$$E = \{(x, y, z, u) \in R^4 | x = 0, y = 0, u = -bz\}.$$

For the equilibrium E, the Jacobian matrix of system (2) is given by

$$\mathcal{J}_E = \begin{pmatrix} -a & a & 0 & 0 \\ c - z & -1 & 0 & 0 \\ 0 & 0 & -b & -1 \\ d & 0 & 0 & 0 \end{pmatrix}.$$

The charasteristic equation of system (2) evaluated at the equilibrium E is

$$\lambda^4 + (a + b + 1)\lambda^3 + (a + b(a + 1) - a(c - z))\lambda^2 + b(a - a(c - z))\lambda = 0,$$

and the eigenvalues are

$$\lambda_1 = 0, \lambda_2 = -b, \ \lambda_3 = -\frac{1}{2}\sqrt{4ac - 2a - 4az + a^2 + 1} - \frac{1}{2}a - \frac{1}{2},$$
$$\lambda_4 = \frac{1}{2}\sqrt{4ac - 2a - 4az + a^2 + 1} - \frac{1}{2}a - \frac{1}{2}.$$

Providing $z > c-1$, we have $\lambda_2 < 0$, $Re(\lambda_3) < 0$ and $Re(\lambda_4) < 0$, then the equilibrium point E is stable. Therefore, if $z < c-1$, the eigenvalue λ_4 is positive, then the equilibrium point E is unstable. Moreover, if $z = c - 1$, the system (2) reduces to the one whose dimension is less than four.

3.4 Bicurcation diagram and Lyapunov exponents

In this subsection, to explore the effect of the parameters a, b, c and d on the behavior of a new 4D system, we fixed the parameters $(a, b, c) = (10, 5, 30)$ and varied d in $[0, 25]$. Now we carry out the dynamic analysis of the system (2) numerically by its Lyapunov exponent spectrum and bifurcation diagram. Figure 2 shows the bifurcation diagram of system (2) with different values of the parameter d and the initial conditions $(x_0, y, z_0, u_0) = (1.8, 0.1, 0.79, 2.8)$. Table 1 represents the Lyapunov exponents and Lyapunov dimensions of the system (2) with different values of the parameter d .

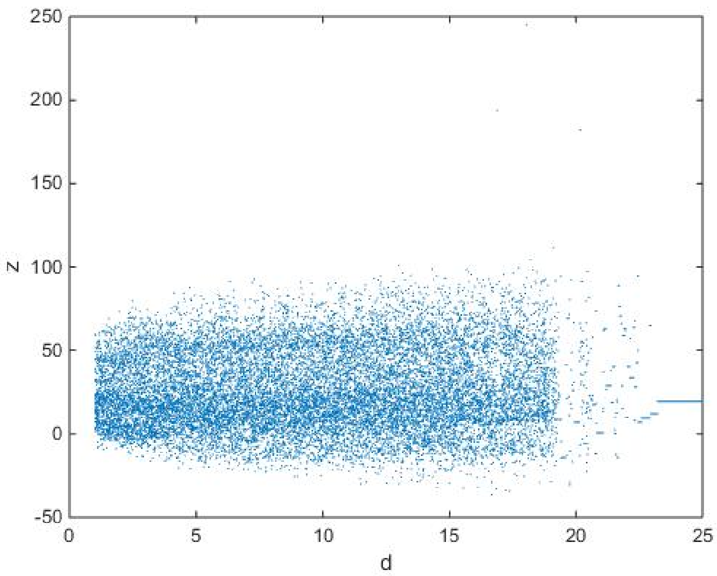

Figure 2: Bifurcation diagrams of hyperchaotic system (2) when $a = 10, b = 5, c = 30$ and d varies in $[0, 25]$.

The dynamics of system (2) changes with the increasing value of the parameter d. We can see that the results obtained from the Lyapunov spectrum agree with the results given by the bifurcation diagram in Figure 2 and the maximal Lyapunov exponents in Table 1.

Table 1: Lyapunov exponents of (2) with $(a, b, c) = (10, 5, 30)$ and different values of d.

d	L_1	L_2	L_3	L_4	Attractor type	D_{KY}
2	0	-0.7873	-0.6745	-14.5233	Periodic attractor	1
12	0.9154	0.0452	-0.0832	-16.7967	Hyperchaotic attractor	3.05
19	0.4400	-0.0128	-0.4658	-15.8353	Chaotic attractor	2.92

3.5 Coexisting attractors

The system (2) shows many complex dynamics behaviors such as hyperchaos, chaos, and periodic. Several coexisting attractors of the system (2) will be present under some appropriate parameters. A system with coexisting attractors is very sensitive to system parameters and initial values. In the event of a sudden disturbance, the state of the system can easily shift from an ideal state to another state that may be undesirable. The coexisting attractors of the system (2) satisfying different initial values can exhibit various dynamic behaviors.

Coexistence of hyperchaotic and chaotic attractors

When we fix $a = 10, b = 5, c = 30, d = 20$, and change the initial values slightly, the dynamical behaviors of the system may produce large variations in the long term:
(a) For the initial values $(0.1, 0.1, 0.1, 0.1)$, the Lyapunov exponents of the system (2) are found to be $L_1 = 0.8289, L_2 = 0.5692, L_3 = -0.2836$, and $L_4 = -16.9904$, while the fractal dimension of the system is estimated to be 3.07. A hyperchaotic attractor with an infinite number of equilibrium points can be obtained, whose 3D phase portrait is shown in Figure 3 (a).

(b) For the initial values $(-2.2, 0.1, 1, 0.1)$, the trajectories of system (2) converge to a chaotic attractor. The Lyapunov exponents of the system (2) are found to be $L_1 = 0.6775, L_2 = -0.1010, L_3 = -2.7889$, and $L_4 = -13.3054$, while the fractal dimension of the system is estimated to be 2.83, whose 3D phase portrait is shown in Figure 3(b).
Therefore, the modification of the inial conditions causes a change of the hyperchaotic behavior of the system (2). The trajectories of system (2) converge to two types of attractors (hyperchaos or chaos).

Coexistence of hyperchaotic and periodic attractors

When we fix $a = 10, b = 5, c = 30, d = 2$, and change the initial values slightly, the dynamical behaviors of the system may produce large variations in the long term:

(a) For the initial values $(-1.8, 0.1, 0.79, 2.8)$ the trajectories of system (2) converge to a hyperchaotic attractor. The Lyapunov exponents of the system (2) are found to be $L_1 = 0.5365, L_2 = 0.0710, L_3 = -0.1822$, and $L_4 = -16.3938$, while the fractal dimension of the system is estimated to be 3.03, whose 3D phase portrait is shown in Figure 4(a).

(b) For the initial values $(1.8, 0.1, 0.79, 2.8)$, the trajectories of system (2) converge to a stable periodic orbit. The Lyapunov exponents of the system (2) are found to be $L_1 = 0, L_2 = -0.7873, L_3 = -0.6745$, and $L_4 = -14.5233$, while the fractal dimension of the system is estimated to be 1, whose 3D phase portrait is shown in Figure 4(b).

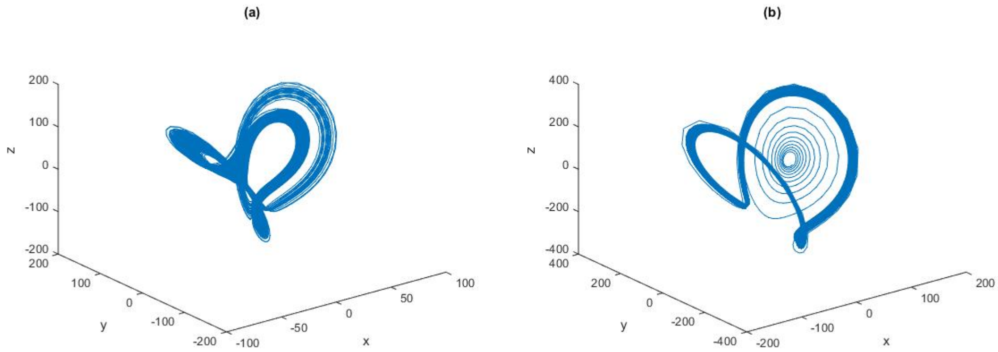

Figure 3: (a) Hyperchaotic attractor with $(a, b, c, d) = (10, 5, 30, 20)$ and $(x_0, y_0, z_0, u_0) = (0.1, 0.1, 0.1, 0.1)$. (b) Chaotic attractor with $(a, b, c, d) = (10, 5, 30, 20)$ and $(x_0, y_0, z_0, u_0) = (-2.2, 0.1, 1, 0.1)$.

A small change in the initial condition of the system causes a wide difference of trajectories. For different initial conditions, the trajectories converge to different attractors: a periodic orbit and hyperchaotic attractor.

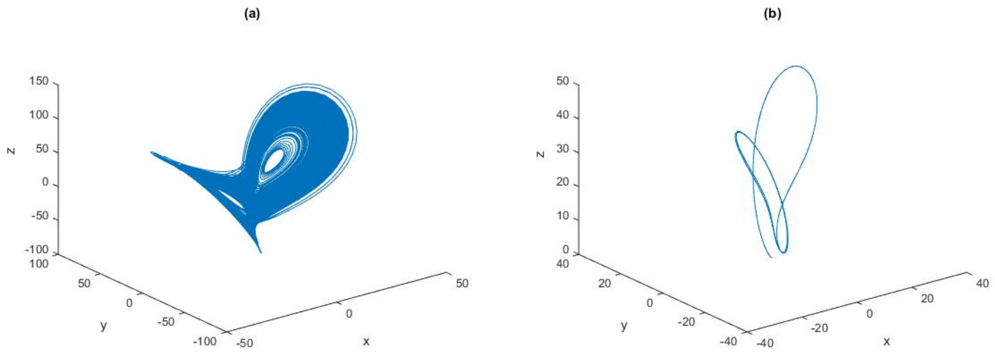

Figure 4: (a) Hyperchaotic attractor with $(a, b, c, d) = (10, 5, 30, 2)$ and $(x_0, y_0, z_0, u_0) = (-1.8, 0.1, 0.79, 2.8)$. (b) Periodic attractor with $(a, b, c, d) = (10, 5, 30, 2)$ and $(x_0, y_0, z_0, u_0) = (1.8, 0.1, 0.79, 2.8)$

4 Synchronization of a New Hyperchaotic System

In this section, the synchronization of the novel 4D hyperchaotic system (2) newly introduced is discussed. Synchronization is done between a system designed as a master one and another system as a slave one. The principle of synchronization is to apply to the slave system a control function such as the error between the two systems tends to zero. We have synchronized two identical new hyperchaotic systems with different initial conditions via the active control method.

As the master system, we consider the novel 4D hyperchaotic system (2) given by

$$\begin{cases} \dot{x_1} = -a(x_1 - x_2), \\ \dot{x_2} = -x_1x_3 + cx_1 - x_2, \\ \dot{x_3} = x_1x_2 - bx_3 - x_4, \\ \dot{x_4} = dx_1. \end{cases} \tag{9}$$

The slave system is the same system (2), with the added to it control signals $U = \{u_1, u_2, u_3, u_4\}$. It is given by

$$\begin{cases} \dot{y_1} = -a(y_1 - y_2) + u_1, \\ \dot{y_2} = -y_1y_3 + cy_1 - y_2 + u_2, \\ \dot{y_3} = y_1y_2 - by_3 - y_4 + u_3, \\ \dot{y_4} = dy_1 + u_4. \end{cases} \tag{10}$$

The synchronization error is defined by

$$\begin{cases} e_1 = y_1 - x_1, \\ e_2 = y_2 - x_2, \\ e_3 = y_3 - x_3, \\ e_4 = y_4 - x_4. \end{cases} \tag{11}$$

Then the error dynamics is expressed by

$$\begin{cases} \dot{e_1} = a(e_2 - e_1) + u_1, \\ \dot{e_2} = ce_1 - e_2 - y_1y_3 + x_1x_3 + u_2, \\ \dot{e_3} = -be_3 - e_4 + y_1y_2 - x_1x_2 + u_3, \\ \dot{e_4} = de_1 + u_4. \end{cases} \tag{12}$$

We choose the active control functions u_1, u_2, u_3, u_4 as shown in equation (13) to eliminate the nonlinear terms in (12):

$$\begin{cases} u_1 = v_1, \\ u_2 = y_1y_3 - x_1x_3 + v_2, \\ u_3 = -y_1y_2 + x_1x_2 + v3, \\ u_4 = v_4. \end{cases} \tag{13}$$

So, the system to be controlled is a linear system with a control input $\{v_1, v_2, v_3, v_4\}$

$$\begin{cases} \dot{e_1} = a(e_2 - e_1) + v_1, \\ \dot{e_2} = ce_1 - e_2 + v_2, \\ \dot{e_3} = -be_3 - e_4 + v_3, \\ \dot{e_4} = de_1 + v_4. \end{cases} \tag{14}$$

The control input v_1, v_2, v_3 and v_4 is used to force the error to converge to zero. So, the two systems (9) and (10) are synchronized.

We choose a constant matrix C which will control the error dynamics (14) such that

$$[\dot{e_1}, \dot{e_2}, \dot{e_3}, \dot{e_4}]^T = C[e_1, e_2, e_3, e_4]^T. \tag{15}$$

For the Routh-Hurwitz criterion, all eigenvalues of the chosen matrix C must be negative [19] to stabilize the synchronization between the master system (9) and the slave system (10):

$$\mathcal{C} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

The eigenvalues $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (-1, -1, -1, -1)$ of the matrix C are negative, which ensures the stability of the dynamic error. Therefore two systems (9) and (10) are synchronized.

Using (14), (15) and (16), we find the control functions

$$\begin{cases} a(e_2 - e_1) + v_1 = -e_1, \\ ce_1 - e_2 + v_2 = -e_2, \\ -be_3 - e_4 + v_3 = -e_3, \\ de_1 + v_4 = -e_4, \end{cases} \tag{16}$$

$$\begin{cases} v_1 = (a-1)e_1 - ae_2, \\ v_2 = -ce_1, \\ v_3 = (b-1)e_3 + e_4, \\ v_4 = -de_1 - e_4. \end{cases} \tag{17}$$

The control function $U = \{u_1, u_2, u_3, u_4\}$ ensures synchronization between the master system (9) and the slave system (10),

$$\begin{cases} u_1 = (a-1)e_1 - ae_2, \\ u_2 = y_1y_3 - x_1x_3 - ce_1, \\ u_3 = -y_1y_2 + x_1x_2 + (b-1)e_3 + e_4, \\ u_4 = -de_1 - e_4. \end{cases} \tag{18}$$

The initial conditions of the master system and the slave system are taken as follows:

$$(x_1, x_2, x_3, x_4) = (3, 4, 10, -8),$$

$$(y_1, y_2, y_3, y_4) = (-2, 32, -6, 12).$$

The results of simulations are depicted in Figure 5 and Figure 6, these figures show that the synchronization occurs and the errors will converge to zero exponentially after applying the active control.

5 Conclusions

In this work, we have introduced a novel hyperchaotic system with an infinite number of equilibrium points. Depending on parameter selection and initial conditions, this system can generate various types of coexisting attractors. Through theoretical analysis

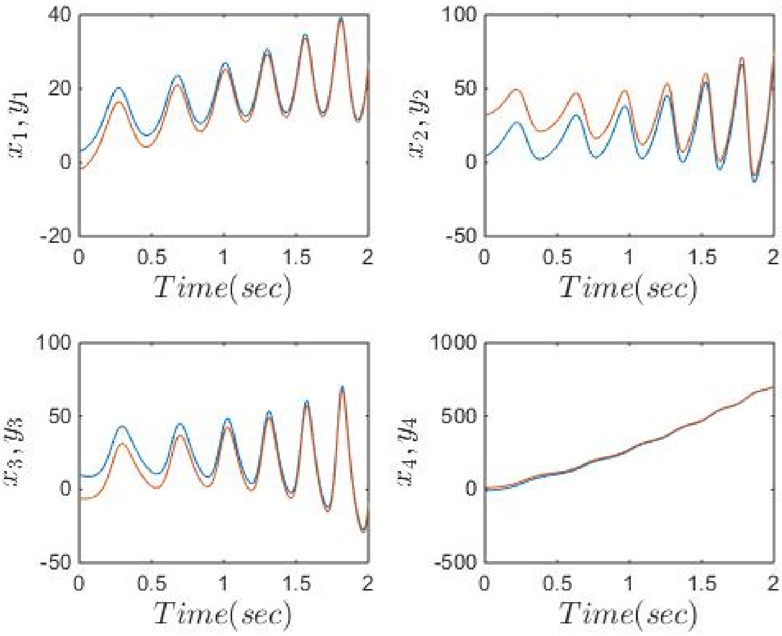

Figure 5: Synchronization of different state variables.

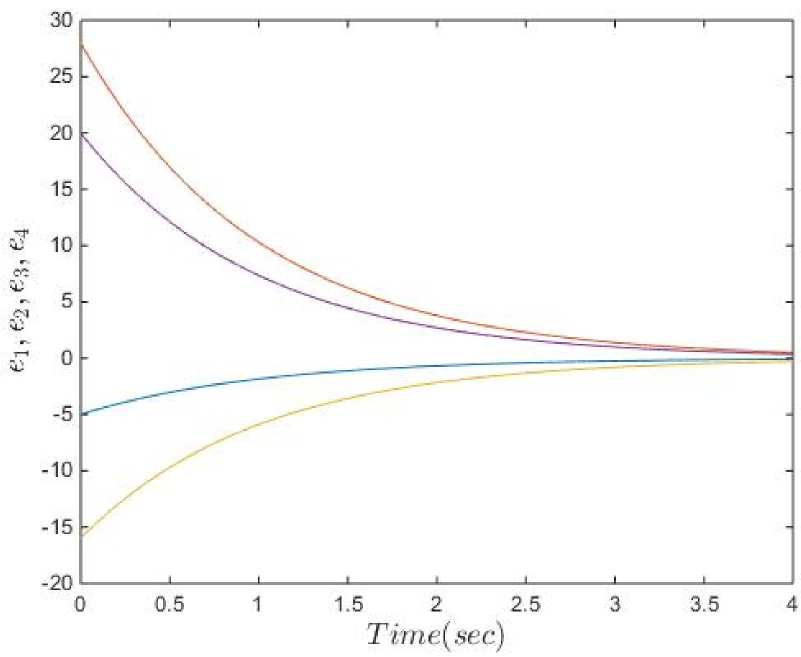

Figure 6: Errors of the synchronization system.

and numerical simulations, we investigate the dynamical behaviors of the new hyperchaotic system. This exploration encompasses dissipativity and invariance, equilibrium points and their stability, periodic orbits, as well as chaotic and hyperchaotic attractors. We conduct numerical simulations, including phase diagrams and the Lyapunov exponent spectrum, to analyze and validate the complex phenomena exhibited by our hyperchaotic system. Furthermore, we successfully achieve synchronization between this system, designed as the master one, and another system acting as the slave one, using the active control techniques.

References

[1] Y. Chen and Q.A. Yang. new Lorenz-type hyperchaotic system with a curve of equilibria. *Mathematics and Computers in Simulation* **112** (2015) 40–55.

[2] C. Dong and J. Wang. Hidden and coexisting attractors in a novel 4D hyperchaotic system with no equilibrium point. *Fractal and Fractional* **6**(6) (2022) 306.

[3] P. Frederickson, J.L. Kaplan, E.D. Yorke and J.A. Yorke. The Liapunov dimension of strange attractors. *Journal of differential equations* **49** (2) (1983) 185–207.

[4] F. Zhang, X. Liao, G. Zhang and C. Mu. Complex dynamics of a new 3D Lorenz-type autonomous chaotic system. *Pramana - J. Phys.* **89** (2017) 84.

[5] A. Ikhlef and N. Mansouri. Hyperchaotification and Synchronization of Chaotic Systems. *International Journal of Control* **2** (4) (2012) 69–74.

[6] J. Qiang. Hyperchaos generated from the Lorenz chaotic system and its control. *Physics Letters A* **366** (3) (2007) 217–222.

[7] Basil H. Jasim, Anwer Hammadi Mjily, Anwer Mossa Jassim AL-Aaragee. A novel 4 dimensional hyperchaotic system with its control, synchronization and implementation. *International Journal of Electrical and Computer Engineering* **11** (4) (2021) 2974.

[8] Sifeu Takougang Kingni, Viet-Thanh Pham, Sajad Jafari and Paul Woafo. A chaotic system with an infinite number of equilibrium points located on a line and on a hyperbola and its fractional-order form. *Chaos, Solitons & Fractals* **99** (2017) 209–218.

[9] Qiang Lai, Tsafack Nestor, Kengne Jacques and Xiao-Wen Zhao. Coexisting attractors and circuit implementation of a new 4D chaotic system with two equilibria. *Chaos, Solitons & Fractals* **107** (2018) 92–102.

[10] E.N. Lorenz. Deterministic nonperiodic flow. *Journal of atmospheric sciences* **20** (2) (1963) 130–141.

[11] L. Meddour and K. Belakrouml. On the Equivalence of Lorenz System and Li System. *Nonlinear Dynamics and Systems Theory* **22 (1)** (2022) 58–65.

[12] T. Nestor, A. Belazi, B. Abd-El-Atty, M.N. Aslam, C. Volos, D. Dieu, J. Nkapkop and A. Abd El-Latif. A new 4D hyperchaotic system with dynamics analysis, synchronization, and application to image encryption, *Symmetry* **14** (2) (2022) 424.

[13] O. Rossler. An equation for hyperchaos. *Physics Letters A* **71** (1979) 155-157.

[14] S. Pang and Y. Liu. A new hyperchaotic system from the Lü system and its control. *Journal of Computational and Applied Mathematics* **235** (8) (2011) 2775–2789.

[15] Pham, Viet-Thanh and Volos, Christos and Vaidyanathan, Sundarapandian and Wang, Xiong and others. A chaotic system with an infinitenumber of equilibrium points: dynamics, horseshoe and synchronization. *Advances in Mathematical Physics* **2016** (2016).

[16] L.M. Pecora and T.L. Carroll. Synchronization in chaotic systems. *Physical review letters* **64** (8) (1990) 821.

[17] R. Rameshbabu, K. Kavitha, P. S. Gomathi and K. Kalaichelvi A New Hidden Attractor Hyperchaotic System and Its Circuit Implementation, Adaptive Synchronization and FPGA Implementation. *Nonlinear Dynamics and Systems Theory* **23** (2) (2023) 214—226.

[18] S. Vaidyanathan and K. Rajagopal. Chaos Synchronization between Fractional-Order Lesser Date Moth Chaotic System and Integer-Order Chaotic System via Active Control. *Nonlinear Dynamics and Systems Theory* **22** (4) (2022) 407–413.

[19] S. Vaidyanathan and K. Rajagopal. Anti-Synchronization of Tigan and Li Systems with Unknown Parameters via Adaptive Control. *An International Journal of Optimization and Control: Theories & Applications* **2** (2012) 17–28.

[20] Z. Wang, F. Min and E. Wang. A new hyperchaotic circuit with two memristors and its application in image encryption. *Aip Advances* **6** (9) (2016) 095316.

[21] H. Wang and X. Li. A novel hyperchaotic system withinfinitely many heteroclinic orbits coined. *Chaos, Solitons & Fractals* **106** (2018) 5–15.

[22] Z. Wei, R. Wang and A. Liu. A new finding of the existence of hidden hyperchaotic attractors with no equilibria. *Mathematics and Computers in Simulation* **100** (2014) 13–23.

[23] S. Zhang and T. Gao. A coding and substitution frame based on hyper-chaotic systems for secure communication. *Nonlinear Dynamics* **84** (2016) 833–849.