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Conformable Fractional Inverse Gamma Distribution

Mamon Abu Hammad, Shameseddin Alshorm *, Sondos Rasem and Lena Abed

Department of Mathematics, Al Zaytoonah University, Amman 11733, Jordan.

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Abstract: In this paper, we will study the Inverse Gamma distribution to introduce the Conformable Fractional Inverse Gamma Distribution (CFIGD) with a study of the entropy measures in the fractional case. The CFIGD's CDF, survival function, and hazard function are defined to shed light on its behavior and suggest possible uses for it in reliability and risk analysis. It is possible to better understand the central tendencies and higher-order characteristics of statistics by using the conformable fractional analogs of statistical measures like the expected values, r-th moments, mean, variance, skewness, and kurtosis. In addition, the conformable fractional analogs of well-known entropy measures like the Shannon, Renyi, and Tsallis entropy are introduced, offering useful instruments for estimating uncertainty and randomness.

Keywords: *probability distribution functions; conformable fractional; conformable derivative; entropy.*

Mathematics Subject Classification (2010): 26A33, 34A08, 34K37, 70K75, 70K99.

1 Introduction

The Conformable Fractional Inverse Gamma Distribution (CFIGD), which offers an avant-garde framework for simulating various real-world phenomena, has emerged as a promising statistical tool [1,2]. The development of sophisticated tools to model complex phenomena and derive meaningful insights is made possible by advancements in probability theory and statistical analysis [3,4]. The CFIGD stands out among these tools as a potent and ground-breaking idea, providing a new viewpoint on probability distributions and their applications. The CFIGD and its numerous applications in various fields will be thoroughly explored in this research paper.

* Corresponding author: alshormanshams@gmail.com

The foundation of the CFIGD is the inverse gamma distribution, which is well-known for its importance in Bayesian statistics. This family of distributions is the reciprocal of a variable that follows a gamma distribution and operates on positive real numbers. By extending conventional statistical measures to conformable fractional analogs, the CFIGD introduces a novel approach. Our comprehension of the distribution's central tendencies, higher-order properties, and quantification of uncertainty is greatly improved by these analogs [5, 7, 8].

The paper aims to clarify the fundamental characteristics of the CFIGD, namely the cumulative distribution function (CDF), survival function, and hazard function. In addition to these, we investigate the conformable fractional analogs of fundamental statistical quantities like the expected values, r-th moments, mean, variance, skewness, and kurtosis. The theoretical framework of the CFIGD is built on this thorough analysis.

Nonlinear dynamics plays a pivotal role in understanding complex systems, where traditional linear models fall short. In recent years, there has been a growing interest in exploring non-conventional probability distributions and statistical measures to enhance our comprehension of intricate systems and phenomena. This paper delves into the realm of nonlinear dynamics by introducing a novel statistical distribution, the Conformable Fractional Inverse Gamma Distribution (CFIGD) and studying its properties in the context of entropy measures in the fractional case.

Nonlinear systems, characterized by their sensitivity to initial conditions and intricate feedback loops, necessitate sophisticated statistical tools for accurate modeling and analysis. The Inverse Gamma distribution has proven to be invaluable in various fields, especially in reliability and risk analysis, where understanding the underlying probabilistic nature of events is paramount. The extension of this distribution to its fractional counterpart, as explored in this paper, opens new avenues for studying nonlinear systems with fractional dynamics, providing a more nuanced understanding of their behavior.

The organization of this paper is listed as follows. In the following section, we will introduce the basic definitions and remarks on conformable fractional derivatives. In Section 3, we will introduce a fractional inverse gamma distribution, this part is followed by Section 4 with the conclusion for this paper and references.

2 Conformable Fractional Derivative

Fractional calculus is a branch of mathematics that deals with fractional derivatives and integrals, and it has numerous applications in a variety of scientific and engineering disciplines. A current framework for describing non-local and memory-dependent complex processes is offered by fractional derivatives. One of the fascinating developments in this field is the idea of a "conformable fractional derivative", introduced by Khalil et al. [6] in 2014.

When applied to phenomena with non-differentiable or non-smooth behavior, integer-order derivatives, which are frequently used in classical calculus, have some limitations. Fractional derivatives, however, overcome these limitations by taking into account non-integer orders, which better capture a variety of natural and artificial systems utilizing the notion of conformable fractional derivative. This new concept provides a different perspective on fractional derivatives and their applications [9–11]. The conformable fractional derivative, while retaining the advantages of fractional calculus, offers an intriguing framework for modeling complex phenomena that exhibit conformability to prevailing physical laws [12–15].

Definition 2.1 Let α be a real nonnegative number. For a positive integer m such that $m-1 < \alpha \leq m$, the Riemann-Liouville fractional-order differential operator of a function f of order α is defined by

$$D_a^\alpha f(x) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dx^m} \int_a^x (x-t)^{m-\alpha-1} f(t)dt. \tag{1}$$

Definition 2.2 Let $\alpha \in \mathbb{R}$ and $m = \lceil \alpha \rceil$. The Caputo fractional-order derivative operator D_a^α is defined by

$$D_a^\alpha f = J_a^{m-\alpha} D^m f. \tag{2}$$

Definition 2.3 Let $\alpha \in \mathbb{R}^+$ and $m = \lceil \alpha \rceil$ such that $m-1 < \alpha \leq m$. Then the Caputo fractional-order deivative operator of order α is given by

$$D_a^\alpha f(x) = \frac{1}{\Gamma(m-\alpha)} \int_a^x (x-t)^{m-\alpha-1} f^{(m)}(t)dt, \ x > a. \tag{3}$$

It should be noted that if $a = 0$ in Equation (3), one can get the most reliable version of the Caputo fractional-order derivative operator. That is,

$$D_*^\alpha f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} f^{(m)}(t)dt, \ x > 0. \tag{4}$$

Definition 2.4 Let $\omega : [0, \infty) \to \mathbb{R}$ and $t > 0$, then given ω of order α, the conformable fractional derivative is defined as

$$D_\alpha(\omega)(t) = \lim_{\epsilon \to 0} \frac{\omega(t + \epsilon t^{1-\alpha}) - \omega(t)}{\epsilon} \tag{5}$$

for all $t > 0$ and $\alpha \in (0,1)$. If ω is α-differentiable in some $(0, a)$, $a > 0$ and $\lim_{t \to 0^+} \omega^{(\alpha)}(t)$ exists, then specify $\omega^{(\alpha)}(0) = \lim_{t \to 0^+} \omega^{(\alpha)}(t)$.

Remark 2.1 The conformable fractional derivatives of ω of order α are denoted by $\omega^{(\alpha)}(t)$ for $D_\alpha(\omega)(t)$. Furthermore, we simply state ω is α-differentiable if the conformable fractional derivative of ω of order α exists, where $D_\alpha(t^p) = pt^{p-\alpha}$.

Remark 2.2 The integral departs from the standard Riemann integral and instead involves an erroneous version. The parameter α takes values in the interval (0, 1). The conformable derivative, while maintaining all the conventional characteristics of the ordinary first derivative, is under consideration. Furthermore, the derivative gives rise to correct propositions according to the context presented.

- $D_\alpha(a\omega + b\varphi) = aD_\alpha(\omega) + bD_\alpha(\varphi)$,
- $D_\alpha(t^p) = pt^{p-\alpha}$, for all $p \in \mathbb{R}$,
- $D_\alpha(\omega\varphi) = \omega D_\alpha(\varphi) + \varphi D_\alpha(\omega)$,
- $D_\alpha(\frac{\omega}{\varphi}) = \frac{\varphi D_\alpha(\omega) - \omega D_\alpha(\varphi)}{\varphi^2}$.

3 Fractional Inverse Gamma Distribution

The Inverse Gamma distribution, denoted as $IG(\delta, \beta)$, is a two-parameter family of continuous probability distributions defined on the positive real line. The Cumulative Distribution Function (CDF) of the Inverse Gamma distribution is given by

$$G(x;\delta,\beta) = 1 - \frac{\beta^{\delta}}{\Gamma(\delta)} \int_0^x t^{\delta+1} e^{-\beta/t} dt, \tag{6}$$

where $\delta > 0$ and $\beta > 0$ are the shape and scale parameters, respectively, and $\Gamma(\delta)$ represents the Gamma function.

The Probability Density Function (PDF) of the Inverse Gamma distribution is defined as

$$g(x;\delta,\beta) = \frac{\beta^{\delta}}{\Gamma(\delta)} x^{-(\delta+1)} e^{-\beta/x}, \tag{7}$$

where $\delta > 0$ and $\beta > 0$ are the shape and scale parameters, respectively, and $\Gamma(\delta)$ represents the Gamma function.

In this section, we will introduce the CFIGD defined as follows:

$$\begin{aligned} &x^{2\alpha} D^{\alpha} y + (bx^{\alpha} - \beta + x^{\alpha}) y = 0, \\ &g_{\alpha}(x) = \frac{\alpha^2 (\frac{\beta}{\alpha})^{(b+1)/\alpha}}{\beta \Gamma(\frac{(b-\alpha-1)}{\alpha})} x^{-(b+1)} e^{\frac{-\beta}{(\alpha x^{\alpha})}}. \end{aligned} \tag{8}$$

Consequently,

$$\lim_{\alpha \to 1^-} g_{\alpha}(x) = \frac{\beta^b}{\Gamma(b)} x^{-(b+1)} e^{\frac{-\beta}{x}}. \tag{9}$$

This represents the PDF of the Inverse Gamma distribution. Consequently, the CPDF $g_{\alpha}(x)$ can be seen as an extension or generalization of the PDF for the Inverse Gamma distribution as the following Figure 1 shows, and for more about fractional distributions, see [16–21]. For the Inverse Gamma distribution, α refers to the shape parameter. The

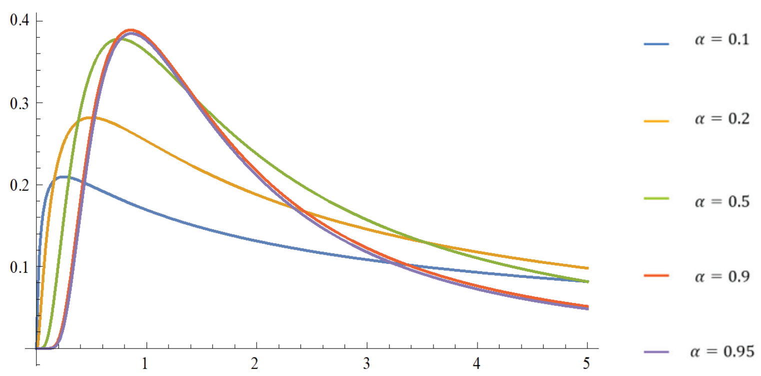

Figure 1: The CPDF of the inverse gamma distribution for different values of α.

shape parameter α is a crucial parameter that governs the shape of the distribution. It determines the skewness and tail behavior of the distribution.

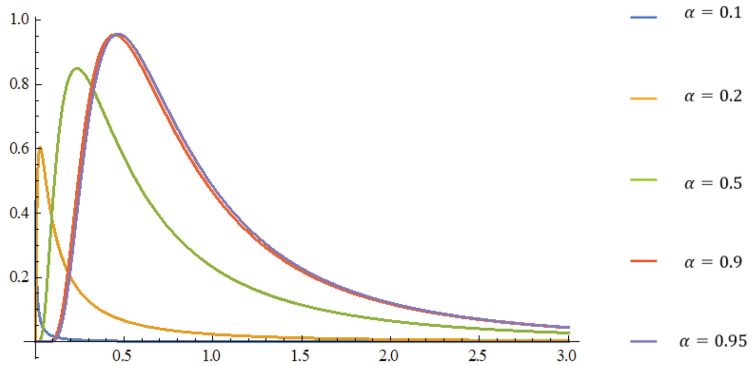

Figure 2: The CFSF of the inverse gamma distribution for different values of α.

A higher value of α results in a distribution that is more concentrated around its mean, with lighter tails. On the other hand, a lower value of α leads to a more spread-out distribution with heavier tails. Note that α must be greater than 0.

Definition 3.1 The Conformable α-Inverse Gamma Distribution (CFCDF) is defined as

$$\begin{aligned} G_\alpha(x) &= \int_0^x g_\alpha(t) d^\alpha t, \\ G_\alpha(x) &= \frac{(\frac{\alpha}{\beta})^{\frac{(2+b-\alpha)(-1+\alpha)}{\alpha}} \Gamma(2+b-\frac{1}{\alpha}-\alpha, \frac{x^{-\alpha\beta}}{\alpha})}{\Gamma(\frac{1+b-\alpha}{\alpha})}. \end{aligned} \tag{10}$$

Definition 3.2 The Conformable Fractional Survival Function (CFSF) of X, (S_α) is defined as

$$\begin{aligned} S_\alpha(x) &= 1 - G_\alpha(X) \\ S_\alpha(x) &= 1 - \frac{(\frac{\alpha}{\beta})^{\frac{(2+b-\alpha)(-1+\alpha)}{\alpha}} \Gamma(2+b-\frac{1}{\alpha}-\alpha, \frac{x^{-\alpha\beta}}{\alpha})}{\Gamma(\frac{1+b-\alpha}{\alpha})}. \end{aligned} \tag{11}$$

Definition 3.3 The Conformable Fractional Hazard Function (CFHF) X, (H_α) is defined as

$$\begin{aligned} H_\alpha &= \frac{S_\alpha(x)}{g_\alpha(x)}, \\ H_\alpha(x) &= \frac{e^{\frac{-x^{-\alpha\beta}}{\alpha}} x^{-1-b} \alpha^{\frac{-1-b+2\alpha}{\alpha}} \beta^{\frac{1+b-2\alpha}{\alpha}}}{\Gamma(\frac{1+b-\alpha}{\alpha}) - (\frac{\alpha}{\beta})^{\frac{(2+b-\alpha)(-1+\alpha)}{\alpha}} \Gamma(2+b-\frac{1}{\alpha}-\alpha, \frac{x^{-\alpha\beta}}{\alpha})}. \end{aligned} \tag{12}$$

Definition 3.4 The conformable fractional expectation E_α for the function $v(x)$ is defined as follows:

$$\begin{aligned} E_\alpha v(X) &= \int v(x) g_\alpha(x) d^\alpha x, \\ E_\alpha(X^r) &= \frac{\alpha^{\frac{-r}{\alpha}} \beta^{\frac{r}{\alpha}} \Gamma(\frac{1+b-r-\alpha}{\alpha})}{\Gamma(\frac{1+b-\alpha}{\alpha})}. \end{aligned} \tag{13}$$

To find the conformable fractional variance, conformable fractional standard deviation, conformable fractional skewness, and conformable fractional kurtosis, we will find the $E_\alpha(X^r)$ for $r = 1, 2, 3, 4$ as follows:

$$\begin{aligned} E_\alpha(X) = \mu_\alpha &= \frac{(\frac{\alpha}{\beta})^{\frac{-1}{\alpha}}\Gamma(-1+\frac{b}{\alpha})}{\Gamma(\frac{1+b-r-\alpha}{\alpha})}, \\ E_\alpha(X^2) &= \frac{(\frac{\alpha}{\beta})^{\frac{-2}{\alpha}}\Gamma(\frac{-1+b-\alpha}{\alpha})}{\Gamma(\frac{1+b-\alpha}{\alpha})}, \\ E_\alpha(X^3) &= \frac{(\frac{\alpha}{\beta})^{1-\frac{1+b}{\alpha}}\beta(\frac{\beta}{\alpha})^{\frac{2-b}{\alpha}}\Gamma(-1+\frac{-2+b}{\alpha})}{\alpha\Gamma(-1+\frac{1+b}{\alpha})}, \\ E_\alpha(X^4) &= \frac{(\frac{\alpha}{\beta})^{1-\frac{1+b}{\alpha}}\beta(\frac{\beta}{\alpha})^{\frac{3-b}{\alpha}}\Gamma(-1+\frac{-3+b}{\alpha})}{\alpha\Gamma(-1+\frac{1+b}{\alpha})}. \end{aligned} \tag{14}$$

Definition 3.5 The conformable fractional variance is defined as follows:

$$\begin{aligned} \alpha\sigma^2 &= E_\alpha(X^2) - (\mu_\alpha)^2, \\ \alpha\sigma^2 &= \frac{\alpha^{\frac{-2}{\alpha}}\beta^{\frac{2}{\alpha}}\Big(-\Gamma(-1+\frac{b}{\alpha})^2 + \Gamma(\frac{-1+b-\alpha}{\alpha})\Gamma(\frac{1+b-\alpha}{\alpha})\Big)}{\Gamma(\frac{1+b-\alpha}{\alpha})^2}. \end{aligned} \tag{15}$$

Definition 3.6 The conformable fractional standard deviation is defined as follows:

$$\alpha\sigma = \sqrt{\frac{\alpha^{\frac{-2}{\alpha}}\beta^{\frac{2}{\alpha}}\Big(-\Gamma(-1+\frac{b}{\alpha})^2 + \Gamma(\frac{-1+b-\alpha}{\alpha})\Gamma(\frac{1+b-\alpha}{\alpha})\Big)}{\Gamma(\frac{1+b-\alpha}{\alpha})^2}}. \tag{16}$$

Definition 3.7 The conformable fractional skewness is defined as follows:

$$\alpha sk = \frac{E_\alpha(X-\mu)^3}{\alpha\sigma^3}, \tag{17}$$

and the conformable fractional skewness for CFIGD is defined as

$$sk = \frac{\Big(2\Gamma(-1+\frac{b}{\alpha})^3 - 3\Gamma(-1+\frac{b}{\alpha})\Gamma(\frac{-1+b-\alpha}{\alpha})\Gamma(\frac{1+b-\alpha}{\alpha}) + \Gamma(\frac{-2+b-\alpha}{\alpha})\Gamma(\frac{1+b-\alpha}{\alpha})^2\Big)}{\Big(-\Gamma(-1+\frac{b}{\alpha})^2 + \Gamma(\frac{-1+b-\alpha}{\alpha})\Gamma(\frac{1+b-\alpha}{\alpha})\Big)^{\frac{3}{2}}}. \tag{18}$$

Definition 3.8 The conformable fractional kurtosis is defined as follows:

$$\alpha ku = \frac{E_\alpha(X-\mu)^4}{\alpha\sigma^4}, \tag{19}$$

and the conformable fractional kurtosis for CFIGD is defined as

$$\begin{aligned} \alpha ku = &\frac{-3\Gamma(-1+\frac{b}{\alpha})^4 + 6\Gamma(-1+\frac{b}{\alpha})^2\Gamma(\frac{-1+b-\alpha}{\alpha})\Gamma(\frac{1+b-\alpha}{\alpha})}{\Big(\Gamma(-1+\frac{b}{\alpha})^2 - \Gamma(\frac{-1+b-\alpha}{\alpha})\Gamma(\frac{1+b-\alpha}{\alpha})\Big)^2} \\ &- \frac{4\Gamma(-1+\frac{b}{\alpha})\Gamma(\frac{-2+b-\alpha}{\alpha})\Gamma(\frac{1+b-\alpha}{\alpha})^2 + \Gamma(\frac{-3+b-\alpha}{\alpha})\Gamma(\frac{1+b-\alpha}{\alpha})^3}{\Big(\Gamma(-1+\frac{b}{\alpha})^2 - \Gamma(\frac{-1+b-\alpha}{\alpha})\Gamma(\frac{1+b-\alpha}{\alpha})\Big)^2}. \end{aligned} \tag{20}$$

Definition 3.9 The conformable fractional Shannon entropy of a random variable x for the CFIGD is defined as follows:

$$SH_\alpha(x) = -E_\alpha \log g_\alpha(X),$$
$$SH_\alpha(x) = \frac{1+b-\alpha-2\alpha\log(\alpha)+\alpha\log(\beta)+\alpha\log(\Gamma(\frac{1+b-\alpha}{\alpha}))-(1+b)\Psi(0,\frac{1+b-\alpha}{\alpha})}{\alpha}. \tag{21}$$

Definition 3.10 The conformable fractional Tsallis entropy of a random variable x for the CFIGD is defined as follows:

$$SH_{T,\xi}(x) = \frac{1}{1-\xi}\log(E_\alpha(g_\alpha(X))^{\xi-1}-1),$$
$$SH_{T,\xi}(x) = \frac{-1+\alpha^{-2+2\xi}\beta^{1-\xi}\xi^{1-\frac{(1+b)\xi}{\alpha}}\Gamma(\frac{1+b-\alpha}{\alpha})^{-\xi}\Gamma(\frac{-\alpha+\xi+b\xi}{\alpha})}{1-\xi}. \tag{22}$$

In Table 1, the quantiles of the distribution are classified by α in the rows and q in the columns, the parameters of the distribution are $\beta = 0.85$ and $b = 2.51$.

$\alpha \setminus q$	25	50	75
0.1	0.037	0.181	0.298
0.2	0.144	0.393	0.534
0.3	0.363	0.646	0.771
0.4	0.767	0.952	1.023
0.5	1.5	1.333	1.302
0.6	2.85	1.823	1.624
0.7	5.5	2.491	2.019
0.8	11.456	3.49	2.546
0.9	29.883	5.34	3.394

Table 1: Quantiles of the distribution.

The percentiles of the distribution are classified by α in the rows and q in the columns, the parameters of the distribution are $\beta = 0.70$ and $b = 1.51$.

Corollary 3.1 *Hence* $\lim_{\xi\to 1} SH_{T,\alpha,\xi}(x) = SH_\alpha(x)$. *We get that the limit of the conformable fractional Tsallis entropy is equal to the conformable fractional Shannon entropy.*

Definition 3.11 The conformable fractional Renyi entropy of a random variable x for the CFIGD is defined as follows:

$$SH_{R,\alpha,\xi} = \frac{1}{1-\xi}\log(E_\alpha(g_\alpha(X))^{\xi-1}),$$
$$SH_{R,\alpha,\xi} = \frac{1}{\alpha(-1+\xi)}\Big(-2\alpha(-1+\xi)\log(\alpha)+\alpha(-1+\xi)\log(\beta)-\alpha\log(\xi)$$
$$+\xi\log(\xi)+b\xi\log(\xi)+\alpha\xi\log(\Gamma(-1+\frac{1+b}{\alpha}))-\alpha\log(\Gamma(-1+\frac{(1+b)\xi}{\alpha}))\Big). \tag{23}$$

Corollary 3.2 *Hence* $\lim_{\xi\to 1} SH_{R,\alpha,\xi} = SH_\alpha(x)$. *We get that the limit of the conformable fractional Renyi entropy is equal to the conformable fractional Shannon entropy.*

$\alpha \setminus q$	10	20	30	40	50	60	70	80	90
0.1	0.001	0.006	0.023	0.05	0.084	0.12	0.158	0.196	0.233
0.2	0.002	0.022	0.069	0.128	0.19	0.249	0.306	0.357	0.404
0.3	0.004	0.053	0.138	0.23	0.315	0.391	0.458	0.516	0.567
0.4	0.01	0.104	0.236	0.36	0.464	0.551	0.622	0.681	0.73
0.5	0.023	0.187	0.374	0.526	0.644	0.734	0.804	0.859	0.903
0.6	0.052	0.322	0.57	0.746	0.868	0.953	1.014	1.059	1.093
0.7	0.12	0.556	0.866	1.049	1.159	1.227	1.269	1.296	1.313
0.8	0.301	1.004	1.353	1.508	1.575	1.601	1.606	1.601	1.59
0.9	0.982	2.112	2.359	2.359	2.294	2.215	2.139	2.068	2.004

Table 2: Percentiles of the distribution.

4 Conclusion

The paper defines the Cumulative Distribution Function (CDF), survival function, and hazard function of the CFIGD, offering valuable insights into its behavior and potential applications in reliability and risk analysis. Statistical measures such as the expected values, r-th moments, mean, variance, skewness, and kurtosis are introduced as conformable fractional analogs, facilitating a deeper understanding of its central tendencies and higher-order characteristics. Additionally, this research reveals the conformable fractional analogs of popular entropy measures like the Shannon, Renyi, and Tsallis entropy, providing useful tools for quantifying uncertainty and randomness.

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