Nonlinear Dynamics and Systems Theory, **24** (2) (2024) 205–216

Analysis and Numerical Approximation of the Variable-Order Time-Fractional Equation

Abdelwahab Zarour * and Mohamed Dalah

University Frères Mentouri Constantine 1,
Faculty of Exact Sciences, Department of Mathematics,
P.O. Box, 325 Ain El Bey Way, Constantine 25017, Algeria,
MAM and Differential Equations Laboratory.

Received: October 1, 2023; Revised: March 3, 2024

Abstract: In this paper, we investigate a fully implicit finite scheme approximation equation (IFSAE) of the 1-D linear variable-order time-fractional diffusion equation (VOTFDE). The numerical method of solving differential equations by approximating them with difference equations is called the implicit finite difference method (IFDM). The first-order numerical scheme, stability, consistency and convergence of the method are proven. Moreover, the scheme is implemented on two test problems and some graphical results are offered to verify the theoretical analysis of the above scheme and illustrate the effectiveness of the suggested schemes.

Keywords: *fractional derivatives; discretization; implicit numerical scheme; stability; convergence.*

Mathematics Subject Classification (2010): 70K75, 93A30, 34K37, 65N06.

1 Introduction

Applied mathematics is the application of mathematical methods in various fields such as physics, engineering, medicine, biology, finance, economics, computer science and industry. Thus, applied mathematics is a combination of mathematics and engineering. Operational calculus, also called operational analysis, is a technique used to transform analytical problems, especially differential equations, into algebraic problems, usually the problem of solving a polynomial equation. Numerical analysis is the study of algorithms that use numerical approximation (as opposed to symbolic manipulations) to the problems of mathematical analysis (as distinct from discrete mathematics). This is the study

* Corresponding author: mailto:zarour.abdelwahab@umc.edu.dz

of numerical methods that attempt to find approximate solutions rather than exact solutions to problems. Numerical analysis is used in all areas of engineering and natural sciences and, in the 21st century, also in life and social sciences, medicine, economics and even in the arts.

Nonlinear dynamics is a hot topic in physics, complexity science, and theoretical biology. Nonlinear dynamics offers new ways of studying the numerical solution of the variable-order time-fractional reaction-diffusion equation (VOTFRDE), namely implicit and explicit finite difference methods. Motivated by the above facts and the literature [3–8], namely in this work, we consider the numerical solution of the variable-order time-fractional reaction-diffusion equation (VOTFRDE) in one-dimensional space, where the parameter alpha is a function depending on x and t. A numerical scheme was given in [5]– [12] for approximating the variable-order time-fractional reaction-diffusion equation (VOTFRDE). An Implicit Finite Difference Scheme method was applied for the variable-order time-fractional reaction-diffusion equation (VOTFRDE) with the Coimbra derivative. Moreover, the scheme is implemented on two test problems and some graphical results are offered to verify the theoretical analysis of the above scheme and illustrate the effectiveness of the suggested schemes.

The paper is organized in a clear and comprehensive manner, as follows. The construction of our mathematical model with boundary conditions is presented in Section 2. Section 3 develops the implicit finite difference scheme, which utilizes forward finite difference approximations for space derivatives and Caputo's concept for time-fractional derivatives. In Section 4, we study the stability of the approximate scheme by using the method of Fourier. Next, in Section 5, we prove the convergence of the approximate scheme obtained in Section 3. The final Section 6 completes and summarizes several numerical problems addressed using the method developed in Section 2. The numerical solutions are obtained using MATLAB and graphically visualized to provide a clear understanding of the results.

2 Implicit Finite Difference Scheme

For the numerical solution of the space fractional diffusion equations, implicit and explicit finite difference methods have been proposed in the literature [7], [8], [11], [16], [18] and [20] for the numerical solution of the time-fractional diffusion equation. We augment the implicit numerical scheme in this section. Let us take a variable-order time-fractional diffusion equation as an example

$$\begin{cases} \frac{\partial^\beta u}{\partial t^\beta} + c\frac{\partial u}{\partial x} = 0 \;\; 0 < x < L, 0 < t < T, 0 < \beta < 1, \\ u(0,t) = u(L,t) = 0, \\ u(x,0) = u_0(x). \end{cases} \tag{1}$$

3 Discretization and Development of the Scheme

Let $[0, L]$ be the clomain of inerest, we discretise the domain first.We define

$$x_i = ih, \text{ where } i = \overline{0, M}, \text{ and } t_j = jk, \text{ where } j = \overline{0, N},$$

where k represents the time step size and h represents the space step length.

Let us assume that

$$u(x_i, t_j) = u_i^j,$$

note that u_i^j is the numerical approximation of $u(x_i, t_j)$. Next, we consider the fractional-order diffusion Equation (1), where β is the fractional order. The variable-order fractional derivative of order $\beta(x,t)$ is defined by the Coimbra derivative and is written as

$$\frac{\partial^{\beta(x,t)}u(x,t)}{\partial t^{\beta(x,t)}} = \begin{cases} \frac{1}{\Gamma(1-\beta(x,t))}\int_0^t \frac{u_\xi(x,\xi)}{(t-\xi)^{\beta(x,t)}}d\xi \ \text{ if } \ 0<\beta<1, \\ u_t(x,t) \ \text{ if } \ \beta(x,t)=1. \end{cases} \tag{2}$$

Initially, as the boundary value problem needs to be discretized to be able to solve (1), it is first necessary to discretize the variable-order time-fractional derivative (2) as follows:

$$\begin{aligned} \frac{\partial^{\beta(x_i,t_{j+1})}u(x_i,t_{j+1})}{\partial t^{\beta(x_i,t_{j+1})}} &= \frac{1}{\Gamma(1-\beta(x_i,t_{j+1}))}\int_0^{t_{j+1}} \frac{u_\xi(x_i,\xi)}{(t_{j+1}-\xi)^{\beta(x_i,t_{j+1})}}d\xi \\ &= \frac{1}{\Gamma(1-\beta(x_i,t_{j+1}))}\sum_{s=0}^{j}\int_{sk}^{(s+1)k} \frac{u_\xi(x_i,\xi)}{(t_{j+1}-\xi)^{\beta(x_i,t_{j+1})}}d\xi, \end{aligned}$$

then we obtain

$$\frac{\partial^{\beta(x_i,t_{j+1})}u(x_i,t_{j+1})}{\partial t^{\beta(x_i,t_{j+1})}} = \frac{1}{\Gamma(1-\beta(x_i,t_{j+1}))}\sum_{s=0}^{j}\int_{sk}^{(s+1)k} \left(\frac{\partial u}{\partial \xi}\right)_i^{s+1} \frac{d\xi}{(t_{j+1}-\xi)^{\beta(x_i,t_{j+1})}}.$$

The first-order spatial derivative can be approximated by the following expression:

$$\left(\frac{\partial u}{\partial \xi}\right)_i^{s+1} = \frac{u_i^{s+1}-u_i^s}{k} + \Delta(k). \tag{3}$$

Adopting the discrete scheme given in (7), we discretize the variable-order time-fractional derivative as

$$\begin{aligned} \frac{\partial^{\beta(x_i,t_{j+1})}u(x_i,t_{j+1})}{\partial t^{\beta(x_i,t_{j+1})}} &= \frac{1}{\Gamma(1-\beta(x_i,t_{j+1}))}\sum_{s=0}^{j}\frac{u_i^{s+1}-u_i^s}{k}\int_{(j-s)k}^{(j-s+1)k}\frac{dy}{y^{\beta(x_i,t_{j+1})}} \\ &= \frac{1}{\Gamma(1-\beta(x_i,t_{j+1}))}\sum_{n=0}^{j}\frac{u_i^{j-n+1}-u_i^{j-n}}{k}\int_{nk}^{(n+1)k}\frac{dy}{y^{\beta(x_i,t_{j+1})}} \\ &= \frac{k^{-\beta(x_i,t_{j+1})}}{(1-\beta(x_i,t_{j+1}))\Gamma(1-\beta(x_i,t_{j+1}))}\sum_{n=0}^{j}(u_i^{j-n+1}-u_i^{j-n})[(n+1)^{1-\beta(x_i,t_{j+1})}-n^{1-\beta(x_i,t_{j+1})}]. \end{aligned}$$

Let $y = t_{j+1}-\xi$. We have $\Gamma(1+\beta)=\beta\Gamma(\beta)$, and expanding the summation for $n=0$, we reach

$$\begin{aligned} \frac{\partial^{\beta(x_i,t_{j+1})}u(x_i,t_{j+1})}{\partial t^{\beta(x_i,t_{j+1})}} = \frac{k^{-\beta(x_i,t_{j+1})}}{\Gamma(2-\beta(x_i,t_{j+1}))}\Bigg[\, & u_i^{j+1}-u_i^j+\sum_{n=1}^{j}(u_i^{j-n+1}-u_i^{j-n}) \\ & [(n+1)^{1-\beta(x_i,t_{j+1})}-n^{1-\beta(x_i,t_{j+1})}]\Bigg], \end{aligned} \tag{4}$$

where $b_i^{j+1}(n)=(n+1)^{1-\beta(x_i,t_{j+1})}-n^{1-\beta(x_i,t_{j+1})}, \ \forall j=\overline{0,N-1}$.

We will use the forward difference approximation of space derivative as follows:

$$\frac{\partial u(x_i,t_{j+1})}{\partial x} = \frac{u_{i+1}^{j+1} - u_i^{j+1}}{h} + \Delta(h). \tag{5}$$

Using approximations (3) and (5), the semi-linear diffusion equation (1), we obtain

$$\frac{k^{-\beta(x_i,t_{j+1})}}{\Gamma(2-\beta(x_i,t_{j+1}))}\left[u_i^{j+1} - u_i^j + \sum_{n=1}^{j}(u_i^{j-n+1} - u_i^{j-n})b_i^{j+1}(n)]\right] + c\frac{u_{i+1}^{j+1} - u_i^{j+1}}{h} = 0, \tag{6}$$

$\forall i = \overline{1,M}$ and $\forall j = \overline{1,N}$, where

$$a_i^{j+1} = \frac{c.k^{\beta(x_i,t_{j+1})}.\Gamma(2-\beta(x_i,t_{j+1}))}{h}.$$

The initial and boundary conditions are

$$u_0(x_i) = u_i^0, \ \ \forall i = \overline{0,M},$$

$$u_0^{j+1} = u_M^{j+1} = 0, \ \ \forall j = \overline{0,N-1}.$$

We obtain the following approximate scheme for equation (1):

$$\begin{cases} a_i^{j+1}u_{i+1}^{j+1} + (1-a_i^{j+1})u_i^{j+1} = u_i^j - \sum_{n=1}^{j}(u_{i+1}^{j-n+1} - u_i^{j-n})b_i^{j+1}(n), \\ \qquad \forall i = \overline{1,M-1}, \ \ \forall j = \overline{1,N-1}, \\ u_0^{j+1} = u_M^{j+1} = 0, \ \ \forall j = \overline{0,N-1}, \\ u_i^0 = u_0(x_i), \ \ \forall i = \overline{0,M}. \end{cases} \tag{7}$$

3.1 Properties

The coefficients $b_i^{j+1}(n)$ for all $i = \overline{0,M}$ and $j = \overline{0,N-1}$, satisfy

- **P1:** $b_i^{j+1}(0) = 1$.
- **P2:** $0 < b_i^{j+1}(0) < 1$.

4 Stability of the Approximate Scheme

In this section, we use the method of the Fourier analysis to discuss the stability of the approximate scheme (7). Consider the following equation:

$$a_i^{j+1}u_{i+1}^{j+1} + (1-a_i^{j+1})u_i^{j+1} = u_i^j - \sum_{n=1}^{j}(u_{i+1}^{j-n+1} - u_i^{j-n})b_i^{j+1}(n), \tag{8}$$

$$\forall i = \overline{1,M-1}, \ \ \forall j = \overline{1,N-1}.$$

Now, we define the following function:

$$\begin{cases} u^j(x) = u_i^j, \ \text{ if } \ x_{i-\frac{1}{2}} < x < x_{i+\frac{1}{2}}, \ \ \forall i = \overline{1,M-1}, \\ 0, \ \ \text{otherwise}, \end{cases}$$

where $u^j(x)$ has the Fourier series expansion

$$u^j(x) = \sum_{m=-\infty}^{+\infty} \xi_j(m) e^{\frac{2\pi m}{L} x}, \ \ \forall j = \overline{1, N-1},$$

where

$$\xi_j(m) = \frac{1}{L} \int_0^L u^j(x) e^{-\frac{2\pi m}{L} x} dx.$$

After that, using Parseval's theorem, we get

$$\int_0^L |u^j(x)|^2 dx = \sum_{m=-\infty}^{+\infty} |\xi_j(m)|^2, \ \ \forall j = \overline{1, N-1}.$$

Then, we obtain the following expression:

$$\int_0^L |u^j(x)|^2 dx = \sum_{i=1}^{M-1} |h u_i^j|^2, \ \ \forall j = \overline{0, N},$$

and

$$\|u^j\|_2^2 = \sum_{i=1}^{M-1} |h u_i^j|^2 = \sum_{m=-\infty}^{+\infty} |\xi_j(m)|^2, \ \ \forall j = \overline{0, N}.$$

Now, assume that the solution of the equation (8) has the form

$$u_i^j = \xi_j e^{\nu \tau h i}, \tag{9}$$

where $\tau = \frac{2\pi m}{L}$ and $\nu^2 = -1$. Next, we replace (9) in equation (8), then we have

$$\xi_{j+1}(1 + a_i^{j+1}(e^{\nu \tau h} - 1)) = \xi_j - \sum_{n=1}^{j} (\xi_{j-n+1} - \xi_{j-n}) b_i^{j+1}(n). \tag{10}$$

Equation (10) can be rewritten as

$$\xi_{j+1} = \frac{\xi_j - \sum_{n=1}^{j} (\xi_{j-n+1} - \xi_{j-n}) b_i^{j+1}(n)}{(1 + a_i^{j+1}(e^{\nu \tau h} - 1))}, \ \ \forall j = \overline{0, N-1}. \tag{11}$$

We have the following first result.

Theorem 4.1 *The implicit finite difference scheme* (7) *is unconditionally stable for* $0 < \beta < 1$ *if*

$$\exists C > 0, \ \ \|u^j\|_2 = |\xi_j| \leq C \|u^0\|_2 = C|\xi_0|, \ \ \forall j = \overline{0, N-1}.$$

Proof. We use the proof by recurrence for $j = 1$, in view of (11), we obtain the following majoration:

$$\begin{aligned} |\xi_1| &= \left| \frac{\xi_0}{(1 + a_i^{j+1}(e^{\nu \tau h} - 1))} \right| = \frac{|\xi_0|}{\sqrt{[1 + a_i^1(cos(\tau h) - 1)]^2 + (a_i^1 \sin(\tau h))^2}} \\ &= \frac{|\xi_0|}{|1 - 2a_i^1 \sin(\frac{\tau h}{2})|} \leq C|\xi_0|. \end{aligned}$$

Now, let C^0 be given by

$$C^0 = C = \frac{1}{|1 - 2a^1 \sin(\frac{\tau h}{2})|} \text{ such that } a^1 = \min_{0\leq i\leq M}(-a_i^1).$$

We assume that the statement defined in (12) is true,

$$|\xi_j| \leq C|\xi_0|, \ \ \forall j = \overline{1, N}, \tag{12}$$

and we prove that the statement defined by (13) is true,

$$|\xi_{j+1}| \leq C|\xi_0|, \ \ \forall j = \overline{1, N}. \tag{13}$$

$$\begin{aligned}
|\xi_{j+1}| &= \left| \frac{\xi_j - \sum_{n=1}^{j}(\xi_{j-n+1} - \xi_{j-n})b_i^{j+1}(n)}{(1 + a_i^{j+1}(e^{\nu\tau h} - 1))} \right| \\
&= \frac{|\xi_j - \sum_{n=1}^{j}(\xi_{j-n+1} - \xi_{j-n})b_i^{j+1}(n)|}{\sqrt{[1 + a_i^{j+1}(cos(\tau h) - 1)]^2 + (a_i^{j+1}\sin(\tau h))^2}} \\
&\leq \frac{|\xi_j| + \sum_{n=1}^{j}|\xi_{j-n+1} - \xi_{j-n}||b_i^{j+1}(n)|}{\sqrt{[1 + a_i^{j+1}(cos(\tau h) - 1)]^2 + (a_i^{j+1}\sin(\tau h))^2}} \\
&= \frac{|\xi_j| + \sum_{n=1}^{j}(|\xi_{j-n+1}| + |\xi_{j-n}|)}{\sqrt{[1 + a_i^{j+1}(cos(\tau h) - 1)]^2 + (a_i^{j+1}\sin(\tau h))^2}} \\
&= \frac{2N - 1}{|1 - 2a_i^{j+1}\sin(\frac{\tau h}{2})|}|\xi_0| \\
&\leq (2N - 1)C^j|\xi_0| \leq [(2N - 1)\max_{0\leq j\leq N-1} C^j]|\xi_0| \\
&= C|\xi_0|,
\end{aligned}$$

where

$$C^j = \frac{1}{|1 + 2a^{j+1}\sin(\frac{\tau h}{2})|} \text{ such that } a^{j+1} = \min_{0\leq i\leq M}(-a_i^{j+1}), \ \ \forall j = \overline{0, N-1},$$

then we obtain $C = (2N - 1)\max_{0\leq j\leq N-1} C^j$. Finally, we get

$$|\xi_{j+1}| \leq C|\xi_0|, \ \ \forall j = \overline{0, N-1},$$

and the approximate scheme (7) is unconditionally stable, which concludes the proof of Theorem 4.1.

5 Convergence of the Approximate Scheme

In this section, we use the method of the Fourier analysis to discuss the convergence of the error e_i^j, which is given by

$$e_i^j = u(x_i, t_j) - u_i^j. \tag{14}$$

Replacing (14) in equation (8), we obtain

$$a_i^{j+1}e_{i+1}^{j+1} + (1 - a_i^{j+1})e_i^{j+1} = e_i^j - \sum_{n=1}^{j}(e_{i+1}^{j-n+1} - e_i^{j-n})b_i^{j+1}(n) + r_i^j \tag{15}$$

$\forall i = \overline{1, M-1}$ and $\forall j = \overline{1, N-1}$. Then we obtain

$$r_i^j = k^{\beta(x_i,t_{j+1})}\Gamma(2 - \beta(x_i, t_{j+1}))[\Delta(k) + \Delta(h)].$$

Next, we define the following grid functions as follows:

$$e^j(x) = \begin{cases} e_i^j, & \text{if } x_{i-\frac{1}{2}} < x < x_{i+\frac{1}{2}}, \ \forall i = \overline{1, M-1}, \\ 0, & \text{otherwise}, \end{cases}$$

and

$$r^j(x) = \begin{cases} r_i^j, & \text{if } x_{i-\frac{1}{2}} < x < x_{i+\frac{1}{2}}, \ \forall i = \overline{1, M-1}, \\ 0, & \text{otherwise}. \end{cases}$$

Then $e^n(x)$ and $r^n(x)$ have the Fourier series expansions as follows:

$$e^j(x) = \sum_{m=-\infty}^{+\infty}\gamma_j(m)e^{\frac{2\pi m}{L}x}, \quad r^j(x) = \sum_{m=-\infty}^{+\infty}\lambda_j(mp)e^{\frac{2\pi m}{L}x}, \ \forall j = \overline{0, N}, \tag{16}$$

and

$$\gamma_j(m) = \frac{1}{L}\int_0^L e^j(x)e^{-\frac{2\pi m}{L}x}dx,, \quad \lambda_j(m) = \frac{1}{L}\int_0^L r^j(x)e^{-\frac{2\pi m}{L}x}dx. \tag{17}$$

After that, using Parseval's thorem, we obtain

$$\int_0^L |e^j(x)|^2dx = \sum_{m=-\infty}^{+\infty}|\gamma_j(m)|^2, \quad \int_0^L |r^j(x)|^2dx = \sum_{m=-\infty}^{+\infty}|\gamma_j(m)|^2, \ \forall j = \overline{0, N},$$

then the errors e^j and r^j take the following form:

$$\|e^j\|_2^2 = \sum_{i=1}^{M-1}|he_i^j|^2 = \sum_{m=-\infty}^{+\infty}|\gamma_j(m)|^2, \ \forall j = \overline{0, N}, \tag{18}$$

and

$$\|r^j\|_2^2 = \sum_{i=1}^{M-1}|hr_i^j|^2 = \sum_{m=-\infty}^{+\infty}|\lambda_j(m)|^2, \ \forall j = \overline{0, N}. \tag{19}$$

Next, we suppose that

$$e_i^j = \gamma_j e^{\nu\tau hi}\ , r_i^j = \lambda_j e^{\nu\tau hi}, \tag{20}$$

we replace (20) in equation (15), then we get

$$\gamma_{j+1}(1 + a_i^{j+1}(e^{\nu\tau h} - 1)) = \gamma_j - \sum_{n=1}^{j}(\gamma_{j-n+1} - \gamma_{j-n})b_i^{j+1}(n) + \lambda_j,$$

then

$$\gamma_{j+1} = \frac{\gamma_j - \sum_{n=1}^{j}(\gamma_{j-n+1} - \gamma_{j-n})b_i^{j+1}(n) + \lambda_j}{(1 + a_i^{j+1}(e^{\nu\tau h} - 1))}, \ \forall j = \overline{0, N-1}. \tag{21}$$

We get the following result.

Theorem 5.1 *The implicit finite difference scheme* (7) *is convergent for* $0 < \beta < 1$ *if*

$$\|e^j\|_2 = |\gamma_j| \leq C_1(k+h), \ \ \forall j = \overline{1,N}.$$

Proof. We use the proof by recurrence for $j = 1$, we have

$$|\gamma_1| = \left| \frac{\gamma_0 + \lambda_0}{(1 + a_i^1(e^{\nu\tau h} - 1))} \right| \leq \frac{|\gamma_0| + |\lambda_0|}{|1 - 2a_i^1 \sin(\frac{\tau h}{2})|} \leq \frac{|\lambda_0|}{|1 - 2a_i^1 \sin(\frac{\tau h}{2})|},$$

where $\gamma_0 = e_i^0 = u(x_i, 0) - u_i^0 = 0$.

By the convergence of the series on the right-hand side of (19), there is a positive constant C_2 such that

$$\exists C_2 > 0, \ \ |r_i^0| \leq C_2(k+h), \ \ \forall i = \overline{0,M},$$

then we obtain

$$\exists C_2 > 0, \ \ \|r^0\|_2 = |\lambda_0| \leq C_2\sqrt{L}(k+h),$$

so

$$|\gamma_1| \leq \frac{C_2\sqrt{L}(k+h)}{|1 - 2a_i^1 \sin(\frac{\tau h}{2})|} \leq \frac{C_2\sqrt{L}(k+h)}{|1 - 2a^1 \sin(\frac{\tau h}{2})|} = C_1(k+h)$$

such that

$$a^1 = \min_{0 \leq i \leq M}(-a_i^1),$$

and

$$C^0 = \frac{1}{|1 - 2a^1 \sin(\frac{\tau h}{2})|} \ \text{ such that } \ C_1 = C^0 C_2 \sqrt{L},$$

then we obtain

$$|\gamma_1| \leq C_1(k+h).$$

We assume that the following statement is true:

$$\|e^j\|_2 = |\gamma_j| \leq C_1(k+h), \ \ j = \overline{1,N}, \tag{22}$$

and we prove that the following statement is true:

$$\|e^{j+1}\|_2 = |\gamma_{j+1}| \leq C_1(k+h), \ \ j = \overline{0,N-1}. \tag{23}$$

One can see that $|\gamma_{j+1}|$ satisfies

$$\begin{aligned}
|\gamma_{j+1}| &= \left| \frac{\gamma_j - \sum_{n=1}^{j}(\gamma_{j-n+1} - \gamma_{j-n})b_i^{j+1}(n) + \lambda_j}{(1 + a_i^{j+1}(e^{\nu\tau h} - 1))} \right| \\
&= \left| \frac{\gamma_j - \sum_{n=1}^{j}(\gamma_{j-n+1} - \gamma_{j-n})b_i^{j+1}(n) + \lambda_j}{1 - 2a_i^{j+1}\sin(\frac{\tau h}{2})} \right| \\
&\leq \frac{|\gamma_j| + \sum_{n=1}^{j}(|\gamma_{j-n+1}| + |\gamma_{j-n}|)|b_i^{j+1}(n)| + |\lambda_j|}{|1 - 2a_i^{j+1}\sin(\frac{\tau h}{2})|} \\
&\leq \frac{2N-1}{|1 - 2a_i^{j+1}\sin(\frac{\tau h}{2})|}C_1(k+h) + \frac{|\lambda_j|}{|1 - 2a_i^{j+1}\sin(\frac{\tau h}{2})|}.
\end{aligned}$$

By the convergence of the series on the right-hand side of (19), there is a positive constant C_2 such that

$$\exists C_2 > 0,\ \ |r_i^j| \le C_2(k+h),\ \ i = \overline{0,M},\ \ \forall j = \overline{0,N}.$$

Thereafter, we obtain

$$\exists C_2 > 0,\ \ \|r^j\|_2 = |\lambda_j| \le C_2\sqrt{L}(k+h),\ \ \forall j = \overline{0,N},$$

then $|\gamma_{j+1}|$ becomes as follows:

$$\begin{aligned}|\gamma_{j+1}| &\le \frac{2N-1}{|1-2a_i^{j+1}\sin(\frac{\tau h}{2})|}C_1(k+h) + \frac{1}{|1-2a_i^{j+1}\sin(\frac{\tau h}{2})|}C_2\sqrt{L}(k+h)\\ &\le CC_1(k+h) + \frac{CC_2\sqrt{L}}{2N-1}(k+h)\end{aligned}$$

such that $C = (2N-1)\max_{0\le j\le N-1} C^j$, where C^j is given by

$$C^j = \frac{1}{|1+2a^{j+1}\sin(\frac{\tau h}{2})|} \ \text{ such that } \ a^{j+1} = \min_{0\le i\le M}(-a_i^{j+1}),\ \ j = \overline{0,N-1}.$$

Let

$$L = \left(\frac{C_1(1-C)(2N-1)}{CC_2}\right)^2.$$

Subsequently, we can obtain

$$|\gamma_{j+1}| \le CC_1(k+h) + \frac{CC_2\sqrt{L}}{2N-1}(k+h) = C_1(k+h).$$

Finally, we obtain

$$\|e^j\|_2 \le C_1(k+h),\ \ \forall j = \overline{0,N-1}.$$

So the implicit finite difference scheme (7) is convergent, which concludes the proof of Theorem 5.1.

5.1 Solvability of the approximate scheme

Theorem 5.2 *The approximate scheme* (24) *is uniquely solvable.*

It can be seen that the corresponding homogeneous linear algebraic equations for the approximate scheme (24) are

$$\begin{cases} a_i^{j+1}u_{i+1}^{j+1} + (1-a_i^{j+1})u_i^{j+1} = u_i^j - \sum_{n=1}^{j}(u_{i+1}^{j-n+1} - u_i^{j-n})b_i^{j+1}(n),\\ \forall i = \overline{1,M-1} \ \text{ and } \ \forall j = \overline{1,N-1},\\ u_0^{j+1} = u_M^{j+1} = 0,\ \ \forall i = \overline{0,M} \ \text{ and } \ \ \forall j = \overline{0,N-1},\\ u_i^0 = 0,\ \text{ for all } \ i = \overline{0,M}. \end{cases} \tag{24}$$

Proof. Similar to the proof of Theorem 4.1, we can also verify the solutions of the equations (24) satisfy $\|u^j\|_2 \le C\|u^0\|$, for all $j = \overline{1,N}$, we have $u^0 = 0$, so we get $u^j = 0$ for all $j = \overline{1,N}$.

This indicates that the equations (24) have only zero solutions, the approximate scheme (24) is uniquely solvable. We now easily conclude the proof of Theorem 5.2.

6 Numerical Experiments

In this section, two numerical examples are discussed to confirm the effectiveness of the developed implicit difference scheme (IFDS).

Example 1. This first example is concerned with the numerical solution of the linear variable-order time-fractional diffusion equation with initial and boundary conditions, where the first order derivative is substituted by a Caputo fractional derivative of order β $(0 < \beta < 1)$. Consider the 1-D linear variable-order time-fractional diffusion equation with initial and boundary conditions:

$$\begin{cases} \frac{\partial^\beta u}{\partial t^\beta} + c\frac{\partial u}{\partial x} = 0 \ \ 0 < x < L, 0 < t < T, 0 < \beta < 1, \\ u(0,t) = u(L,t) = 0, \\ u(x,0) = u_0(x). \end{cases} \tag{25}$$

Specifically, we consider the model problem in (25) over $x \in [0;1]$, $t \in [0;1]$, $L = 1$, $T = 1$, $\beta = e^{-x-t}$, $c = 0.5$, $N = M = 50, 100, 150, 200$ and $u_0(x) = 1-0.3*\cos(pi*x)$. We present several numerical experiments to support the theoretical and numerical analyses of the previous sections.

Example 2. We first fix in the mathematical model defined by (26), the values of $x \in [0;1]$, $t \in [0;1]$, $L = 1$, $T = 1$, $\beta = e^{-x*t}$, $c = 0.75$, $N = M = 20, 30, 40, 50$ and $u_0(x) = 1 - 0.5 * pi * x$. We present several numerical experiments to support the theoretical and numerical analyses of the previous sections.

$$\begin{cases} \frac{\partial^\beta u}{\partial t^\beta} + c\frac{\partial u}{\partial x} = 0 \ \ 0 < x < L, 0 < t < T, 0 < \beta < 1, \\ u(0,t) = u(L,t) = 0, \\ u(x,0) = u_0(x). \end{cases} \tag{26}$$

In this example, we present different numerical experiments to support the theoretical and numerical analyses of the previous sections.

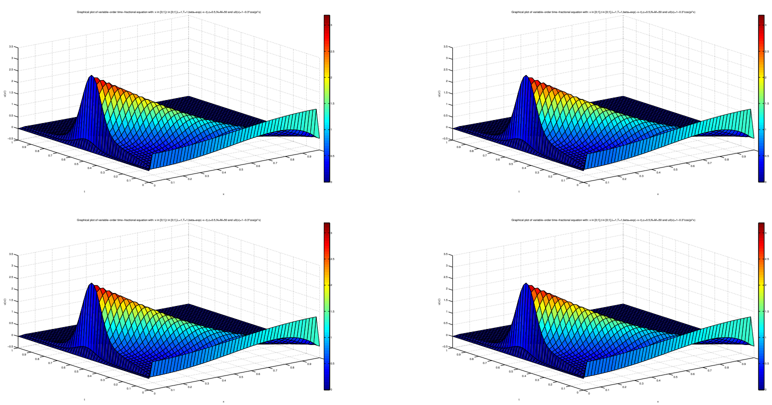

Figure 1. Example 1: Plot of LVTFDE: $\beta = e^{-x-t}$, $c = 0.50$, $u_0(x) = 1 - 0.3 * cos(pi * x)$ $N = M = 50, 100, 150, 200$, $L = T = 1$, $0 < \beta < 1$.

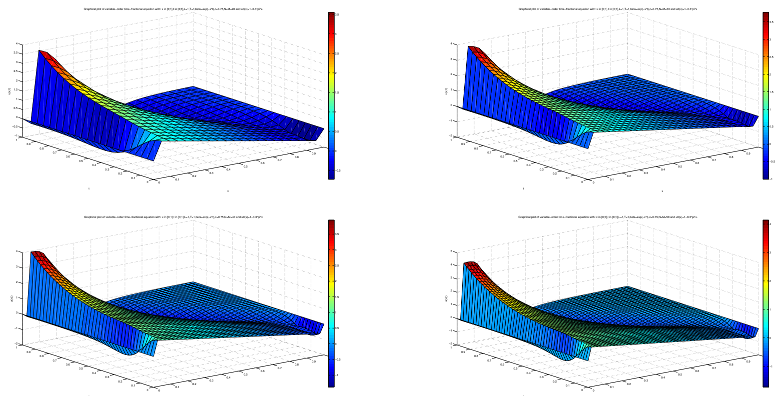

Figure 2. Example 2: Plot of LVTFDE: $\beta = e^{-x*t}$, $c = 0.75$, $u_0(x) = 1 - 0.5 * pi * x$, $N = M = 20, 30, 40, 50$, $L = T = 1$, $0 < \beta < 1$.

Conclusion

In this paper, we consider a numerical approximation method to solve a fractional model for the advection-reaction-diffusion equation with the Coimbra derivative. By using finite difference schemes and obtaining the operational matrix, all that remains is to solve fractional partial differential equations. Some examples are given to illustrate the effectiveness of the proposed numerical algorithm. In several cases, the capabilities of the program developed with MATLAB were reached in terms of meshes and calculation times.

Acknowledgment

We thank the referee for the careful and insightful review of our manuscript. We address all of the concerns of the referee here. The author thanks the anonymous referees for their helpful comments that improved the quality of the manuscript. We revised the manuscript according to your respected advice and the referee's detailed suggestions.

References

[1] K. B. Oldman and J. Spanier. *The Fractional Calculus.* Academic Press, 1974.

[2] K. S. Miller and B. Ross. *An Introduction to the Fractional Calculus and Fractional Differential Equations.* John Wiley, New York, London, 1993.

[3] S. G. Samko, A. A. Kilbas and O. I. Marichev. *Fractional Integrals and Derivatives.* Gordon and Breach Sciences Publishers, 1993.

[4] I. Podlubny. *Fractional Differential Equations.* Academic Press. 1999.

[5] R. D. Richtmyer and K. W. Morton. *Difference Methods for Initial-Value Problems.* Interscience Publishers, New York, 1967.

[6] K. Diethelm and A. D. Freed. On the solution of nonlinear fractional order differential equations used in the modelling of viscoplasticity Scientific Computing in Chemical Engineering II: In: *Computational Fluid Dynamics, Reaction Engineering and Molecular Properties, Springer Verlag, Heidelberg*, 1999, 217–224.

[7] F. Liu, P. Zhuang, V. Anh and I. Turner. A fractional-order implicit difference approximation for the space-time fractional diffusion equation. *ANZIAM J.* **47** (2006) 48–68.

[8] T. A. M. Langlands and B. I. Henry. The accuracy and stability analysis of an implicit solution method for the fractional diffusion equation. *J. Comp. Phys.* **205** (2005) 719–736.

[9] D. Y. Goos and G. F. Reyero. Mathematical analysis of a cauchy problem for the time-fractional diffusion-wave equation with α in [0, 2]. *J. Fourier. Anal.* (2016) 1–23.

[10] Z. Wang and S. Vong. Compact difference schemes for the modified anomalous fractional sub-diffusion equation and the fractional diffusion-wave equation. *J. Comput. Phy.* **277** (2014) 1–15.

[11] E. Afshari, B. Sepehrian and A. M. Nazari. Finite difference method for solving the space-time fractional wave equation in the Caputo form, *Fract. Diff. Eqn.* **5** (1) (2015) 55–63.

[12] U. Ali, M. Sohail, M. Usman, F. A. Abdullah, I. Khan and K. S. Nisar. Fourth-Order Difference Approximation for Time-Fractional Modified Sub-Diffusion Equation. *Syemmetry* **12** (2020) 1–11.

[13] M. Dehghan, J. Manafian and A. Saadatmandi. Solving nonlinear fractional partial differential equations using the homotopy analysis method. *Numerical Methods Partial Diff. Eqs.: Int. J.* **26** (2) (2010) 448–479.

[14] M. Dehghan, J. Manafian and A. Saadatmandi. The solution of the linear fractional partial differential equations using the homotopy analysis method. *Zeitschrift für Naturforschung-A* **65** (11) (2010) 935–949.

[15] J. Singh, D. Kumar, Z. Hammouch and A. Atangana. A fractional epidemiological model for computer viruses pertaining to a new fractional derivative. *Appl. Math. Comput.* **316** (2018) 504–515.

[16] S. A. Osman and T. A. M. Langlands. An implicit Keller Box numerical scheme for the solution of fractional sub-diffusion equations. *Appl. Math. Comput.* **348** (2019) 609–626.

[17] K. D. Dwivedi, S. Das and D. Baleanu. Numerical Solution of Nonlinear Space-Time Fractional-Order Advection-Reaction-Diffusion Equation, *J. Comput. Nonlinear Dyn.* **15** (6) (2020) 1–8.

[18] M. Meerschaert and C. Tadjeran. Finite difference approximations for two-sided space-fractional partial differential equations. *Appl. Numer. Math.* **56** (2006) 80–90.

[19] J. H. He. Approximate analytical solution for seepage flow with fractional derivatives in porous media. *Comput. Methods Appl. Mech. Eng.* **167** (1–2) (1998) 57–68.

[20] C. Tadjeran and M. M. Meerschaert. A second-order accurate numerical method for the two dimensional fractional diffusion equation. *J. Comput. Phys.* **220** (2007) 813–823.