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Properties of MDTM and RDTM for Nonlinear Two-Dimensional Lane-Emden Equations

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Abstract: In this paper, we study the linear and the nonlinear forms of two–dimensional Lane–Emden type equations by proving and applying new product and quotient properties for different differential transform methods (RDTM and MDTM), in order to minimize computation to the maximum. We will obtain exact analytic solutions without linearization, discretization or perturbation, even with less computation.

Keywords: *two-dimensional Lane–Emden equation; two-dimensional Lane-Emden system of equations; reduced differential transform method; modified differential transform method; initial value problems.*

Mathematics Subject Classification (2010): 35J15, 35J47, 70F15, 35J75. 70K99.

1 Introduction

The linear and nonlinear two-dimensional Lane-Emden type equations are first introduced by Wazwaz, Rach and Duan in [8], as follows:

$$u_{xx} + \frac{\alpha}{x}u_x + u_{yy} + \frac{\beta}{y}u_y + g(x,y)f(u) = 0, \tag{1}$$

$$x > 0,\ y > 0,\ \ \alpha > 0,\ \beta > 0,$$

$$u(x,0) = h(x), u_y(x,0) = 0, \tag{2}$$

where $g(x,y)f(u)$ is a linear or nonlinear term.

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In this paper, we apply reduced and modified differential transform methods for solving this kind of elliptic problems, with singularities in both x and y, for obtaining exact solutions, not by multiplying singular equation by xyu^n, $n \in \mathbb{N}$, as we have done in other paper, but by proving new product and quotient properties for the RDTM and MDTM, which implies a minimum of computation.

L. Maia, G. Nornberg and F. Pacella in [7], introduce a dynamical system approach for second-order Lane–Emden type problems by defining some new variables that allow us to transform the radial fully nonlinear Lane-Emden equations into a quadratic dynamical system. For the one-dimension Lane-Emden equation, the original formal conservation of specific entropy along streamlines was given by a PDE in the function of t (time) and r (radius).

2 Definition and Properties of Reduced Differential Transform Method (RDTM)

We introduce the basic definitions of the reduced differential transform method as follows.

Definition 2.1 If the function $u(x,y)$ is analytic and differentiated continuously with respect to x and y, in the domain of interest, then let

$$U_k(x) = \frac{1}{k!}\left(\frac{\partial^k}{\partial y^k}u(x,y)\right)_{y=0}, \qquad k \in \mathbb{N}, \tag{3}$$

where the y-dimensional spectrum function $U_k(x)$ is the reduced transformed function. In this paper, the lowercase $u(x,y)$ represents the original function, while the uppercase $U_k(x)$ stands for the transformed function.

Definition 2.2 The reduced differential inverse transform $U_k(x)$ of $u(x,y)$ is defined as follows:

$$u(x,y) = \sum_{k=0}^{\infty} U_k(x) y^k. \tag{4}$$

Then, combining equation (3)and (4), we write

$$u(x,y) = \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{\partial^k}{\partial y^k}u(x,y)\right)_{y=0} y^k.$$

Some important properties of RDTM used in this paper, can be readily obtained and are listed in Table 1.

3 Definition and Properties of Modified Differential Transform Method (MDTM)

We introduce the basic definitions of the modified differential transform method as follows

Definition 3.1 The modified differential transform of $u(x,y)$ with respect to the variable y at y_0 is defined as

$$U(x,h) = \frac{1}{h!}\left(\frac{\partial^h}{\partial x^h}u(x,y)\right)_{y=y_0}, \qquad k \in \mathbb{N}, \tag{5}$$

where $u(x,y)$ is the original function and U(x,h) is the transformed function.

Original functions	Transformed functions
$w(x,y)=\alpha u(x,y)\pm\beta v(x,y)$	$W_k(x)=\alpha U_k(x)\pm\beta V_k(x)$
$w(x,y)=x^m y^n$	$W_k(x)=x^m\delta(k-n)$
$w(x,y)=x^m y^n u(x,y)$	$W_k(x)=x^m U_{k-n}(x)$
$w(x,y)=u(x,y)v(x,y)$	$W_k(x)=\sum_{r=0}^{k} U_r(x)U_{k-r}(x)$
$w(x,y)=[u(x,y)]^3$	$W_k(x)=\sum_{r=0}^{k}\sum_{s=0}^{r} U_{k-r}(x)U_s(x)U_{r-s}(x)$
$w(x,y)=\frac{\partial u(x,y)}{\partial x}$	$W_k(x)=\frac{\partial}{\partial x}U_k(x)$
$w(x,y)=\frac{\partial^2 u(x,y)}{\partial x^2}$	$W_k(x)=\frac{\partial^2}{\partial x^2}U_k(x)$
$w(x,y)=\frac{\partial^r u(x,y)}{\partial y^r}$	$W_k(x)=\frac{(k+r)!}{k!}U_{k+r}(x)$
$w(x,y)=e^{au(x,y)}$	$W_k(x)=\begin{cases} e^{aU_0(x)} & k=0 \\ a\sum_{r=0}^{k-1}\frac{r+1}{k}U_{r+1}(x)W_{k-r-1}(x) & k\geq 1\end{cases}$

Table 1: Fundamental properties of the RDTM.

Definition 3.2 The modified inverse differential transform $U(x,h)$ of $u(x,y)$ is defined as

$$u(x,y)=\sum_{h=0}^{\infty}U(x,h)(y-y_0)^h. \tag{6}$$

Then, combining equations (5) and (7), we write

$$u(x,y)=\sum_{h=0}^{\infty}\frac{1}{h!}\left(\frac{\partial^h}{\partial x^h}u(x,y)\right)_{y=y_0}(y-y_0)^h. \tag{7}$$

When (x,y_0) is taken as $(x,0)$, then (6) can be expressed as

$$u(x,y)=\sum_{h=0}^{\infty}U(x,h)y^h.$$

Some important properties of MDTM used in this paper, are listed in Table 2.

4 Theorems and Corollaries

Theorem 4.1

$$w(x,y)=\frac{u(x,y)}{v(x,y)}$$

and $V(x,0)\neq 0$, *then the modified differential transform version is*

$$W(x,h)=\begin{cases}\dfrac{U(x,0)}{V(x,0)}, & h=0,\\ \dfrac{U(x,h)-\sum_{i=0}^{h-1}W(x,i)V(x,h-i)}{V(x,0)} & h\geq 1.\end{cases}$$

Original functions	Transformed functions
$w(x,y)=\alpha u(x,y)\pm\beta v(x,y)$	$W(x,h)=\alpha U(x,h)+\beta V(x,h)$
$w(x,y)=x^m y^n$	$W(x,h)=x^m\delta(h-n)$
$w(x,y)=x^m y^n u(x,y)$	$W(x,h)=x^m U(x,h-n)$
$w(x,y)=u(x,y)v(x,y)$	$W(x,h)=\sum_{s=0}^{k} U(x,s)V(x,h-s)$
$w(x,y)=[u(x,y)]^3$	$W(x,h)=\sum_{r=0}^{h}\sum_{s=0}^{r} U(x,h-r)U(x,s)U(x,r-s)$
$w(x,y)=\dfrac{\partial u(x,y)}{\partial x}$	$W(x,h)=\dfrac{\partial U(x,h)}{\partial x}$
$w(x,y)=\dfrac{\partial^2 u(x,y)}{\partial x^2}$	$W(x,h)=\dfrac{\partial^2 U(x,h)}{\partial x^2}$
$w(x,y)=\dfrac{\partial u(x,y)}{\partial y}$	$W(x,h)=(h+1)U(x,h+1)$
$w(x,y)=\dfrac{\partial^2 u(x,y)}{\partial y^2}$	$W(x,h)=(h+1)(h+2)U(x,h+2)$
$w(x,y)=e^{au(x,y)}$	$W(x,h)=\begin{cases} e^{aU(x,0)} & h=0 \\ a\sum_{s=0}^{h-1}\dfrac{s+1}{h}U(x,s+1)W(x,h-s-1), & h\geq 1\end{cases}$

Table 2: Fundamental properties of the MDTM.

Proof.

$$v(x,y)w(x,y)=u(x,y).$$

So, by applying the modified differential transform, we get

$$\sum_{r=0}^{k}\sum_{s=0}^{h} V(x,s)W(x,h-s)=U(x,h).$$

For $h=0$, $V(x,0)W(x,0)=U(x,0)$ gives $W(x,0)=\dfrac{U(x,0)}{V(x,0)}$.
For $h=1$, $V(x,1)W(x,0)+V(x,0)W(x,1)=U(x,1)$.
Then

$$W(x,1)=\frac{U(x,1)-V(x,1)W(x,0)}{V(x,0)}.$$

For $h=2$,

$$W(x,2)=\frac{U(x,2)-V(x,2)W(x,0)-V(x,1)W(x,1)}{V(0,0)}.$$

Finally,

$$W(x,h)=\frac{U(x,h)-\sum_{i=0}^{h-1}W(x,i)V(x,h-i)}{V(x,0)}, \qquad h\geq 1.$$

Corollary 4.1 (MDTM) *If*

$$w(x,y)=\frac{x^m y^n}{v(x,y)}$$

and $V(x,0)\neq 0$, *then the modified differential transform version is*

$$W(x,h)=\begin{cases}\dfrac{x^m\delta(n)}{V(x,0)}, & h=0,\\ \dfrac{x^m\delta(h-n)-\sum_{i=0}^{h-1}W(x,i)V(x,h-i)}{V(x,0)}, & h\geq 1.\end{cases}$$

Corollary 4.2 (MDTM) *If*

$$w(x,y)=\frac{1}{v(x,y)}$$

and $V(x,0)\neq 0$, *then the modified differential transform version is*

$$W(x,h)=\begin{cases}\dfrac{1}{V(x,0)}, & h=0,\\ \dfrac{-\sum_{i=0}^{h-1}W(x,i)V(x,h-i)}{V(x,0)}, & h\geq 1.\end{cases}$$

Theorem 4.2 (MDTM) *If*

$$w(x,y)=\frac{u(x,y)}{x^m y^n},\quad (x,y)\neq(0,0),$$

then the modified differential transform version of $w(x,y)$ *is*

$$W(x,h)=\frac{U(x,h+n)}{x^m}.$$

Proof. We have

$$x^m y^n w(x,y)=u(x,y),\qquad (x,y)\neq(0,0).$$

From Table 2, we get $x^m W(x,h-n)=U(x,h)$. Then

$$W(x,h)=\frac{U(x,h+n)}{x^m}.$$

Corollary 4.3 (MDTM) *If*

$$w(x,y)=\frac{u(x,y)}{x},\qquad x\neq 0,$$

then the modified differential transform version is

$$W(x,h)=\frac{U(x,h)}{x}.$$

Corollary 4.4 (MDTM) *If*

$$w(x,y)=\frac{u(x,y)}{y},\qquad y\neq 0.$$

then the modified differential transform version is

$$W(x,h)=U(x,h+1).$$

Theorem 4.3

$$w(x,y)=\frac{u(x,y)}{v(x,y)}.$$

Then the reduced differential transform version is

$$W_k(x)=\begin{cases}\dfrac{U_0(x)}{V_0(x))}, & k=0,\\ \dfrac{U_k(x)-\sum_{i=0}^{k-1}W_i(x)V_{k-i}(x)}{V_0(x))}, & k\geq 1.\end{cases}$$

Proof. The proof is similar to the previous one (Theorem 4.2).

Corollary 4.5 (MDTM) *If*

$$w(x,y)=\frac{x^m y^n}{v(x,y)}$$

and $V_0(x)\neq 0$, *then the reduced differential transform version is*

$$W_k(x)=\begin{cases}\dfrac{x^m\delta(n)}{V_0(x)}, & k=0,\\ \dfrac{x^m\delta(k-n)-\sum_{i=0}^{k-1}W_i(x)V_{k-i}(x)}{V_0(x))}, & k\geq 1.\end{cases}$$

Corollary 4.6 (MDTM) *If*

$$w(x,y)=\frac{1}{v(x,y)}$$

and $V_0(x)\neq 0$, *then the reduced differential transform version is*

$$W_k(x)=\begin{cases}\dfrac{1}{V_0(x)}, & k=0,\\ \dfrac{-\sum_{i=0}^{k-1}W_i(x)V_{k-i}(x)}{V_0(x))} & k\geq 1.\end{cases}$$

Theorem 4.4 *If*

$$w(x,y)=\frac{u(x,y)}{x^m y^n},\qquad (x,y)\neq(0,0),$$

then the reduced differential transform version of $w(x,y)$ *is*

$$W_k(x)=\frac{U_{k+n}(x)}{x^m}.$$

Proof. We have

$$x^m y^n w(x,y) = u(x,y), \qquad (x,y) \neq (0,0).$$

From Table 3, we get $\quad x^m W_{k-n}(x) = U_k(x)$.
Then

$$W_k(x) = \frac{U_{k+n}(x)}{x^m}.$$

Corollary 4.7 (MDTM) *If*

$$w(x,y) = \frac{u(x,y)}{x}, \qquad x \neq 0,$$

then the reduced differential transform version is

$$W_k(x) = \frac{U_k(x)}{x}.$$

Corollary 4.8 (MDTM) *If*

$$w(x,y) = \frac{u(x,y)}{y}, \qquad y \neq 0,$$

then the reduced differential transform version is

$$W_k(x) = \frac{U_{k+1}(x)}{x}.$$

5 Applications

5.1 Modified differential transform method

Example 5.1 First, we consider the nonlinear Lane-Emden equation

$$u_{xx} + \frac{3}{x}u_x + u_{yy} + \frac{4}{y}u_y - 14e^{-u} = 0 \tag{8}$$

subject to the initial conditions

$$u(x,0) = 2lnx, \qquad u_y(x,0) = 0. \tag{9}$$

From Table 2 and Corollaries 4.7, 4.8, the modified differential transform version of (8) is

$$U(x,h+2) = \frac{-1}{(h+5)(h+2)}\left[\frac{\partial^2 U(x,h)}{\partial x^2} + \frac{3}{x}\frac{\partial U(x,h)}{\partial x} - 14W(x,h)\right],$$

where $W(x,h)$ is the modified differential transform version of e^{-u} (Table 2) such that

$$W(x,h) = \begin{cases} e^{-U(x,0)} = \frac{1}{x^2}\ , & h = 0, \\ -\displaystyle\sum_{s=0}^{h-1} \frac{s+1}{h} U(x,s+1)W(x,h-s-1), & h \geq 1. \end{cases}$$

Also, the modified differential transform version of initial conditions (9) is

$$U(x,0) = 2lnx \ , \qquad U(x,1) = 0,$$

then, for $h = 0$,

$$U(x,2) = \frac{-1}{10x}\left[\frac{\partial^2 U(x,0)}{\partial x^2} + \frac{3}{x}\frac{\partial U(x,0)}{\partial x} - 14W(x,0)\right] = \frac{1}{x^2}.$$

For $h = 1$ ($W(x,1) = 0$),

$$U(x,3) = \frac{-1}{18x}\left[\frac{\partial^2 U(x,1)}{\partial x^2} + \frac{3}{x}\frac{\partial U(x,1)}{\partial x} - 14W(x,1)\right] = 0.$$

For $h = 2$,

$$U(x,4) = \frac{-1}{28}\left[\frac{\partial^2 U(x,2)}{\partial x^2} + \frac{3}{x}\frac{\partial U(x,2)}{\partial x} - 14W(x,2)\right] = \frac{-1}{2x^4},$$

where $W(x,2) = \dfrac{-1}{x^4}$.

For $h = 3$,

$$U(x,5) = \frac{-1}{40}\left[\frac{\partial^2 U(x,3)}{\partial x^2} + \frac{3}{x}\frac{\partial U(x,3)}{\partial x} - 14W(x,3)\right] = 0.$$

For $h = 4$,

$$U(x,6) = \frac{-1}{54}\left[\frac{\partial^2 U(x,4)}{\partial x^2} + \frac{3}{x}\frac{\partial U(x,4)}{\partial x} - 14W(x,4)\right] = \frac{1}{3x^6}.$$

Then, by substituting the quantities $U(x,h)$ in Eq.(7), we get the series solution

$$u(x,y) = 2lnx + \left(\frac{y}{x}\right)^2 - \frac{1}{2}\left(\frac{y}{x}\right)^4 + \frac{1}{3}\left(\frac{y}{x}\right)^6 + \dots \tag{10}$$

and the exact solution is

$$u(x,y) = \ln\left(x^2 + y^2\right). \tag{11}$$

The solution is also obtained by the Adomian decomposition method in [8].

Example 5.2 Second, we consider the nonlinear Lane-Emden equation

$$u_{xx} + \frac{3}{x}u_x + u_{yy} + \frac{3}{y}u_y - 7u^{-1} = 0 \tag{12}$$

subject to the initial conditions

$$u(x,0) = x \quad , \quad u_y(x,0) = 0. \tag{13}$$

From Table 2 and Corollaries 4.7, 4.8, the modified differential transform version of (12) is

$$\frac{\partial^2 U(x,h)}{\partial x^2} + \frac{3}{x}\frac{\partial U(x,h)}{\partial x} + (h+4)(h+2)U(x,h+2) - 7W(x,h) = 0.$$

where $W(x,h)$ is the modified differential transform version of u^{-1} (Corollary 4.2) such that

$$W(x,h) = \begin{cases} \dfrac{1}{x}, & h = 0, \\ \dfrac{-\sum_{i=0}^{h-1} W(x,i)U(x,h-i)}{x}, & h \geq 1. \end{cases}$$

From (13), we get $U(x,0) = x, \quad U(x,1) = 0,$ then $W(x,1) = 0$.

$$U(x,h+2) = \frac{-1}{(h+4)(h+2)} \left[\frac{\partial^2 U(x,h)}{\partial x^2} + \frac{3}{x} \frac{\partial U(x,h)}{\partial x} - 7W(x,h) \right].$$

For h=0, we get

$$U(x,2) = \frac{-1}{8} \left[\frac{\partial^2 U(x,0)}{\partial x^2} + \frac{3}{x} \frac{\partial U(x,0)}{\partial x} - \frac{7}{x} \right] = \frac{4}{8x} = \frac{1}{2x}.$$

For $h = 1$, because $U(x,1) = 0$, we have $W(x,1) = 0$, so

$$U(x,3) = \frac{-1}{15} \left[\frac{\partial^2 U(x,1)}{\partial x^2} + \frac{3}{x} \frac{\partial U(x,1)}{\partial x} - 7W(x,1) \right] = 0.$$

For $h = 2$, we have $\mathrm{W}(x,2) = \frac{-1}{2x^3}$, so

$$U(x,4) = \frac{-1}{24} \left[\frac{\partial^2 U(x,2)}{\partial x^2} + \frac{3}{x} \frac{\partial U(x,2)}{\partial x} - 7W(x,2) \right] = \frac{-3}{24x^3} = \frac{-1}{8x^3}.$$

And

$$U(x,5) = 0, \; U(x,6) = \frac{1}{16x^5}.$$

Then, by substituting the quantities $U(x,h)$ in (7), we get the following series solution:

$$u(x,y) = x + \frac{1}{2} \left(\frac{y^2}{x} \right) - \frac{1}{8} \left(\frac{y^4}{x^3} \right) + \frac{1}{16} \left(\frac{y^6}{x^5} \right) + \dots \tag{14}$$

And the exact solution is

$$u(x,y) = \sqrt{x^2 + y^2}. \tag{15}$$

Example 5.3 We consider now the nonlinear Lane-Emden equation

$$u_{xx} + \frac{4}{x} u_x + u_{yy} + \frac{4}{y} u_y - \left(5 + 4x^2 y^2\right) \left(x^2 + y^2\right) u^{-3} = 0 \tag{16}$$

subject to the initial conditions

$$u(x,0) = 1 \;, u_y(x,0) = 0. \tag{17}$$

From (17), we get

$$U(x,0) = 1 \qquad U(x,1) = 0.$$

From Table 2 and Corollaries 4.7, 4.8, the modified differential transform version of (16) gives

$$\begin{aligned} U(x,h+2) \quad = \quad & \frac{1}{(h+5)(h+2)} \Bigg[\frac{\partial^2 U(x,h)}{\partial x^2} + \frac{4}{x} \frac{\partial U(x,h)}{\partial x} - (5x^2 W(x,h) + 5W(x,h-2) \\ & +4x^4 W(x,h-2) + 4x^2 W(x,h-4) \Bigg], \end{aligned}$$

where $W(x,h)$ is the modified differential transform version of u^{-3} (Table 2) such that

$$W(x,h)=\begin{cases} \dfrac{1}{U^3(x,0)}=1, & h=0,\\ -\displaystyle\sum_{i=0}^{h-1} W(x,i)U_1(x,h-i), & h\geq 1\end{cases}$$

$$U_1(x,h)=\sum_{r=0}^{h}\sum_{s=0}^{r} U(x,h-r)U(x,s)U(x,r-s).$$

Then

$$W(x,h)=\begin{cases} \dfrac{1}{U^3(x,0)}=1, & h=0,\\ -\displaystyle\sum_{i=0}^{h-1} W(x,i)\sum_{r=0}^{h-i}\sum_{s=0}^{r} U(x,h-i-r)U(x,s)U(x,r-s), & h\geq 1.\end{cases}$$

For $h=0$, *we get*

$$\frac{\partial^2 U(x,0)}{\partial x^2}+\frac{4}{x}\frac{\partial U(x,0)}{\partial x}+10U(x,2)-5x^2W(x,0)=0.$$

Then

$$U(x,2)=\frac{5x^2}{10}=\frac{x^2}{2}.$$

For $h=1$,

$$\frac{\partial^2 U(x,1)}{\partial x^2}+\frac{4}{x}\frac{\partial U(x,1)}{\partial x}+18U(x,3)-5x^2W(x,1)=0,$$

where

$$W(x,1)=-\sum_{i=0}^{0} W(x,i)\sum_{r=0}^{1-i}\sum_{s=0}^{r} U(x,1-i-r)U(x,s)U(x,r-s)$$

$$=-W(x,0)\left[U(x,1)U(x,0)U(x,0)+U(x,0)U(x,0)U(x,1)+U(x,0)U(x,1)U(x,0)\right]=0$$

Then $18U(x,3)=0$. So, $U(x,3)=0$.

For $h=2$,

$$\frac{\partial^2 U(x,2)}{\partial x^2}+\frac{4}{x}\frac{\partial U(x,2)}{\partial x}+28U(x,4)-5x^2W(x,2)-5W(x,0)-4x^4W(x,0)=0. \quad (18)$$

We first calculate

$$W(x,2)=-\sum_{i=0}^{1} W(x,i)\sum_{r=0}^{2-i}\sum_{s=0}^{r} U(x,2-i-r)U(x,s)U(x,r-s)$$

$$=-W(x,0)\left[U(x,2)U(x,0)U(x,0)+U(x,1)U(x,0)U(x,1)+U(x,0)U(x,0)U(x,2)\right.$$

$$\left.+U(x,0)U(x,1)U(x,1)+U(x,0)U(x,2)U(x,0)\right]=-\left(\frac{x^2}{2}+\frac{x^2}{2}+\frac{x^2}{2}\right)=\frac{-3x^2}{2}.$$

Then Eq.(18) becomes

$$1 + 4 + 28U(x,4) + 5x^2\left(\frac{3x^2}{2}\right) - 5 - 4x^4 = 0.$$

So,

$$28U(x,4) = \frac{-7x^4}{2} \Rightarrow U(x,4) = \frac{-x^4}{8}.$$

And

$$U(x,5) = \ 0, \ U(x,6) = \ \frac{x^6}{16}, \ \ U(x,7) = \ 0, \ \ U(x,8) = \ \frac{-5x^8}{128}, \ \ \ldots$$

Then, by substituting the quantities $U(x,h)$ in Eq.(7), we get the following series solution:

$$u(x,y) = 1 + \frac{x^2y^2}{2} - \frac{x^4y^4}{8} + \frac{x^6y^6}{16} - \frac{5x^8y^8}{128} + \ldots.$$

And the exact solution is

$$u(x,y) = \sqrt{1 + x^2y^2}.$$

The exact solution is not obtained by the Adomian method in [8], only the approximate solution is mentioned.

Remark 5.1 The application of the reduced differential transform method gives the same coefficients of power series and same steps of computation.

Remark 5.2 The nonlinear two-dimensional Lane–Emden system of equations can be introduced and solved by the RDTM and MDTM using the same computation steps.

$$\begin{cases} u_{xx} + \dfrac{\alpha}{x}u_x + u_{yy} + \dfrac{\beta}{y}u_y + f(x,y,v) = 0, \\ \\ v_{xx} + \dfrac{\gamma}{x}v_x + v_{yy} + \dfrac{\theta}{y}v_y + g(x,y,u) = 0, \end{cases}$$

$$x > 0, \quad y > 0, \quad \alpha > 0, \quad \beta > 0, \quad \gamma > 0, \quad \theta > 0,$$

$$u(x,0) = h(x), \quad u_y(x,0) = 0, \quad v(x,0) = k(x), \quad v_y(x,0) = 0.$$

6 Conclusion

In this paper, the Reduced Differential Transform Method (RDTM) and Modified Differential Transformation Method (MDTM) have been successfully applied for obtaining exact solutions to the nonlinear forms of two–dimensional Lane–Emden type equations, by proving and applying new product and quotient properties for different differential transform methods (RDTM and MDTM). This paper is the first to provide an exact solution by the DTM (Differential transform method) in the case, where the Adomian method gives an approximative solution. We also introduce the two–dimensional Lane–Emden system of equations and obtain exact analytic solutions.

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