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Numerical Solution of Neutral Double Delay Volterra Integral Equations Using Taylor Collocation Method

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Abstract: In this paper, we employ a direct collocation method using Taylor polynomials to estimate the solution of linear Volterra integral equations with two constant delays within the polynomial spline $S_{m-1}^{(-1)}(\Pi_N)$. This approach is well-suited for capturing the dynamics inherent in age and size-structured population models (Gurtin–MacCamy model), epidemiological models, chemical engineering processes (heat equation with delay in control and in state) and control theory (Roesser model). We derive an iterative formula to compute the approximate solution and prove its convergence. We confirm the validity and efficacy of this convergent algorithm by presenting numerical results.

Keywords: *neutral double delay Volterra integral equation; collocation method; Taylor polynomials; error analysis.*

Mathematics Subject Classification (2010): 34K28, 45L05, 65R20, 70K99, 93A99.

1 Introduction

Delay Volterra integral equations pose a significant mathematical challenge across various scientific and engineering domains, encompassing biology [1], epidemiology [2,3], chemical engineering [4], control theory [5], physics [6] and social sciences [7]. In [8,9], double delay Volterra integral equations (DDVIEs), which incorporate memory and delayed effects, apply to model age-structured populations, where two distinct age groups within a single population are considered. The dual delays in equation (1) correspond to the time required for maturation and reaching the maximum age, rendering analytical solutions impractical for many problems. As a result, developing efficient and precise numerical

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methods for solving such equations has become a significant area of research in recent years.

The study of DDVIEs is driven by their widespread applicability, addressing problems that involve multiple delays and intricate memory effects. Including two distinct delays introduces additional complexity to the equations, making their numerical analysis intellectually stimulating and practically crucial.

The numerical investigation of equation (1) has only recently garnered attention, and the existing specialized literature on this subject still needs to be expanded. A numerical method developed to resolve equation (1) is elaborated in [10].

This research explores the Taylor collocation method [11] applied to the solution of linear DDVIEs. The method leverages the versatility of Taylor polynomials and the precision of collocation schemes, providing an advantageous approach for addressing the challenges posed by this class of integral equations.

We consider the linear Volterra integral equation with two constant delays τ_1, τ_2 of the form

$$x(t) = g(t) + x(t-\tau_1) + x(t-\tau_2) + \int_{t-\tau_2}^{t-\tau_1} K(t,s)x(s)ds \tag{1}$$

for $t \in [\tau_2, T]$ and $x(t) = \phi(t)$ for $t \in [0, \tau_2]$. In the following, we assume that the given functions g, K and ϕ are sufficiently smooth. Furthermore, we suppose that

$$\phi(\tau_2) = g(\tau_2) + \phi(\tau_2 - \tau_1) + \phi(0) + \int_0^{\tau_2-\tau_1} K(\tau_2, s)\phi(s)ds.$$

Bellour et al. [12] provide a proof regarding the existence and uniqueness of results for equation (1).

The structure of this paper is as follows. Section 2 presents a description of the Taylor collocation method for our problem (1). Section 3 introduces the analysis of convergence. Subsequently, in Section 4, we present the results obtained using our proposed approach. Finally, in Section 5, we discuss the implications of our work and suggest potential avenues for the future.

2 Description of the Method

We suppose that $T = (r+1)\tau_2$, where $r \in \{1, 2, 3, ...\}$. Let Π_N be a uniform partition of the interval $I = [\tau_2, T]$ defined by $t_n^i = (i+1)\tau_2 + nh$, $n = 0, 1, .., N,\ i = 0, 1, .., r-1$, where the step size is given by $h = t_{n+1}^i - t_n^i$, and assume that $h = \frac{\tau_1}{N_1} = \frac{\tau_2}{N}$ with N and N_1 being positive and integer. Define the subintervals $\sigma_n^i = [t_n^i; t_{n+1}^i), n = 0, 1, .., N-1,\ i = 0, 1, .., r-2$ and $\sigma_{N-1}^{r-1} = [t_{N-1}^{r-1}, t_N^{r-1}]$. Moreover, denote by π_m the set of all real polynomials of degree not exceeding m. We define the real polynomial spline space of degree $m-1$ as follows:

$$S_{m-1}^{(-1)}(\Pi_N) = \{u(I, \mathbb{R}) : u_n^i = u|_{\sigma_n^i} \in \pi_{m-1}, n = 0, ..., N-1,\ i = 0, 1, .., r-1\}.$$

This is the space of piecewise polynomials of degree (at most) $m-1$. Its dimension is rNm, i.e., the same as the total number of the coefficients of the polynomials $u_n^i, n = 0, ..., N-1, i = 0, 1, .., r-1$. To find these coefficients, we use the Taylor polynomial on each subinterval.

First, to approximate x by u_n^0 ($n \in \{0,1,...,N-1\}$) on the interval σ_n^0, x must be approximated by u_k^0 ($0 \le k < n$) on each interval σ_k^0 so that

$$u_n^0(t) = \sum_{j=0}^{m-1} \frac{\hat{u}_{n,0}^{(j)}(t_n^0)}{j!}(t-t_n^0)^j; \quad t \in \sigma_n^0, \tag{2}$$

where $\hat{u}_{0,0}(\tau_2) = x(\tau_2)$ and $\hat{u}_{n,0}$ is the exact solution of the integral equations

$$\hat{u}_{n,0}(t) = g(t) + \phi(t-\tau_1) + \phi(t-\tau_2) + \int_{t-\tau_2}^{t-\tau_1} K(t,s)\phi(s)ds, \;\; n \in \{0,...,N_1-1\}, \tag{3}$$

and

$$\begin{aligned}\hat{u}_{n,0}(t) &= g(t) + u_{n-N_1}^0(t-\tau_1) + \phi(t-\tau_2) + \int_{t-\tau_2}^{\tau_2} K(t,s)\phi(s)ds \\ &+ \sum_{i=0}^{n-N_1-1} \int_{t_i^0}^{t_{i+1}^0} K(t,s)u_i^0(s)ds + \int_{t_n^0-\tau_1}^{t-\tau_1} K(t,s)u_{n-N_1}^0(s)ds, \;\; n \in \{N_1,...,N-1\}.\end{aligned} \tag{4}$$

Now, for all $j = 1,...,m-1$, the formula for computing the values of the coefficients $\hat{u}_{n,0}^{(j)}(t_n^0)$ can be obtained by differentiating (3) and (4), respectively, we get the following formulas:

$$\begin{aligned}\hat{u}_{n,0}^{(j)}(t_n^0) &= g^{(j)}(t_n^0) + \phi^{(j)}(t_n^0-\tau_1) + \phi^{(j)}(t_n^0-\tau_2) \\ &+ \left(\int_{t-\tau_2}^{t-\tau_1} K(t,s)\phi(s)ds\right)^{(j)}(t_n^0), \;\; n \in \{0,...,N_1-1\},\end{aligned}$$

and

$$\begin{aligned}\hat{u}_{n,0}^{(j)}(t_n^0) &= g^{(j)}(t_n^0) + \hat{u}_{n-N_1,0}^{(j)}(t_{n-N_1}^0) + \phi^{(j)}(t_n^0-\tau_2) + \left(\int_{t-\tau_2}^{\tau_2} K(t,s)\phi(s)ds\right)^{(j)}(t_n^0) \\ &+ \sum_{i=0}^{n-N_1-1}\sum_{l=0}^{m-1} \frac{\hat{u}_{i,0}^{(l)}(t_i^0)}{l!} \int_{t_i^0}^{t_{i+1}^0} \partial_1^{(j)}K(t_n^0,s)(s-t_i^0)^l ds \\ &+ \sum_{i=0}^{j-1}\sum_{l=0}^{i} \binom{i}{l} \hat{u}_{n-N_1,0}^{(l)}(t_{n-N_1}^0)[\partial_1^{(j-1-i)}K(t,t-\tau_1)]^{(i-l)}(t_n^0), \;\; n \in \{N_1,...,N-1\}.\end{aligned}$$

Second, for x to be approximated by u_n^p ($n \in \{0,1,...,N-1\}$ and $p \in \{1,2,...,r-1\}$) on the interval σ_n^p, x must be approximated by u_k^j ($0 \le k < n$ and $0 \le j \le p$) on each interval σ_k^j so that

$$u_n^p(t) = \sum_{j=0}^{m-1} \frac{\hat{u}_{n,p}^{(j)}(t_n^p)}{j!}(t-t_n^p)^j; \quad t \in \sigma_n^p, \tag{5}$$

where $\hat{u}_{n,p}$ is the exact solution of the integral equations

$$\begin{aligned}\hat{u}_{n,p}(t) &= g(t) + u_{N-N_1+n}^{p-1}(t-\tau_1) + u_n^{p-1}(t-\tau_2) \\ &+ \int_{t-\tau_2}^{t_{n+1}^{p-1}} K(t,s)u_n^{p-1}(s)ds + \sum_{d=n+1}^{N-N_1+n-1} \int_{t_d^{p-1}}^{t_{d+1}^{p-1}} K(t,s)u_d^{p-1}(s)ds \\ &+ \int_{t_n^p-\tau_1}^{t-\tau_1} K(t,s)u_{N-N_1+n}^{p-1}(s)ds, \;\; n \in \{0,...,N_1-1\},\end{aligned} \tag{6}$$

and

$$
\begin{aligned}
\hat{u}_{n,p}(t) &= g(t) + u^p_{n-N_1}(t-\tau_1) + u^{p-1}_n(t-\tau_2) + \int_{t-\tau_2}^{t^{p-1}_{n+1}} K(t,s)u^{p-1}_n(s)ds \\
&+ \sum_{d=n+1}^{N-1} \int_{t^{p-1}_d}^{t^{p-1}_{d+1}} K(t,s)u^{p-1}_d(s)ds + \sum_{i=0}^{n-N_1-1} \int_{t^p_i}^{t^p_{i+1}} K(t,s)u^p_i(s)ds \\
&+ \int_{t^p_n-\tau_1}^{t-\tau_1} K(t,s)u^p_{n-N_1}(s)ds, \;\; n \in \{N_1,...,N-1\}.
\end{aligned} \tag{7}
$$

The coefficients $\hat{u}^{(j)}_{n,p}(t^p_n)$ are given by the following formulas. For $n \in \{0,...,N_1-1\}$,

$$
\begin{aligned}
\hat{u}^{(j)}_{n,p}(t^p_n) &= g^{(j)}(t^p_n) + \hat{u}^{(j)}_{N-N_1+n,p-1}(t^{p-1}_{N-N_1+n}) + \hat{u}^{(j)}_{n,p-1}(t^{p-1}_n) \\
&- \sum_{i=0}^{j-1}\sum_{l=0}^{i}\binom{i}{l}\hat{u}^{(l)}_{n,p-1}(t^{p-1}_n)[\partial_1^{(j-1-i)}K(t,t-\tau_2)]^{(i-l)}(t^p_n) \\
&+ \sum_{l=0}^{m-1}\frac{\hat{u}^{(l)}_{n,p-1}(t^{p-1}_n)}{l!}\int_{t^p_n-\tau_2}^{t^{p-1}_{n+1}} \partial_1^{(j)}K(t^p_n,s)(s-t^{p-1}_n)^l ds \\
&+ \sum_{d=n+1}^{N-N_1+n-1}\sum_{l=0}^{m-1}\frac{\hat{u}^{(l)}_{d,p-1}(t^{p-1}_d)}{l!}\int_{t^{p-1}_d}^{t^{p-1}_{d+1}} \partial_1^{(j)}K(t^p_n,s)(s-t^{p-1}_d)^l ds \\
&+ \sum_{i=0}^{j-1}\sum_{l=0}^{i}\binom{i}{l}\hat{u}^{(l)}_{N-N_1+n,p-1}(t^{p-1}_{N-N_1+n})[\partial_1^{(j-1-i)}K(t,t-\tau_1)]^{(i-l)}(t^p_n),
\end{aligned}
$$

and for $n \in \{N_1,...,N-1\}$,

$$
\begin{aligned}
\hat{u}^{(j)}_{n,p}(t^p_n) &= g^{(j)}(t^p_n) + \hat{u}^{(j)}_{n-N_1,p}(t^p_{n-N_1}) + \hat{u}^{(j)}_{n,p-1}(t^{p-1}_n) \\
&- \sum_{i=0}^{j-1}\sum_{l=0}^{i}\binom{i}{l}\hat{u}^{(l)}_{n,p-1}(t^{p-1}_n)[\partial_1^{(j-1-i)}K(t,t-\tau_2)]^{(i-l)}(t^p_n) \\
&+ \sum_{l=0}^{m-1}\frac{\hat{u}^{(l)}_{n,p-1}(t^{p-1}_n)}{l!}\int_{t^p_n-\tau_2}^{t^{p-1}_{n+1}} \partial_1^{(j)}K(t^p_n,s)(s-t^{p-1}_n)^l ds \\
&+ \sum_{d=n+1}^{N-1}\sum_{l=0}^{m-1}\frac{\hat{u}^{(l)}_{d,p-1}(t^{p-1}_d)}{l!}\int_{t^{p-1}_d}^{t^{p-1}_{d+1}} \partial_1^{(j)}K(t^p_n,s)(s-t^{p-1}_d)^l ds \\
&+ \sum_{i=0}^{n-N_1-1}\sum_{l=0}^{m-1}\frac{\hat{u}^{(l)}_{i,p}(t^p_i)}{l!}\int_{t^p_i}^{t^p_{i+1}} \partial_1^{(j)}K(t^p_n,s)(s-t^p_i)^l ds \\
&+ \sum_{i=0}^{j-1}\sum_{l=0}^{i}\binom{i}{l}\hat{u}^{(l)}_{n-N_1,p}(t^p_{n-N_1})[\partial_1^{(j-1-i)}K(t,t-\tau_1)]^{(i-l)}(t^p_n).
\end{aligned}
$$

3 Analysis of Convergence

The following lemmas will be used in this section.

Lemma 3.1 *(Discrete Gronwall-type inequality [13]) Let* $\{k_j\}_{j=0}^{n}$ *be a given nonnegative sequence, and the sequence* $\{\varepsilon_n\}$ *satisfy* $\varepsilon_n \leq p_0 + \sum_{i=0}^{n-1} k_i \varepsilon_i, \quad n \geq 0,$ *with* $p_0 \geq 0$. *Then* ε_n *can be bounded by* $\varepsilon_n \leq p_0 exp\left(\sum_{j=0}^{n-1} k_j\right), \quad n \geq 0.$

Lemma 3.2 *Let* g *and* K *be* m *times continuously differentiable on their respective domains. Then there exists a positive number* $\alpha(m)$ *such that for all* $n = 0, 1, ..., N-1$, $p = 0, 1, ..., r-1$, *and* $j = 0, ..., m$, *we have* $\| \hat{u}_{n,p}^{(j)} \|_{L^\infty(\sigma_n^p)} \leq \alpha(m)$, *where* $\hat{u}_{0,0}(t) = x(t)$ *for* $t \in \sigma_0^0$.

Proof. The proof is split into two steps.
Claim 1. There exists a positive constant $\alpha_1(m)$ such that $\| \hat{u}_{n,0}^{(j)} \|_{L^\infty(\sigma_n^p)} \leq \alpha_1(m)$ for all $n = 0, 1, ..., N-1$, $j = 0, ..., m$. Let $a_n^j = \| \hat{u}_{n,0}^{(j)} \|_{L^\infty(\sigma_n^0)}$, then, for $j = 0, ..., m$,

$$a_0^j \leq max\left\{\| x^{(j)} \|_{L^\infty(\sigma_0^0)}, j = 0, ..., m\right\} = \alpha_1^1(m). \tag{8}$$

Now, for all $n = 1, 2, ..., N-1$ and $j = 1, ..., m$, by differentiating (3) and (4) j times, we obtain

$$\begin{aligned} a_n^j &\leq c + d_1^2 h \sum_{l=j}^{m-1} a_{n-N_1}^l + h d_1^1 \sum_{i=0}^{n-N_1} \sum_{l=0}^{m-1} a_i^l + m^2 h d_1^2 \sum_{l=0}^{m-1} a_{n-1-N_1}^l + h d_1^1 \sum_{l=0}^{m-1} a_{n-N_1}^l \\ &\leq c + \underbrace{\left(d_1^2(1+m^2) + d_1^1\right)}_{d_1} h \sum_{l=0}^{m-1} a_{n-N_1}^l + h d_1^1 \sum_{i=0}^{n-1} \sum_{l=0}^{m-1} a_i^l, \end{aligned} \tag{9}$$

where the constants c, d_1^1 and d_1^2 are positive and independent of N.

On the other hand, if $j = 0$, then, from (3) and (4), we have

$$a_n^0 \leq c + (d_1^1 + d_1^2) h \sum_{l=0}^{m-1} a_{n-N_1}^l + h d_1^1 \sum_{i=0}^{n-1} \sum_{l=0}^{m-1} a_i^l. \tag{10}$$

From (9) and (10), for each fixed $n \geq 1$, we consider the sequence a_n^j for $j = 0, ..., m$,

$$a_n^j \leq c + 2 h d_1 \sum_{i=0}^{n-1} \sum_{l=0}^{m-1} a_i^l. \tag{11}$$

Consider the sequence $z_n = \sum_{j=0}^{m} a_n^j$ for $n \geq 0$. Then, from (11), we have

$$z_n \leq \underbrace{(m+1)c}_{c_1} + h \underbrace{2(m+1)d_1}_{d_2} \sum_{i=0}^{n-1} z_i. \tag{12}$$

Moreover, from (8), we obtain

$$z_0 \leq (m+1)\alpha_1^1(m) = c_2. \tag{13}$$

Then, from (12) and (13), we have for all $n = 0, 1, ..., N-1$,

$$z_n \leq c_3 + h d_2 \sum_{i=0}^{n-1} z_i$$

such that $c_3 = max(c_1, c_2)$. Using Lemma 3.1, for all $n = 0, 1, ..., N-1$, we obtain

$$z_n \leq \underbrace{c_3 exp(\tau_2 d_2)}_{\alpha_1(m)}.$$

Hence, the first claim is true.

Claim 2. There exists a positive constant $\alpha_2(m)$ such that $\| \hat{u}^{(j)}_{n,p} \|_{L^\infty(\sigma^p_n)} \leq \alpha_2(m)$ for all $n = 0, 1, ..., N-1$, $j = 0, ..., m$ and $p = 1, 2, ..., r-1$.

Let $a^{(j)}_{n,p} = \| \hat{u}^{(j)}_{n,p} \|_{L^\infty(\sigma^p_n)}$ and $\varepsilon_p = max \left\{ a^j_{i,p}, j = 0, .., m, i = 0, ..., N-1 \right\}$ for $p = 1, ..., r-1$. Similarly to Claim 1, by differentiating (6) and (7) j times, we obtain for all $n = 1, ..., N-1$,

$$a^j_{n,p} \leq c_4 + b_1 \varepsilon_{p-1} + d_3 h \sum_{i=0}^{n-1} \sum_{l=0}^{m-1} a^l_{i,p},$$

where c_4, b_1, and d_3 are positive numbers. Consider the sequence $y_n = \sum_{j=0}^{m} a^j_{n,p}$, $n = 0, 1, ..., N-1$, using Lemma 3.1, for all $n = 0, ..., N-1$, we obtain

$$\begin{aligned} y_n &\leq ((m+1)c_4 + (m+1)b_1 \varepsilon_{p-1})\, exp \underbrace{\sum_{i=0}^{n-1} (m+1) d_4 h}_{d_4} \\ &\leq \underbrace{(m+1)c_4 exp\,(d_4)}_{c_5} + \underbrace{(m+1)b_1 exp\,(d_4)}_{b_2} \varepsilon_{p-1} \\ &\leq \underbrace{c_6 + b_3 \alpha_1(m)}_{\alpha_2(m)}, \end{aligned}$$

where c_6 and b_3 are positive numbers.

Hence, the proof of Lemma 3.2 is completed by setting $\alpha(m) = \max\{\alpha_1(m), \alpha_2(m)\}$.

The following theorem gives the convergence of the presented method.

Theorem 3.1 *Let g and K be m times continuously differentiable on their respective domains. Then (2) and (5) define a unique approximation $u \in S^{(-1)}_{m-1}(\Pi_N)$, and the resulting error function $e = x - u$ satisfies $\| e \|_{L(l)^\infty} \leq C h^m$, provided that h is sufficiently small, where C is a finite constant independent of h.*

Proof. The proof is split into two steps.

Claim 1. There exists a constant C_1 independent of h such that $\| e^0 \|_{L^\infty(\sigma^0)} \leq C_1 h^m$, where the error $e^0 = e \mid_{\sigma^0}$ is defined on σ^0_n by $e^0(t) = e^0_n(t) = \mid x(t) - u^0_n(t) \mid$ for all $n = 0, 1, ..., N-1$. Let $t \in \sigma^0_0$. Then we have for sufficiently small h,

$$\mid e^0_0(t) \mid = \mid x(t) - u^0_0(t) \mid \leq \frac{\| x^m \|_{L^\infty(\sigma^0_0)}}{m!} h^m \leq \frac{\alpha(m)}{m!} h^m.$$

From (3) and (4), for $n = 1, ..., N-1$, we have

$$\| x - \hat{u}_{n,0} \|_{L^\infty(\sigma^0_n)} \leq hk \sum_{i=0}^{n-1} \| e^0_i \|_{L^\infty(\sigma^0_i)} + \| e^0_{n-N_1} \|_{L^\infty(\sigma^0_{n-N_1})},$$

where $k = \parallel k \parallel_{L^\infty(I)}$, therefore, by Lemma 3.2,

$$\begin{aligned}\parallel e_n^0 \parallel_{L^\infty(\sigma_n^0)} &\leq \parallel x - \hat{u}_{n,0} \parallel_{L^\infty(\sigma_n^0)} + \parallel \hat{u}_{n,0} - u_n^0 \parallel_{L^\infty(\sigma_n^0)} \\ &\leq hk \sum_{i=0}^{n-1} \parallel e_i^0 \parallel_{L^\infty(\sigma_i^0)} + \parallel e_{n-N_1}^0 \parallel_{L^\infty(\sigma_{n-N_1}^0)} + \frac{\alpha(m)}{m!} h^m \\ &\leq hk \sum_{i=0}^{n-1} \parallel e_i^0 \parallel_{L^\infty(\sigma_i^0)} + 2\frac{\alpha(m)}{m!} h^m.\end{aligned}$$

Hence, by Lemma 3.1, for all $n = 1, ..., N_1 - 1$,

$$\parallel e_n^0 \parallel_{L^\infty(\sigma_n^0)} \leq \underbrace{\left(\frac{2\alpha(m)}{m!} exp\,(\tau_2 k) \right)}_{C_1^1} h^m,$$

by taking $C_1 = max\{\frac{\alpha(m)}{m!}, C_1^1\}$, Claim 1 is true.

Claim 2. There exists a constant C_2 independent of h such that $\parallel e \parallel_{L^\infty(l)} \leq C_2 h^m$. Define the error $e^p(t)$ on σ^p by $e^p(t) = x(t) - u^p(t)$ and on σ_n^p by $e^p(t) = e_n^p(t) = x(t) - u_n^p(t)$ for all $n = 0, ..., N-1$ and $p = 1, ..., r-1$.

Let $t \in \sigma_n^p$ for $n = 0, 2, ..., N-1$. Then we have from (6) and (7),

$$\begin{aligned}\parallel x - \hat{u}_{n,p} \parallel_{L^\infty(\sigma_n^p)} &\leq \parallel e_{N-N_1+n}^{p-1} \parallel_{L^\infty(\sigma_{N-N_1+n}^{p-1})} + \parallel e_n^{p-1} \parallel_{L^\infty(\sigma_n^{p-1})} + \parallel e_{n-N_1}^p \parallel_{L^\infty(\sigma_{n-N_1}^p)} \\ &+ 2hk \sum_{i=0}^{N-1} \parallel e_i^{p-1} \parallel_{L^\infty(\sigma_d^{p-1})} + hk \sum_{i=0}^{n-1} \parallel e_i^p \parallel_{L^\infty(\sigma_i^p)},\end{aligned}$$

hence,

$$\begin{aligned}\parallel x - \hat{u}_{n,p} \parallel_{L^\infty(\sigma_n^p)} &\leq 2(1 + \tau_2 k) \parallel e^{p-1} \parallel_{L^\infty(\sigma^{p-1})} + \parallel e_{n-N_1}^p \parallel_{L^\infty(\sigma_{n-N_1}^p)} \\ &+ hk \sum_{i=0}^{n-1} \parallel e_i^p \parallel_{L^\infty(\sigma_i^p)} .\end{aligned}$$

Therefore, by Theorem 3.2, for $n = 1, 2, ..., N-1$,

$$\begin{aligned}\parallel e_n^p \parallel_{L^\infty(\sigma_n^p)} &\leq \parallel x - \hat{u}_{n,p} \parallel_{L^\infty(\sigma_n^p)} + \parallel \hat{u}_{n,p} - u_n^p \parallel_{L^\infty(\sigma_n^p)} \\ &\leq \parallel x - \hat{u}_{n,p} \parallel_{L^\infty(\sigma_n^p)} + \frac{\alpha(m)}{m!} h^m.\end{aligned}$$

Then we repeat the next step a times for $n = 1, 2, ..., aN_1 - 1, a = \{1, 2, .., \frac{\tau_2}{\tau_1}\}$,

$$\begin{aligned}\parallel e_n^p \parallel_{L^\infty(\sigma_n^p)} &\leq 2(1 + \tau_2 k) \parallel e^{p-1} \parallel_{L^\infty(\sigma^{p-1})} + hk \sum_{i=0}^{n-1} \parallel e_i^p \parallel_{L^\infty(\sigma_i^p)} \\ &+ \parallel e_{n-N_1}^p \parallel_{L^\infty(\sigma_{n-N_1}^p)} + \frac{\alpha(m)}{m!} h^m,\end{aligned}$$

hence, by Lemma 3.1, for all $n = 0, 1, ..., N-1$,

$$\parallel e_n^p \parallel_{L^\infty(\sigma_n^p)} \leq \frac{\alpha(m) C_1 exp(\tau_2 k)}{m!} h^m + 2(1 + \tau_2 k) exp(\tau_2 k) \parallel e^{p-1} \parallel_{L^\infty(\sigma^{p-1})},$$

then, for all $p = 1, ..., r-1$, $\parallel e_n^p \parallel_{L^\infty(\sigma_n^p)} \leq C_2 h^m$.

Thus, the proof is completed by taking $C = max\{C_1, C_2\}$.

4 Numerical Illustrations

In this section, we present numerical experiments that evaluate the effectiveness of the Taylor collocation method (TCM) in solving problems described by (1). We demonstrate convergence of order m by providing numerical examples that exhibit convergence rates of up to m when applying this method. Additionally, we record the computational time (in CPU time/s) for each example. Our numerical experiments were conducted using Maple version 17 on a personal computer equipped with an Intel Core i7-1165G7 CPU @ 2.80GHz and 16 GB of RAM, running the MS Windows 7 operating system. As the values of both n and m increase, the absolute error function decreases, thereby confirming the theoretical estimates outlined in Section 3.

Example 4.1 Consider the neutral double delay Volterra integral equation

$$x(t) = g(t) + x(t - \frac{1}{2}) + x(t-1) + \int_{t-1}^{t-\frac{1}{2}} (ts + \cos(t+s))x(s)ds$$

for $t \in [1, 5]$, and g is chosen so that the exact solution $x(t) = \cos(t) + 1$, $x(t) = \Phi(t)$ for $t \in [0, 1]$. The absolute errors and computational time for $(N, m) = \{(2, 2), (4, 4), (6, 6), (8, 8)\}$ are presented in Table 1. The exact and approximate solutions for $N = m = 8$ are shown in Figure 1(a).

t	$N=m=2$	$N=m=4$	$N=m=6$	$N=m=8$
1.00	0.0	0.0	0.0	0.0
2.00	$1.58e-02$	$1.02e-5$	$1.29e-9$	$7.08e-11$
3.00	$1.27e-02$	$2.89e-6$	$2.52e-9$	$9.70e-10$
4.00	$4.45e-02$	$2.53e-4$	$1.17e-7$	$2.53e-08$
5.00	$1.44e-01$	$7.31e-3$	$3.81e-6$	$8.50e-07$
CPU	1.79 s	1.95 s	3.18 s	5.29 s

Table 1: Absolute errors for Example 4.1.

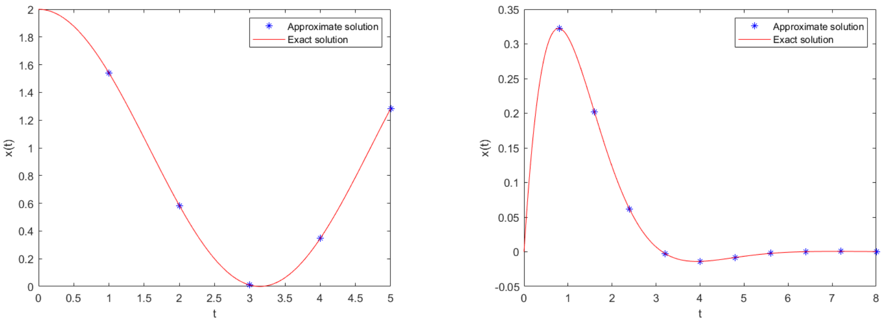
Example 4.2 Consider the neutral double delay Volterra integral equation

$$x(t) = g(t) + x(t-0.4) + x(t-0.8) + \int_{t-0.8}^{t-0.4} (t-s)e^{t-s}x(s)ds$$

for $t \in [1, 8]$, and g is chosen so that the exact solution $x(t) = e^{-t}\sin(t)$, $\Phi(t) = x(t)$ for $t \in [0, 0.8]$. The absolute errors and computational time for $(N, m) = \{(2, 6), (2, 8), (2, 10), (4, 4), (4, 6), (4, 8)\}$ are presented in Table 2. Figure 1(b) displays the exact and approximate solutions for $N = 4, m = 8$.

	$N=2$			$N=4$		
t	$m=6$	$m=8$	$m=10$	$m=4$	$m=6$	$m=8$
0.8	0	0	0	0	0	0
1.6	$5.22e-07$	$2.26e-09$	$2.82e-10$	$6.28e-06$	$9.24e-09$	$2.81e-10$
2.4	$1.57e-06$	$9.66e-09$	$1.05e-09$	$2.59e-05$	$2.64e-08$	$1.08e-09$
3.2	$5.31e-06$	$3.57e-08$	$3.53e-09$	$9.19e-05$	$8.57e-08$	$3.75e-09$
4.0	$1.88e-05$	$1.30e-07$	$1.18e-08$	$3.21e-04$	$2.95e-07$	$1.29e-08$
4.8	$6.69e-05$	$4.67e-07$	$3.97e-08$	$1.12e-03$	$1.02e-06$	$4.46e-08$
5.6	$3.65e-04$	$1.66e-06$	$1.34e-07$	$3.91e-03$	$3.55e-06$	$1.54e-07$
6.4	$8.30e-04$	$5.86e-06$	$4.56e-07$	$1.35e-02$	$1.23e-05$	$5.32e-07$
7.2	$2.90e-03$	$2.06e-05$	$1.55e-06$	$4.71e-02$	$4.25e-05$	$1.83e-06$
8.0	$1.07e-02$	$7.81e-05$	$4.98e-06$	$1.65e-01$	$1.49e-04$	$6.28e-06$
CPU	1.57 s	1.90 s	3.85 s	1.75 s	2.56 s	3.82 s

Table 2: Absolute errors of Example 4.2.



(a) Example 1 ($N=m=8$)   (b) Example 2 ($N=4, m=8$)

Figure 1: The exact and approximate solutions.

5 Conclusion

In the concluding remarks of our study, we have addressed the reviewer's suggestion to bolster the description of the novelty arising from our results. Our exploration of the Taylor collocation method for approximating the solution of linear DDVIEs has yielded noteworthy outcomes. Our findings underscore this method's effectiveness and ease of implementation, emphasizing its reliability in generating approximate solutions through iterative formulas without necessitating the resolution of algebraic equation systems. The numerical examples presented in this paper demonstrate the method's convergence with high accuracy. The results obtained from these examples validate our theoretical estimates and highlight the proposed approach's practical applicability. It is worth noting that the numerical outcomes consistently align with the expected convergence patterns, affirming the reliability of the Taylor collocation method in the context of linear DDVIEs. Our future research endeavors will extend beyond the scope of this paper. We plan to generalize the proposed method to tackle DDVIEs in two dimensions and a system of DDVIEs, further expanding its applicability to a broader class of problems. This extension is anticipated to contribute to the method's versatility and potential to address complex systems exhibiting intricate temporal dependencies.

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