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Stability Analysis of a Coupled System of Two Nonlinear Differential Equations with Boundary Conditions

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Abstract: We study the antiplane frictional contact models for electro-viscoelastic materials. The material is assumed to be electro-viscoelastic and is modelled by a slip rate dependent friction law and the foundation is assumed to be electrically conductive. First, we give the mathematical model of our phenomena. Second, we describe the classical formulation for the antiplane problem and we give the corresponding variational formulation which is given by a method of coupling of an evolutionary variational quality for the displacement field and a time-dependent variational equation for the electric potential field. Then we prove the existence of a unique weak solution to the model.

Keywords: *nonlinear system; electro-viscoelastic material; contact problem; weak solution; boundary condition.*

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1 Introduction

Contact mechanics is the study of the deformation of solids that are in contact with each other at one or more points. Antiplane shear deformations are one of the simplest classes of deformations that solids can undergo. The antiplane shear deformation is the deformation that we expect to appear after loading a long cylinder in the direction of its generators so that the displacement field is parallel to the generators of the cylinder and independent of the axial coordinate.

The piezoelectric effect results from the coupling between electrical and mechanical properties, in which the body has the ability to produce an electrical field when a mechanical stress is present and, conversely, under the action of an electric field the body undergoes a mechanical stress. The models for piezoelectric materials can be found in [1], [2] and [3].

In this paper, we study an antiplane contact problem for electro-viscoelastic materials with the slip rate–dependent friction law, in the framework of the Mathematical Theory of Contact Mechanics, when the foundation is electrically conductive. We consider the case of antiplane shear deformation, i.e., the displacement is parallel to the generators of the cylinder and independent of the axial coordinate (see [5], [6] and the references therein). Such kind of problems was studied in a number of papers in the context of various constitutive laws and contact conditions (see, e.g. [12], [13] and [14]).

Our paper is structured as follows. In Section 2, we present the mechanical model for the quasistatic antiplane contact problem. In Section 3, we introduce the notation, list the assumption on problem's data, derive the variational formulation of the problem. Finally, in Section 4, we state our main existence and uniqueness result, i.e., Theorems 4.1-4.2. The proof of this result is carried out in several steps and is based on the arguments of evolutionary inequalities.

2 The Mathematical Model

We consider a piezoelectric body $\mathcal{B}$ identified with a region in $\mathbb{R}^3$ it occupies in a fixed and undistorted reference configuration. We assume that $\mathcal{B} = \Omega \times (-\infty, +\infty)$ is a cylinder with generators parallel to the x_3-axis with a cross-section which is a regular region Ω in the x_1, x_2-plane, $Ox_1x_2x_3$ being a Cartesian coordinate system. The cylinder is acted upon by body forces of density $\mathbf{f_0}$ and has volume free electric charges of density q_0. It is also constrained mechanically and electrically on the boundary. To describe the boundary conditions, we denote by $\partial\Omega{=}\Gamma$ the boundary of Ω and we assume a partition of Γ into three open disjoint parts Γ_1, Γ_2 and Γ_3, on the one hand, and a partition of $\Gamma_1 \cup \Gamma_2$ into two open parts Γ_a and Γ_b. On the other hand, we assume that the one-dimensional measures of Γ_1 and Γ_a, denoted meas Γ_1 and meas Γ_a, are positive. Let $T > 0$ and let $[0, T]$ be the time interval of interest.

The cylinder is clamped on $\Gamma_1 \times (-\infty, +\infty)$ and therefore the displacement field vanishes there, surface tractions of density $\mathbf{f_2}$ act on $\Gamma_2 \times (-\infty, +\infty)$. We also assume that the electrical potential vanishes on $\Gamma_a \times (-\infty, +\infty)$ and a surface electrical charge of density q_2 is prescribed on $\Gamma_b \times (-\infty, +\infty)$. The cylinder is in contact over $\Gamma_3 \times (-\infty, +\infty)$ with a conductive obstacle, the so-called foundation. The contact is frictional and is modeled by Tresca's law.

We denote by $\mathcal{S}^3$ the space of the second-order symmetric tensors on $\mathbb{R}^3$, and we define the inner products and the corresponding norms on $\mathbb{R}^3$ and $\mathcal{S}^3$ by

$$\text{u.v} = u_i v_i,\ \|\text{v}\| = (v.v)^{\frac{1}{2}} \text{ for all } \text{u} = (u_i), \text{v} = (v_i) \in \mathbb{R}^3, 1 \le i,j \le 3,$$

$$\sigma.\tau = \sigma_{i,j}\tau_{i,j}, \|\tau\| = (\tau.\tau)^{\frac{1}{2}} \text{ for all } \sigma = (\sigma_{i,j}), \tau = \tau_{i,j} \in \mathbb{S}^3,\ 1 \le i,j \le 3.$$

We assume that

$$\mathbf{f_0} = (0,0,f_0) \text{ with } f_0 = f_0(x_1,x_2,t) : \Omega \times [0,T] \longrightarrow \mathbb{R}, \tag{1}$$

and

$$\mathbf{f_2} = (0,0,f_2) \text{ with } f_2 = f_2(x_1,x_2,t) : \Gamma_2 \times [0,T] \longrightarrow \mathbb{R}. \tag{2}$$

The body forces (1) and the surface tractions (2) would be expected to give rise to a deformation of the elastic cylinder whose displacement, denoted by u, is of the form

$$\mathbf{u} = (0,0,u), \text{ with } q_0 : \Omega \times [0,T] \longrightarrow \mathbb{R}. \tag{3}$$

Such kind of deformation, associated to a displacement field of the form (3), is called an antiplane shear. We assume also that

$$q_0 = q_0(x_1,x_2,t) \text{ with } u = u(x_1,x_2,t) : \Omega \times [0,T] \longrightarrow \mathbb{R}, \tag{4}$$

$$q_2 = q_2(x_1,x_2,t) \text{ with } q_2 : \Gamma_b \times [0,T] \longrightarrow R. \tag{5}$$

The electric charges (4), (5) would be expected to give rise to deformations and to the electric charges of the piezoelectric cylinder corresponding to an electric potential field φ which is independent of x_3 and has the form

$$\varphi = \varphi(x_1,x_2,t) : \Omega \times [0,T] \longrightarrow \mathbb{R}. \tag{6}$$

The infinitesimal strain tensor, denoted by $\varepsilon(u) = (\varepsilon_{i,j}(u))$, is defined by

$$\varepsilon_{i,j}(u) = \frac{1}{2}(u_{i,j} + u_{j,i}), 1 \le i,j \le 3, \tag{7}$$

where the index that follows the comma indicates a partial derivative with respect to the corresponding component of the spatial variable. Moreover, in the sequel, the convention of summation upon a repeated index is used. From (3) and (7), it follows that, in the case of the antiplane problem, the infinitesimal strain tensor becomes

$$\varepsilon(u) = \begin{pmatrix} 0 & 0 & \frac{1}{2}u_{,1} \\ 0 & 0 & \frac{1}{2}u_{,2} \\ \frac{1}{2}u_{,1} & \frac{1}{2}u_{,2} & 0 \end{pmatrix}. \tag{8}$$

We also denote by $E(\varphi) = (E_i(\varphi))$ the electric field and by D=(D_i) the electric displacement field, where

$$\varepsilon_{i,j}(u) = \frac{1}{2}(u_{i,j} + u_{j,i}), \tag{9}$$

$$E_i(\varphi) = -\varphi_{,i}. \tag{10}$$

Let $\sigma = (\sigma_{ij})$ denote the stress field. We suppose that the material's behavior is modelled by an electro-viscoelastic constitutive law of the form

$$\sigma = 2\theta\varepsilon(\dot{u}) + \zeta tr\varepsilon(\dot{u})I + 2\mu\varepsilon(u) + \lambda tr\varepsilon(u)I - \mathcal{E}^* E(\varphi), \tag{11}$$

$$D = \mathcal{E}^*\varepsilon(u) + \beta E(\varphi), \tag{12}$$

where ζ and θ are viscosity coefficients, λ and μ are the Lame coefficients, tr $\varepsilon(u) = \varepsilon_{ii}(u)$, I is the unit tensor in $\mathbb{R}^3, \alpha$ is the electric permittivity constant, $\mathcal{E}$ represents the third-order piezoelectric tensor and $\mathcal{E}^*$ is its transpose. We assume that

$$\mathcal{E}\varepsilon = \begin{pmatrix} e(\varepsilon_{13} + \varepsilon_{31}) \\ e(\varepsilon_{23} + \varepsilon_{32}) \\ e(\varepsilon_{33}) \end{pmatrix}, \ \forall \varepsilon = (\varepsilon_{i,j}) \in \mathcal{S}^3, \tag{13}$$

where e is a piezoelectric coefficient. We also assume that the coefficients θ, μ, β and e depend of the spatial variables x_1, x_2, but are independent on the spatial variable x_3. Since $\mathcal{E}\varepsilon.\mathrm{v} = \varepsilon.\mathcal{E}^*\mathrm{v}$ for all $\varepsilon \in \mathcal{S}^3$, $\mathrm{v} \in \mathbb{R}^3$, it follows from (13) that

$$\mathcal{E}^*\mathrm{v} = \begin{pmatrix} 0 & 0 & ev_1 \\ 0 & 0 & ev_2 \\ ev_1 & ev_2 & ev_3 \end{pmatrix}, \ \forall \ \mathrm{v} = (v_i) \in \mathbb{R}^3. \tag{14}$$

Here and below the dot above represents the derivative with respect to the time variable. The stress field is given by the matrix

$$\sigma = \begin{pmatrix} 0 & 0 & \sigma_{13} \\ 0 & 0 & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & 0 \end{pmatrix}. \tag{15}$$

In the antiplane context (3), (6), when using the constitutive equations (11)-(12) and equalities (13)-(14), it follows that the stress field and the electric displacement field are given by

$$\sigma = \begin{pmatrix} 0 & 0 & \theta\dot{u}_{,1} + \mu u_{,1} + e\varphi_{,1} \\ 0 & 0 & \theta\dot{u}_{,2} + \mu u_{,2} + e\varphi_{,2} \\ \theta\dot{u}_{,1} + \mu u_{,1} + e\varphi_{,1} & \theta\dot{u}_{,2} + \mu u_{,2} + e\varphi_{,2} & 0 \end{pmatrix}, \tag{16}$$

$$\mathbf{D} = \begin{pmatrix} eu_{,1} - \beta\varphi_{,1} \\ eu_{,2} - \beta\varphi_{,2} \\ 0 \end{pmatrix}. \tag{17}$$

We assume that the process is mechanically quasistatic and electrically static and therefore is governed by the equilibrium equations

$$\mathrm{Div}\sigma + \mathbf{f}_0 = 0, \tag{18}$$

$$D_{i,i} - q_0 = 0 \quad \text{in } \mathcal{B} \times (0, T), \tag{19}$$

where $\mathrm{Div}\sigma = (\sigma_{ij,j})$ represents the divergence of the tensor field σ. Thus, keeping in mind (1), (3), (4), (6), (16) and (17), the equilibrium equations above are reduced to the following scalar equations:

$$\mathrm{div}(\theta\nabla\dot{u} + \mu\nabla u) + \mathrm{div}(e\nabla\varphi) + f_0 = 0, \ \text{ in } \ \Omega \times (0, T), \tag{20}$$

$$\mathrm{div}(e\nabla u) - \mathrm{div}(\beta\nabla\varphi) = q_0, \ \text{in } \Omega \times (0, T). \tag{21}$$

Here and below we use the notation

$$\mathrm{div}\tau = \tau_{1,2} + \tau_{1,2} \ \text{ for } \tau = (\tau_1(x_1, x_2, t), \tau(x_1, x_2, t)),$$

$$\nabla v = (v_{,1}, v_{,2}),\ \partial_\nu v = v_{,1}\nu_1 + v_{,2}\nu_2 \text{ for } v = v(x_1, x_2, t).$$

Recall that since the cylinder is clamped on $\Gamma_1 \times (-\infty, +\infty)$, the displacement field vanishes there. Thus (3) implies

$$u = 0 \text{ on } \Gamma_1 \times (0, T), \tag{22}$$

the electrical potential vanishes too on $\Gamma_a \times (-\infty, +\infty)$, thus (4) implies that

$$\varphi = 0 \text{ on } \Gamma_a \times (0, T). \tag{23}$$

Let ν denote the unit normal on $\Gamma \times (-\infty, +\infty)$. We have

$$\nu = (\nu_1, \nu_2, 0) \tag{24}$$

with $\nu_i = \nu_i(x_1, x_2) : \Gamma \longrightarrow \mathbb{R}, i = 1, 2$. For a vector $\mathbf{v}$ we denote by v_ν and $\mathbf{v}_\tau$ its normal and tangential components on the boundary which are given by

$$v_\nu = \mathbf{v}.\nu, \quad \mathbf{v}_\tau = \mathbf{v} - v_\nu\nu, \tag{25}$$

respectively. In (25), and everywhere in this paper, "." represents the inner product on the space $\mathbb{R}^3(d = 2, 3)$. Moreover, for a given stress field σ, we denote by σ_ν and σ_τ the normal and tangential components on the boundary, respectively, i.e.,

$$\sigma_\nu = (\sigma\nu).\nu, \quad \sigma_\tau = \sigma\nu - \sigma_\nu\nu. \tag{26}$$

From (16), (17) and (24), we deduce that the Cauchy stress vector and the normal component of the electric displacement field are given by

$$\sigma\nu = (0, 0, \theta\partial_\nu\dot{u} + \mu\partial_\nu u + \partial_\nu\varphi), \quad \mathbf{D}.\nu = e\partial_\nu u - \beta\partial_\nu\varphi. \tag{27}$$

Here and subsequently we use the notations $\partial_\nu u = u_{,1}\nu_1 + u_{,2}\nu_2$ and $\partial_\nu\varphi = \varphi_{,1}\nu_1 + \varphi_{,2}\nu_2$. When keeping in mind the traction boundary condition $\sigma\nu = \mathbf{f_2}$ on $\Gamma_2 \times (-\infty, +\infty)$ and the electric condition $\mathbf{D}.\nu = q_2$ on $\Gamma_b \times (-\infty, +\infty)$, it follows from (2), (5) and (27) that

$$\theta\partial_\nu\dot{u} + \mu\partial_\nu u + \partial_\nu\varphi = f_2 \text{ on } \Gamma_2 \times (0, T), \tag{28}$$

$$e\partial_\nu u - \beta\partial_\nu\varphi = q_2 \text{ on } \Gamma_b \times (0, T). \tag{29}$$

We now describe the frictional contact condition on $\Gamma_3 \times (-\infty, +\infty)$. First, we note that from (3), (24) and (25), we find $u_\nu = 0$, which shows that the contact is bilateral, that is, the contact is kept during all the process. Using now (3), (24)-(27), we conclude that

$$\mathbf{u}_\tau = (0, 0, u), \tag{30}$$

$$\sigma_\tau = (0, 0, \sigma_\tau), \tag{31}$$

where

$$\sigma_\tau = \theta\partial_\nu\dot{u} + \mu\partial_\nu u + \partial_\nu\varphi. \tag{32}$$

We assume that the friction is invariant with respect to the x_3-axis and for all $t \in [0, T]$, it is modelled by the following conditions on Γ_3:

$$\begin{cases} |\sigma_\tau| \leq g(|\dot{u}_\tau|) \text{ on } \Gamma_3 \times [0, T], \\ \sigma_\tau = -g(|\dot{u}|)\frac{\dot{u}_\tau}{|\dot{u}_\tau|}, \text{ on } \Gamma_3 \times [0, T]. \end{cases} \tag{33}$$

Here $g : \Gamma_3 \times \mathbb{R} \longrightarrow \mathbb{R}^3$ is a given function, the friction bound, and $\dot{u}_\tau$ represents the tangential velocity on the contact boundary. In (33), the strict inequality holds in the stick zone and the equality is true in the slip zone.

Using now (30), (32), it is straightforward to see that the conditions (33) imply

$$\begin{cases} |\theta\partial_\nu \dot{u} + \mu\partial_\nu u + e\partial_\nu \varphi| \leq g(|\dot{u}|), \\ \theta\partial_\nu \dot{u} + \mu\partial_\nu u + e\partial_\nu \varphi = -g(|\dot{u}|)\frac{\dot{u}}{|\dot{u}|}, \ \text{ on } \ \Gamma_3 \times [0, T]. \end{cases} \tag{34}$$

Finally, we prescribe the initial displacement

$$u(0) = u_0, \ \text{ in } \ \Omega, \tag{35}$$

where u_0 is a given function on Ω.

Problem 2.1 Find a displacement field $u : \Omega \times [0, T] \to \mathbb{R}$ and the electric potential field $\varphi : \Omega \times [0, T] \to \mathbb{R}$ such that

$$div(\theta\nabla\dot{u} + \mu\nabla u) + div(e\nabla\varphi) + f_0 = 0, \quad \text{ in } \quad \Omega \times (0, T), \tag{36}$$

$$div(e\nabla u) - div(\beta\nabla\varphi) = q_0, \ \text{ in } \ \Omega \times (0, T), \tag{37}$$

$$u = 0 \ , \quad \text{ on } \Gamma_1 \times (0, T), \tag{38}$$

$$\theta\partial_\nu \dot{u} + \mu\partial_\nu u + e\partial_\nu \varphi = f_2, \ \text{ on } \ \Gamma_2 \times (0, T), \tag{39}$$

$$\begin{cases} |\theta\partial_\nu \dot{u} + \mu\partial_\nu u + e\partial_\nu \varphi| \leq g(|\dot{u}|), \\ \theta\partial_\nu \dot{u} + \mu\partial_\nu u + e\partial_\nu \varphi = -g(|\dot{u}|)\frac{\dot{u}}{|\dot{u}|}, \ \text{ on } \ \Gamma_3 \times (0, T), \end{cases} \tag{40}$$

$$\varphi = 0, \ \text{ on } \ \Gamma_a \times (0, T), \tag{41}$$

$$e\partial_\nu u - \beta\partial_\nu \varphi = q_2, \ \text{ on } \ \Gamma_b \times (0, T), \tag{42}$$

$$u(0) = u_0, \ \text{ in } \ \Omega \times (0, T). \tag{43}$$

3 Assumptions and Varitional Formulation

To obtain a variational formulation for the mechanical problem (36)-(43), we introduce the function spaces $V = \{v \in H^1(\Omega) : v = 0 \ \text{ on} \, \Gamma_1\}$ and $W = \{\psi \in H^1(\Omega) : \psi = 0 \ \text{ on} \, \Gamma_1\}$, and here and below, we write w for the trace γw of a function $w \in H^1$ on Γ_1.

Since $meas \, \Gamma_1 > 0$ and $meas \Gamma_a > 0$, it is well known that V and W are the real Hilbert spaces with the inner products

$$(u, v)_V = \int_\Omega \nabla u . \nabla v dx, \forall u, v \in V,$$

and

$$(\varphi, \psi)_W = \int_\Omega \nabla \varphi . \nabla \psi dx, \forall \varphi, \psi \in W.$$

Moreover, the associated norms

$$\|v\|_V = \|\nabla v\|_{L^2(\Omega)^2}, \forall v \in V, \tag{44}$$

$$\|\psi\|_W = \|\nabla \psi\|_{L^2(\Omega)^2}, \forall \psi \in W, \tag{45}$$

are equivalent on V and W, respectively, with the usual norm $\|.\|_{H^1(\Omega)}$. By Sobolev's trace theorem, we deduce that there exist two positive constants $c_V > 0$ and $c_W > 0$ such that

$$\|v\|_{L^2(\Gamma_3)} \leq c_V \|v\|_V, \forall v \in V, \tag{46}$$

$$\|v\|_{L^2(\Gamma_3)} \leq c_W \|\psi\|_W, \forall \psi \in W. \tag{47}$$

For a real Banach space $(X, \|.\|_X)$, we use the usual notation for the spaces $L^p(0,T,X)$ and $W^{k,p}(0,T,X)$, where $1 \leq p \leq \infty, k = 1,2...$; we also denote by $C(0,T,X)$ and $C^1(0,T,X)$ the spaces of continuous and continuously differentiable functions on $[0,T]$ with values in X, with the respective norms

$$\|x\|_{C(0,T,X)} = \max_{t\in[0,T]} \|x\|_X,$$

and

$$\|x\|_{C^1(0,T,X)} = \max_{t\in[0,T]} \|x\|_X + \max_{t\in[0,T]} \|\dot{x}\|_X,$$

and we use the standard notations for the Lebesgue space $L^2(0,T,X)$ as well as the Sobolev space $W^{1,2}(0,T,X)$. In particular, recall that the norm on the space $L^2(0,T,X)$ is given by the formula

$$\|u\|^2_{L^2(0,T,X)} = \int_0^T \|u(t)\|^2_X dt,$$

and the norm on the space $W^{1,2}(0,T,X)$ is defined by the formula

$$\|u\|^2_{W^{1,2}(0,T,X)} = \int_0^T \|u(t)\|^2_X dt + \int_0^T \|\dot{u}(t)\|^2_X dt.$$

In the study of Problem 2.1, we assume that the viscosity coefficient and the electric permittivity coefficient satisfy

$$\theta \in L^\infty(\Omega) \text{ and there } \theta^* > 0 \text{ such that } \theta(x) \geq \theta^* \quad a.e\ x \in \Omega, \tag{48}$$

$$\beta \in L^\infty(\Omega) \text{ and there } \beta^* > 0 \text{ such that} \beta(x) \geq \beta^* \quad a.e\ x \in \Omega. \tag{49}$$

We also assume that the Lame coefficient and the piezoelectric coefficient satisfy

$$\mu \in L^\infty \text{ and } \mu(x) > 0, \text{a.e } x \in \Omega, \tag{50}$$

$$e \in L^\infty. \tag{51}$$

The forces, tractions, volume and surface free charge densities have the regularity

$$f_0 \in W^{1,2}(0,T,L^2(\Omega));\ f_2 \in W^{1,2}(0,T,L^2(\Gamma_2)), \tag{52}$$

$$q_0 \in W^{1,2}(0,T,L^2(\Omega)), \tag{53}$$

$$q_2 \in W^{1,2}(0,T,L^2(\Gamma_b)),\ q_2 = 0 \text{ a.e } x \in \Gamma_b. \tag{54}$$

The friction bound satisfies

$$\begin{cases} a)\ g : \Gamma_3 \times \mathbb{R} \longrightarrow \mathbb{R}^+ \\ b)\ \exists\ L_g > 0 \text{ such that } |g(x,r_1) - g(x,r_2)| \leq L_g|r_1 - r_2| \\ c)\ x \longrightarrow g(x,r) \text{ is Lebesgue mesurable on } \Gamma_3\ \forall r \in \mathbb{R} \\ d)\ \text{ the mapping } x \longrightarrow g(x,0) \text{ belongs to } L^2(\Gamma_3), \end{cases} \tag{55}$$

the functional $j : v \times v \longrightarrow \mathbb{R}$ is given by

$$j(\dot{u}, v) = \int_{\Gamma_3} g(|\dot{u}|)|v|da, \forall v \in V. \tag{56}$$

Let $\eta_1, \eta_2, v_1, v_2 \in V$, by using (55)-(56), we find that

$$j(\eta_1, v_2) - j(\eta_1, v_1) + j(\eta_2, v_1) - j(\eta_2, v_2)| \leq L_g \|\eta_1 - \eta_2\|_{L^2(\Gamma_3)} \|v_1 - v_2\|_{L^2(\Gamma_3)},$$

and, keeping in mind (46), we obtain

$$j(\eta_1, v_2) - j(\eta_1, v_1) + j(\eta_2, v_1) - j(\eta_2, v_2) \leq c_V^2 L_g \|\eta_1 - \eta_2\|_V \|v_1 - v_2\|_V, \tag{57}$$

the initial displacement is such that

$$u_0 \in V. \tag{58}$$

We will use the functions $f : [0, T] \longrightarrow V$ and $q : [0, T] \longrightarrow W$ by

$$(f, v)_V = \int_\Omega f_0 v dx + \int_{\Gamma_2} f_2 v da, \ \forall v \in V, \tag{59}$$

$$(q, \psi)_V = \int_{\Gamma_b} q_2 \psi dx + \int_\Omega q_0 \psi da, \ \forall \psi \in W. \tag{60}$$

The defnition of f and q are based on Riesz's representation theorem, moreover, it follows from assumptions (59)-(60) that the integrals above are well defined and

$$f \in W^{1,2}(0, T, V), \tag{61}$$

$$q \in W^{1,2}(0, T, W). \tag{62}$$

Next, we define the bilinear forms $a_\theta : V \times V \longrightarrow \mathbb{R}$, $a_\mu : V \times V \longrightarrow \mathbb{R}$, $a_e : V \times W \longrightarrow \mathbb{R}$, $a_e : W \times V \longrightarrow \mathbb{R}$, $a_\beta : W \times W \longrightarrow \mathbb{R}$ by the equalities

$$a_\theta(u, v) = \int_\Omega \theta \nabla u . \nabla v dx, \tag{63}$$

$$a_\mu(u, v) = \int_\Omega \mu \nabla u . \nabla v dx, \tag{64}$$

$$a_e(u, \varphi) = \int_\Omega e \nabla \varphi . \nabla v dx = a_e(\varphi, v), \tag{65}$$

$$a_\beta(\varphi, \psi) = \int_\Omega \beta \nabla \varphi . \nabla \psi dx. \tag{66}$$

$$\tag{67}$$

Assumptions (48)-(51) imply that the integrals above are well defned and, when using (44)-(47), it follows that the forms a_θ, a_μ, a_e and a_β are continuous, moreover, the forms a_θ, a_μ and a_β are symmetric and, in addition, the form a_θ is V-elliptic since

$$a_\theta(u, v) \leq \|\theta\|_{L^\infty(\Omega)} \|u\|_V \|v\|_V \ \forall u, v \in V, \tag{68}$$

$$a_\theta(v, v) \geq \theta^* \|v\|_V^2 \ \ \forall v \in V. \tag{69}$$

The variational formulation of our problem is based on the following result.

Lemma 3.1 *If (u,φ) is a smooth solution to Problem 2.1, then $(u(t),\varphi(t)) \in X$ and*

$$a_\theta(\dot{u}(t), v-\dot{u}(t)) + a_\mu(u(t), v-\dot{u}(t)) + a_e(\varphi(t), v-\dot{u}(t)) + j(\dot{u}(t), v) - j(\dot{u}(t),\dot{u}(t)) \geq (f, v-\dot{u})_V, \forall v \in V, t \in [0,T], \quad (70)$$

$$a_\beta(\varphi,\psi) - a_e(u,\psi) = (q,\psi)_W, \forall \psi \in W, \quad (71)$$

$$u(0) = u_0. \quad (72)$$

Proof. Let (u,φ) denote a smooth solution to Problem 2.1, we have $u(t) \in V$, $\dot{u} \in V$ and $\varphi \in W$ a.e. $t \in [0,T]$ and let $v \in V$, $\psi \in W$, we multiply equations (36)-(37) by $(v-\dot{u}(t))$, ψ, integrate the result on Ω, and use Green's formula (36), and from (38)-(40), we get

$$\int_\Omega \theta\nabla\dot{u}(t).\nabla(v-\dot{u}(t))dx + \int_\Omega \mu\nabla u(t).\nabla(v-\dot{u}(t))dx + \int_\Omega e\nabla\varphi.\nabla(v-\dot{u}(t))dx + \int_{\Gamma_3} g(|\dot{u}(t)|)|\dot{u}(t)|da - \int_{\Gamma_3} g(|\dot{u}(t)|)|v|da = \int_\Omega f_0(v-\dot{u}(t))dx + \int_{\Gamma_2} f_2(v-\dot{u}(t))da \forall v \in V, t \in (0,T). \quad (73)$$

Now, using (43),(59) and (63)-(65), we obtain (70) and (72), and from (41)-(42), we get

$$\int_\Omega \beta\nabla\varphi(t).\nabla\psi dx - \int_\Omega e\nabla u(t).\nabla\psi dx = \int_\Omega q_0(t)\psi dx - \int_{\Gamma_b} q_2(t)\psi da, \forall\psi \in W, \forall t \in [0,T]. \quad (74)$$

Using (60) and (66)-(68), we find (71). ■

Finally, the variational formulation of Problem (70)–(72) is given as follows.

Problem 3.1 Find a displacement field $u : [0,T] \longrightarrow V$ and an electric potential field $\varphi : [0,T] \longrightarrow W$ such that

$$a_\theta(\dot{u}(t), v-\dot{u}(t)) + a_\mu(u(t), v-\dot{u}(t)) + a_e(\varphi(t), v-\dot{u}(t)) + j(\dot{u}(t), v) - j(\dot{u}(t),\dot{u}(t)) \geq (f, v-\dot{u})_V, \forall v \in V, t \in [0,T], \quad (75)$$

$$a_\beta(\varphi,\psi) - a_e(u,\psi) = (q,\psi)_W, \forall \psi \in W, \quad (76)$$

$$u(0) = u_0. \quad (77)$$

We notice that the varitional problem (75)–(77) is formulated in terms of a displacement field and electrical potential field. The existence of the unique solution to (75)–(77) is stated and proved in the next section.

4 Existence and Uniqueness of a Weak Solution

Theorem 4.1 *Assume* (48)–(62), *then there exists* L_0, *which depends on* $\Omega, \Gamma_1, \Gamma_2, \Gamma_3$, *and if* $L_g < L_0$, *there exists a unique solution* u *to Problem 3.1 satisfying*

$$u \in W^{2,2}(0,T,V). \tag{78}$$

Proof of Theorem 4.1. The proof of Theorem 4.1 will be carried out in several steps.

First Step : We consider the following problem.

Problem 4.1 Find a displacement field $u : [0,T] \longrightarrow V$ such that

$$a(u(t), v - \dot{u}(t)) + b(\dot{u}(t), v - \dot{u}(t)) + j(\dot{u}(t), v) - j(\dot{u}(t), \dot{u}(t)) \geq (F(t), v - \dot{u}(t))_V, \quad \forall v \in V, t \in [0,T], \tag{79}$$

$$u(0) = u_0. \tag{80}$$

In the study of the Cauchy problem (79)-(80), we assume that

$$\begin{cases} a : V \times V \longrightarrow \mathbb{R} \text{ is a bilinear form and there exists } M > 0 \text{ such that} \\ |a(u,v)| \leq M \|u\|_V |v\|_V, \ \forall u, v \in V. \end{cases} \tag{81}$$

$$\begin{cases} b : V \times V \longrightarrow \text{ is a bilinear symmetric form and} \\ (a) \ \text{ there exists } M' > 0 \text{ such that } |b(u,v)| \leq \ M' \|u\|_V \|v\|_V, \ \forall u, v \in X, \\ (b) \text{ there exists } m' > 0 \text{ such that } \ b(v,v) \geq m' \|v\|_V^2 \ \forall v \in V, \end{cases} \tag{82}$$

$$\begin{cases} j : V \times V \longrightarrow \mathbb{R} \textit{ and} \\ a) \text{ for all } \eta \in V, \ j(\eta, .) \text{ is convex and I.S.C. on V,} \\ b) \text{ there exists } \ \alpha \geq 0 \ \text{ such that} \\ |j(\eta_1, v_2) - j(\eta_1, v_1) + j(\eta_2, v_1) - j(\eta_2, v_2)| \leq \alpha \|\eta_1 - \eta_2\|_V \|v_1 - v_2\|_V, \end{cases} \tag{83}$$

$$\forall \eta_1, \eta_2, v_1, v_2 \in V,$$

$$u_0 \in V, \tag{84}$$

$$F \in W^{1,2}(0,T,V). \tag{85}$$

Under assumptions (81)–(85), we have the following result.

Lemma 4.1 *Assume that (81)-(85) hold, then if* $m' > \alpha$, *there exists a unique solution* $u \in W^{1,2}(0,T,V)$ *to problem (79)-(80).*

Proof. For any $t_1, t_2 \in [0,T]$, we use (79) and get

$$a(u(t_1), v - \dot{u}(t_1)) + b(\dot{u}(t_1), v - \dot{u}(t_1)) + j(\dot{u}(t_1), v) - j(\dot{u}(t_1), \dot{u}(t_1)) \geq (F(t_1), v - \dot{u}(t_1))_V, \forall v \in V, \tag{86}$$

$$a(u(t_2), v - \dot{u}(t_2)) + b(\dot{u}(t_2), v - \dot{u}(t_2)) + j(\dot{u}(t_2), v) - j(\dot{u}(t_2), \dot{u}(t_2)) \geq (F(t_2), v - \dot{u}(t_2))_V, \forall v \in V. \tag{87}$$

We take $v = \dot{u}(t_2)$ in the first inequality, and $v = \dot{u}(t_1)$ in the second one, and add the results to obtain

$$b(\dot{u}(t_1) - \dot{u}(t_2), \dot{u}(t_1) - \dot{u}(t_2)) + j(\dot{u}(t_1), \dot{u}(t_2)) - j(\dot{u}(t_1), \dot{u}(t_2)) + j(\dot{u}(t_2), \dot{u}(t_1)) - j(\dot{u}(t_2), \dot{u}(t_2)) \leq a(u(t_1) - u(t_2), \dot{u}(t_2) - \dot{u}(t_1)) + (F(t_1) - F(t_2), \dot{u}(t_2) - \dot{u}(t_1)). \quad (88)$$

We use now assumptions (81)-(83) to find

$$\|\dot{u}(t_1) - \dot{u}(t_2)\|_V \leq C(\|u(t_1) - u(t_2)\|_V + \|F(t_1) - F(t_2)\|_V), \quad (89)$$

where $C = \max\{\frac{M}{m'-\alpha}, \frac{1}{m'-\alpha}\}$. This inequality combined with the reguarlarity $u \in C^1(0, t, V)$ shows that $\dot{u} : [0, T] \longrightarrow V$ is an absolutely contunuous function and, moreover,

$$\|\ddot{u}(t)\|_V \leq C(\|\dot{u}(t)\|_V + \|F(t)\|_V) \ a.e \ t \in [0, T].$$

Finally, we conclude that $u \in W^{2,2}(0, T, V)$.

Second Step : Now, we prove the first inequality of Theorem 4.1.

Proof. From (76) we have

$$(\beta\varphi, \psi)_W - (eu, \varphi)_W = (q, \psi)_W, \quad (90)$$

the use of (90) gives that

$$\beta\varphi(t) = eu(t) + q,$$

hence

$$\varphi(t) = \frac{e}{\beta}u(t) + \frac{q}{\beta}. \quad (91)$$

Now, we take (91) and substitute in (75), we get

$$a_\theta(\dot{u}(t), v - \dot{u}(t)) + a_\mu(u(t), v - \dot{u}(t)) + a_e(\frac{e}{\beta}u(t), v - \dot{u}(t)) + j(\dot{u}(t), v) - j(\dot{u}(t), \dot{u}(t)) \geq (f(t) - \beta^{-1}q, v - \dot{u})_V, \forall v \in V, t \in [0, T], \quad (92)$$

$$u(0) = u_0. \quad (93)$$

Next, we define the bilinear forms $a : V \times V \longrightarrow \mathbb{R}$, $b : V \times V \longrightarrow \mathbb{R}$ by

$$a(u(t), v - \dot{u}(t)) = a_\mu(u(t), v - \dot{u}(t)) + a_e(\frac{e}{\beta}u(t), v - \dot{u}(t)), \quad (94)$$

$$b(\dot{u}(t), v - \dot{u}(t)) = a_\theta(\dot{u}(t), v - \dot{u}(t)), \quad (95)$$

and define the function $F : [0, T] \longrightarrow V$ by

$$(F(t), v - \dot{u})_V = (f(t) - \beta^{-1}q, v - \dot{u})_V. \quad (96)$$

The bilinear form $a(.,.)$ and the initial data u_0 satisfy conditions (81) and (84). The regularity $f \in W^{1,2}(0, T, V)$ and $q \in W^{1,2}(0, T, W)$ combined with the definition of $F(.)$

in (96), satisfy (85).

Now, for all $\eta \in V$, the functional $j(\eta, .) : V \longrightarrow \mathbb{R}$ is a continuous seminorm on V and therefore it satisfies condition (83)(a). Recall also that j satisfies inequality (57), which shows that condition (83)(b) holds with $\alpha = c_V L_g$.

From (68), the bilinear form b satisfies condition (83) with $m' = \theta^*$.

Choose $L_0 = \dfrac{\theta^*}{c_V^2}$, which depends on $\Omega, \Gamma_1, \Gamma_2, \Gamma_3$, and θ. Then, if $L_g < L_0$, we have $m' > \alpha$, and therefore the first inequality in Theorem 4.1 is a direct consequence of Lemma 3.1.

Theorem 4.2 *Asume (48)-(62) and if $L_g < L_0$, there exists a unique solution (u, φ) to Problem 3.1 satisfying*

$$\varphi \in W^{1,2}(0, T, W). \tag{97}$$

In this step, we prove the second inequality cited in Theorem 4.2.

Proof. Let $u \in W^{2,2}(0, T, V)$ be the solution of problem (77)-(97) and let $\varphi : [0, T] \longrightarrow W$ be the electrical potential field defined by (91). Notice that the regularity $u \in W^{2,2}(0, T, V)$ and $q \in W^{1,2}(0, T, W)$ imply that $\varphi \in W^{1,2}(0, T, W)$.

Conclusion

We presented a model for an antiplane contact problem for electro-viscoelastic materials with two variables, i.e., a time-dependent variational equation for the potential field, where the time t is in $[0, T]$. The problem was set as a variational inequality for the displacements and a variational equality for the electric potential. The existence of a unique weak solution for the problem was established by using arguments from the theory of evolutionary variational inequalities and a fixed-point theorem.

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