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Numerical Solution for Benjamin-Bona-Mahony-Burgers Equation Using Septic B-Spline Galerkin Method

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Abstract: In this research, a numerical solution to the Benjamin-Bona-Mahony-Burger (BBMB) equation utilizing septic B-spline Galerkin method has been proposed. The accuracy of the method has been tested by evaluating L_2 and L_∞error norms. Furthermore, the obtained numerical results are compared with those available in the literature. Finally, some graphical representations have been presented to show the method efficiency.

Keywords: *partial differential equations (PDEs); Benjamin-Bona-Mahony-Burger (BBMB) equation; B-spline Galerkin method.*

Mathematics Subject Classification (2010): 70K25, 70K40, 93Axx.

1 Introduction

Many phenomena in applied sciences such as engineering, physics, and chemistry can be described through mathematical models. Partial differential equations (PDEs) can be considered one of the most important of these models [1], [2]. The Benjamin-Bona-Mahoney-Berger (BBMB) equation is one of the fundamental types of nonlinear dispersive equations that has occurred in various areas of applied mathematics [3], [4]. The BBMB equation is a mathematical model proposed in [5] to study the unidirectional long wave motion with small amplitudes. The BBMB equation represents the mathematical

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model of propagation of small amplitude long waves in the nonlinear dispersive media along with the dissipation, which is alternate to the Korteweg-de Vries equation [6]. This equation incorporates nonlinearity as well as the dispersive and the dissipative effects. The terms u_{xx} and u_{xxt} represent the dissipative and the dispersive effects of the equation, respectively. This equation is used in a number of branches of science and engineering. Many studies have been conducted to find the exact solution to the BBMB equation, such as, the homotropy analysis method [7], the exp-function method [8], the multipliers method [9], (see also [10], [11], [12], [13], [14]). In this research paper, the septic B-spline Galerkin method is used to solve the one-dimensional BBMB equation numerically. Several authors have investigated the numerical solution of the BBMB equation, for example, finite difference method [15, 16], domain decomposition technique [17], [18], [19], cubic spline collocation method [20], finite element method [21], [22], [23], Crank–Nicolson-type finite difference methods [24], [25], [26], [27], and B-spline quadratic finite element method [28]. In this research paper, in Section 2, the septic B-spline Galerkin method has been applied to the BBMB equation. In Section 3, two examples are solved using the proposed method, and the accuracy of the method has been tested by comparing the obtained numerical solutions with the exact solutions and with the methods existing in the literature. Finally, a summary of the main conclusions is presented in Section 4.

2 Septic B-spline Galerkin Method

The BBMB equation is defined as

$$v_t - v_{xxt} - \alpha v_{xx} + \beta v_x + v v_x = 0, \qquad c \leq x \leq d, \quad t > 0. \tag{1}$$

The boundary conditions are given as

$$\begin{aligned} &v(c,t) = v(d,t) = 0, \quad v_x(c,t) = v_x(d,t) = 0, \\ &v_{xx}(c,t) = v_{xx}(d,t) = 0, \quad v_{xxx}(c,t) = v_{xxx}(d,t) = 0, \ \ t > 0. \end{aligned} \tag{2}$$

and the initial condition is

$$v(x,0) = f(x), \qquad x \in [c,d]. \tag{3}$$

Applying the Galerkin technique to (1) with the test functions ψ yields

$$\int_0^h \psi(u_t - u_{xxt} - \alpha u_{xx} + \beta u_x + u u_x)dx = 0\,. \tag{4}$$

Therefore, we have

$$\int_0^h (\psi u_t + \psi_x u_{xt} + \alpha \psi_x u_x + \beta \psi\, u_x + \psi u u_x)dx = 0\,. \tag{5}$$

The approximating solution is defined as follows:

$$\begin{aligned} &u_n(x,t) = \textstyle\sum_{p=-3}^{n+3} \delta_p(t) H_p(x), \\ &\psi(x) = H_p(x), \qquad\qquad p = -3, ..., n+3\,, \end{aligned} \tag{6}$$

where H_p is the septic B-spline function:

$$H_p(z)=\frac{1}{h^7}\begin{cases}(z-z_{p-4})^7 & z\in\ [z_{p-4},z_{p-3}]\\ (z-z_{p-4})^7-8(z-z_{p-3})^7 & z\in\ [z_{p-3},z_{p-2}]\\ (z-z_{p-4})^7-8(z-z_{p-3})^7+28(z-z_{p-2})^7 & z\in\ [z_{p-2},z_{p-1}]\\ (z-z_{p-4})^7-8(z-z_{p-3})^7+28(z-z_{p-2})^7-56(z-z_{p-1})^7 & z\in\ [z_{p-1},z_p]\\ (z_{p+4}-z)^7-8(z_{p+3}-z)^7+28(z_{p+2}-z)^7-56(z_{p+1}-z)^7 & z\in[z_p,z_{p+1}]\\ (z_{p+4}-z)^7-8(z_{p+3}-z)^7+28(z_{p+2}-z)^7 & z\in[z_{p+1},z_{p+2}]\\ (z_{p+4}-z)^7-8(z_{p+3}-z)^7 & z\in[z_{p+2},z_{p+3}]\\ (z_{p+4}-z)^7 & z\in[z_{p+2},z_{p+3}]\\ 0 & otherwise.\end{cases} \tag{7}$$

When using the equation as $h\mu=z-z_p$, the septic B-splines (7) in terms of the variable μ over [0, 1] can be given as follows:

$$\begin{aligned}&H_{p-3}(z)=1-7\mu+21\mu^2-35\mu^3+35\mu^4-21\mu^5+7\mu^6-\mu^7,\\ &H_{p-2}(z)=120-392\mu+504\mu^2-280\mu^3+84\mu^5-42\mu^6+7\mu^7,\\ &H_{p-1}(z)=1191-1715\mu+315\mu^2+665\mu^3-315\mu^4-105\mu^5+105\mu^6-21\mu^7,\\ &H_p(z)=2416-1680\mu+560\mu^4-140\mu^6+35\mu^7,\\ &H_{p+1}(z)=1191+1715\mu+315\mu^2-665\mu^3-315\mu^4+105\mu^5+105\mu^6-35\mu^7,\\ &H_{p+2}(z)=120+392\mu+504\mu^2+280\mu^3-84\mu^5-42\mu^6+21\mu^7,\\ &H_{p+3}(z)=1+7\mu+21\mu^2+35\mu^3+35\mu^4+21\mu^5+7\mu^6-7\mu^7,\\ &H_{p+4}(z)=\mu^7.\end{aligned} \tag{8}$$

Therefore, the approximate solution (6) is reduced over the element $[x_p,x_{p+1}]$ to the following form:

$$v_n^e=\sum_{i=p-3}^{p+4}\delta_i(t)H_i(x)\,. \tag{9}$$

Substituting Eq.(9) into (5), we obtain

$$\begin{aligned}&\textstyle\sum_{j=p-3}^{p+4}\int_0^1 H_iH_j\dot{\delta}_jd\mu+\sum_{j=p-3}^{p+4}\int_0^1 H_i'H_j'\dot{\delta}_jd\mu+\alpha\sum_{j=p-3}^{p+4}\int_0^1 H_i'H_j'\delta_jd\mu+\\ &\textstyle\beta\sum_{j=p-3}^{p+4}\int_0^1 H_iH_j'\delta_jd\mu+\sum_{j=p-3}^{p+4}\sum_{k=p-3}^{p+4}\int_0^1 H_iH_jH_k'\delta_k\delta_jd\mu=0,\qquad p=0,1,...n-1.\end{aligned} \tag{10}$$

Here, "•" refers to the derivative with respect to time. Rewrite (10) in a matrix form

$$A^e\dot{\delta}^e+B^e\dot{\delta}^e+\alpha B^e\delta^e+\beta C^e\delta^e+D^e(\delta^e)^T\delta^e=0, \tag{11}$$

where

$$A_{i,j}^e=\int_0^1 H_iH_jd\mu,\ B_{i,j}^e=\int_0^1 H_i'H_j'd\mu,\ C_{i,j}^e=\int_0^1 H_iH_j'd\mu,\ D_{i,j,k}^e=\int_0^1 H_iH_jH_k'd\mu.$$

The matrices $A_{ij}^e,B_{ij}^e,C_{ij}^e$ are matrices of 8×8, and the matrix D is $8\times8\times8$. The matrix D is designed to be in the form of 8×8 as

$$E_{i,j}^e=\sum_{k=p-3}^{p+4}D_{ijk}\delta_k. \tag{12}$$

Thus Eq.(11) becomes

$$(A^e+B^e)\dot{\delta}^e+(\alpha B^e+\beta C^e+E^e)\delta^e=0. \tag{13}$$

Assembling the element matrix (13), we get

$$(A+B)\dot{\delta}+(\alpha B+\beta C+E)\delta=0, \tag{14}$$

where A, B, C, E are obtained from the element matrices A^e, B^e, C^e, E^e, respectively. Applying the forward finite difference formula, $\overset{\bullet}{\delta} = \frac{\delta^{n+1}-\delta^n}{\Delta t}$, with the Crank-Nicolson scheme $\delta = \frac{\delta^{n+1}+\delta^n}{2}$ to Eq.(14), we obtain

$$[2A + (2+\alpha\Delta t)B + \Delta t(\beta C + E)]\,\delta^{n+1} = [2A + (2-\alpha\Delta t)B - \Delta t(\beta C + E)]\,\delta^n. \qquad (15)$$

3 Numerical Examples

In this section, we explain the effectiveness of the proposed method through two examples. The results have been tested using the error norms

$$L_2 = \left\|v^{exact} - v^{num}\right\|_2 = \sqrt{h\sum_{j=0}^{N}\left|v_j^{exact} - v_j^{num}\right|^2}, \quad L_\infty = \max_j \left|v_j^{exact} - v_j^{num}\right|.$$

Example 1

Take equation (1) at $\alpha = \beta = 1$, with the following initial condition:

$$v(x,0) = \sin(x)\,.$$

The exact solution is given by

$$v(x,t) = e^{-t}\sin(x)\,.$$

The boundary conditions are chosen from the exact solution. The obtained results are compared with the exact solution by evaluating L_2 and L_∞error norms at different values of time. Moreover, the results are compared with the results in [15] [23], which are shown in Table 1. In Figures 1 and 2, we plotted the exact and numerical solutions of Example 1, and it turns out that the two plots are very close to each other.

	T	L_2	L_∞
The proposed method	1	3.02235×10^{-7}	1.811223×10^{-10}
	2	5.9976×10^{-7}	4.55663×10^{-10}
	5	9.5589×10^{-7}	2.996753×10^{-9}
	10	1.86532×10^{-6}	8.76521×10^{-9}
[15]	1	2.976712×10^{-6}	1.906127×10^{-9}
	2	4.675453×10^{-6}	3.745931×10^{-9}
	5	1.876409×10^{-5}	2.996753×10^{-8}
	10	2.318245×10^{-5}	4.906127×10^{-8}
[23]	1	4.238861×10^{-3}	8.732152×10^{-4}
	2	2.74893×10^{-3}	4.720312×10^{-4}
	5	1.009326×10^{-2}	3.258217×10^{-3}
	10	3.112871×10^{-2}	2.531897×10^{-3}

Table 1: Errors between the numerical and exact solutions in Example 1, at different times.

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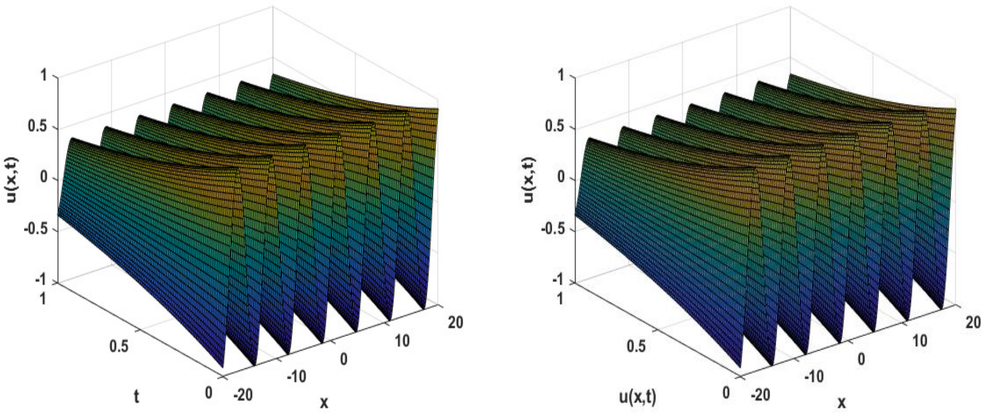

Figure 1: Numerical and exact solutions in Example 1 at $-20 \leq x \leq 20$, and $0 \leq t \leq 1$. Left: Exact solution; right: Numerical solution.

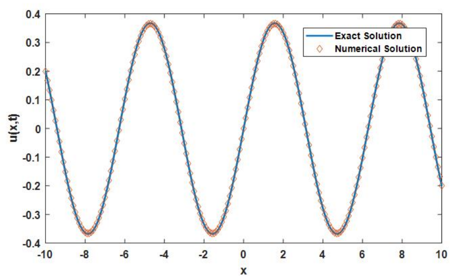

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Figure 2: Comparison between the exact and approximate solution in Example 1 at $t = 1$.

Example 2

Take equation (1) at $\alpha = 0, \ \ \beta = 1$, the initial condition is given by

$$v(x,0) = \sec h^2(\frac{x}{4}) \, .$$

The exact solution is

$$v(x,t) = e^{-t} \sin(\frac{x}{4} - \frac{t}{3}) \, .$$

In Figure 3, the numerical and exact solutions are plotted. The obtained numerical results for Example 2 are compared with the exact solution at different values of h, k and t and are presented in Table 2. Again, it turns out that the results obtained using the proposed method are more accurate when compared with the results in [13], [15], [22].

	h	k	t	Absolute error
The proposed method	0.1	0.1	1	3.78786×10^{-9}
	0.1	0.1	5	9.746302×10^{-9}
	0.05	0.1	1	3.911113×10^{-11}
	0.05	0.1	5	7.55322×10^{-10}
[15]	0.1	0.1	1	1.906512×10^{-8}
	0.1	0.1	5	6.746302×10^{-8}
	0.05	0.1	1	5.650921×10^{-10}
	0.05	0.1	5	1.0774219×10^{-9}
[22]	0.1	0.1	1	6.479231×10^{-2}
	0.1	0.1	5	4.209547×10^{-2}
	0.05	0.1	1	1.226197×10^{-3}
	0.05	0.1	5	2.626195×10^{-3}
[13]	0.1	0.1	1	4.783312×10^{-4}
	0.1	0.1	5	1.622166×10^{-4}
	0.05	0.1	1	2.040573×10^{-6}
	0.05	0.1	5	1.192831×10^{-6}

Table 2: Absolute errors between the numerical and exact solutions in Example 2 at different values for h, k and t.

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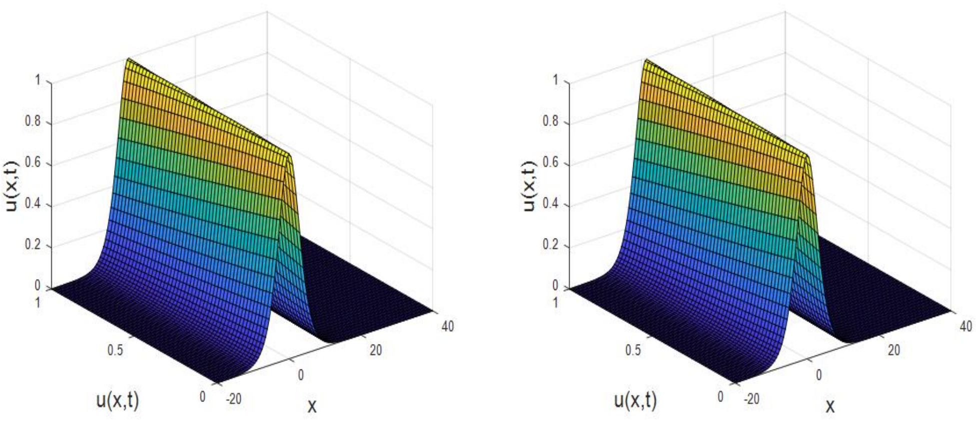

Figure 3: Numerical and exact solutions in Example 2 at $-20 \leq x \leq 40$, and $0 \leq t \leq 1$. Left: Exact solution; right: Numerical solution.

4 Conclusions

In this research paper, the BBMB equation has been solved successfully by using the septic B-spline Galerkin method. From Tables 1 and 2, the obtained numerical results show that the results are more accurate and close to the exact solutions. The results achieved from the numerical approach show that proposed technique is better than the methods used in [13], [15], [22], [23]. Thus, the results illustrated that the proposed method is novel, powerful and efficient. So, we advise to utilize the method to solve several types of partial differential equations.

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