Nonlinear Dynamics and Systems Theory, **24** (3) (2024) 286–297

On Unique Solvability and a Generalized Newton Method for Solving New General Absolute Value Equations

M. Khaldi * and M. Achache

Fondamental and Numerical Mathematics Laboratory, Ferhat Abbas University Sétif 1, Sétif 19000, Algeria.

Received: October 11, 2023; Revised: April 15, 2024

Abstract: In this paper, we consider some sufficient conditions to guarantee the unique solvability of the new general absolute value equations (NGAVE), $Ax-|Bx| = b$, $(A,\ B \in \mathbb{R}^{n\times n}, b \in \mathbb{R}^n)$. Besides Picard's iterative method for solving the NGAVE, a generalized Newton method is also proposed for solving the NGAVE. Moreover, under suitable assumptions, we show that the proposed methods are globally linearly convergent. We also report some numerical results of the proposed method for solving the NGAVE, which show the efficiency of our proposed methods.

Keywords: *absolute value equations; Picard's iterative method; generalized Newton method; global convergence.*

Mathematics Subject Classification (2010): 65F08, 90C33, 93C05.

1 Introduction

In this paper, we consider new general absolute value equations (abbreviated as NGAVE) of the type

$$Ax - |Bx| = b, \tag{1}$$

where $A, B \in \mathbb{R}^{n\times n}$ are given matrices, $b \in \mathbb{R}^n$, and $|Bx|$ is a vector whose i-th entry is the absolute value of the i-th entry of Bx. If $B = I$ is the identity matrix, then the NGAVE (1) can be reduced to the type

$$Ax - |x| = b. \tag{2}$$

* Corresponding author: mailto:Khaldi_m2007@yahoo.fr

Moreover, the system (2) is a special case of the generalized absolute value equation (GAVE) of the following form:

$$Ax - B|x| = b, \tag{3}$$

where $B \in \mathbb{R}^{n\times n}$, the last one was introduced by Rohn [14] and investigated in a more general context by Mangasarian and Meyer (see [9]). Other studies for the AVE can be found in [1,5,6,11,13,16]. The AVE (2), GAVE (3) and NGAVE (1) have received much attention from the optimization community. It is currently an active research topic due to its broad application in many areas of scientific computing and engineering. For instance, linear complementarity, linear programming, convex quadratic programming and bi-matrix games can be equivalent to the NGAVE(1). The research effort can be summarized to the following two aspects. One is purely theoretical analysis. Authors focus on the reformulations of AVE (2), GAVE (3) and NGAVE (1) as different equivalent problems because determining the existence and uniqueness of a solution of the NGAVE (1) is an NP-hard problem because of the nonlinear and non-differentiable term $|Bx|$ in the NGAVE (1). For the unique solution of NGAVE (1), some necessary and sufficient conditions were presented in [17]. The other one is the numerical solvability of AVEs. Recently, several algorithms have been designed to solve the AVE and GAVE, see e.g., [4,7,12] and the references therein. For example, Mangasarian in [10] proposed a semi-smooth Newton method for solving the AVE, and under suitable conditions, he showed the finite and linear convergence to a solution of the AVE. However, other numerical approaches focus on reformulating the AVE as a horizontal linear complementarity problems (HLCP) (see Achache [3]), where they introduce an infeasible path-following interior-point method for solving the AVE by using equivalent reformulations as an HLCP. Recently, Achache and Anane [2] have presented Picard's iterative fixed point method for getting the solution of the uniquely solvable GAVE. Under some suitable conditions, they showed that the proposed method is globally linearly convergent.

The goal of this paper is twofold. First, we present some weaker sufficient conditions that guarantee the unique solvability of the NGAVE (1). Second, for its numerical solution, we propose a generalized Newton method. In particular, under a mild assumption, we show that this method is always well-defined and the generated sequence converges globally and linearly to the unique solution of the NGAVE from any starting initial point. Finally, numerical results are provided to illustrate the efficiency of our proposed algorithm for solving the NGAVE. In addition, a numerical comparison is made with an available method.

The outline of this paper is as follows. The main results of the unique solvability of the NGAVE are stated in Section 2. In Section 3, Picard's iterative method for solving the NGAVE is presented. In Section 4, a generalized Newton method is proposed for solving the NGAVE. Moreover, under suitable conditions, the global convergence to the unique solution is proved. In Section 5, some numerical results are provided to show the efficiency of the proposed algorithm. Finally, a conclusion and some remarks are drawn in the last section of the paper.

At the end of this section, some notations are presented. Let $\mathbb{R}^{n\times n}$ be the set of all $n\times n$ real matrices. The scalar product and the Euclidean norm are denoted, respectively, by $x^T y, x, y \in \mathbb{R}^n$ and $\|x\| = \sqrt{x^T x}$. Recall that a subordinate matrix norm for $A \in \mathbb{R}^{n\times n}$is defined as follows: $\|A\| := \max\{\|Ax\| : x \in \mathbb{R}^n, \|x\| = 1\}$, this definition implies

$$\|Ax\| \leq \|A\| \|x\|, \ \|AB\| \leq \|A\| \|B\|, \ \forall A, B \in \mathbb{R}^{n\times n} \text{ and } x \in \mathbb{R}^n.$$

The $sign(x)$ denotes a vector with the components equal to -1, 0 or 1 depending on whether the corresponding component is negative, zero or positive. In addition, $D(x) := Diag(sign(x))$ will denote a diagonal matrix corresponding to $sign(x)$. The absolute value of a matrix $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ and the vector of all ones are denoted by $|A| = (|a_{ij}|) \in \mathbb{R}^{n\times n}$ and $e \in \mathbb{R}^n$, respectively. $\sigma_{\min}(A)$, $\sigma_{\max}(A)$ represent, respectively, the smallest and the largest singular value of the matrix A. As is well known, $\sigma^2_{\min}(A) = \min_{\|x\|=1} x^T A^T Ax$, and $\sigma^2_{\max}(A) = \max_{\|x\|=1} x^T A^T Ax$. Finally, a matrix $A \in \mathbb{R}^{n\times n}$ is positive definite if for a nonzero vector x, $x^T Ax > 0$, and the inverse of a non singular matrix A is denoted by A^{-1}.

2 The Main Results

In this section, some conditions to guarantee the unique solution of the NGAVE (1) are presented. First, for given matrices $A, B \in \mathbb{R}^{n\times n}$ and for any diagonal matrix $D \in \mathbb{R}^{n\times n}$ whose diagonal elements are ± 1 and 0, we define the matrix $(A - DB) \in \mathbb{R}^{n\times n}$. Then to achieve our main results, the following lemma is required.

Lemma 2.1 *Each of three conditions below implies the non singularity of* $(A-DB)$.
1. $\sigma_{\min}(A) > \sigma_{\max}(B)$,
2. $\|A^{-1}\| \|B\| < 1$, *provided* A *is non singular,*
3. the matrix $A^T A - \|B\|^2 I$ *is positive definite.*

Proof. For the first claim, assume that $(A - DB)$ is singular, then

$$(A - DB)x = 0, \text{for some}\, x \neq 0.$$

We then have

$$\begin{aligned} \sigma^2_{\min}(A) = & \min_{\|y\|=1} y^T A^T Ay \leq x^T A^T Ax = x^T B^T DDBx \\ \leq & \max_{\|z\|=1} z^T B^T DDBz = \|DB\|^2 \\ \leq & \|D\|^2 \|B\|^2 \leq \|B\|^2 = \max_{\|z\|=1} z^T B^T Bz \\ = & \sigma^2_{\max}(B), \end{aligned}$$

which contradicts the first condition. Hence $(A - DB)$ is non singular. Next, by the same argument, assume that A is non singular and let a nonzero vector x with $\|x\| = 1$ be such that

$$(A - DB)x = 0.$$

Next, because $x = A^{-1}DBx$, we then have

$$\begin{aligned} 1 = & \|x\| = \|A^{-1}DBx\| \\ \leq & \|A^{-1}\| \|D\| \|B\| \|x\| \\ \leq & \|A^{-1}\| \|B\|, \end{aligned}$$

which leads to a contradiction and hence $(A - DB)$ is non singular. For the last claim, assume on the contrary that $(A - DB)$ is singular, then for a nonzero vector x with $\|x\| = 1$, we have

$$(A - DB)x = 0.$$

As $Ax = DBx$, we then have

$$\begin{aligned} x^T A^T Ax - \|B\|^2 x^T x = & \quad x^T (DB)^T DBx - \|B\|^2 x^T x \\ = & \quad \|DBx\|^2 - \|B\|^2 x^T x \\ \leq & \quad \|D\|^2 \|B\|^2 \|x\|^2 - \|B\|^2 x^T x \\ \leq & \quad \|B\|^2 - \|B\|^2 = 0, \end{aligned}$$

and consequently,

$$x^T A^T Ax - \|B\|^2 x^T x \leq 0.$$

This contradicts the fact that the matrix $A^T A - \|B\|^2 I$ is positive definite. Hence $(A - DB)$ is non singular for any diagonal matrix D whose elements are ± 1 and 0. This completes the proof.

The following result guarantees the unique solvability of the NGAVE.

Theorem 2.1 *The matrices A and B satisfy*
1. $\sigma_{\min}(A) > \sigma_{\max}(B)$,
2. $\|A^{-1}\| \, \|B\| < 1$, *provided A is non singular,*
3. The matrix $A^T A - \|B\|^2 I$ is positive definite, then the NGAVE (1) is uniquely solvable for any b.

Proof. Let $y = Bx$, then the equation NGAVE (1) is equivalent to

$$\begin{cases} Ax + |y| = b, \\ -Bx + y = 0. \end{cases} \tag{4}$$

The latter system can be expressed as

$$\begin{pmatrix} A & -D(y) \\ -B & I \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b \\ 0 \end{pmatrix}, \tag{5}$$

where $D(y) := diag\,(sign\,(y))\,,\ y \in \mathbb{R}^n$. Once we find the unique solution of the two-by-two linear Eq.(5), naturally, the unique solution of NGAVE (1) is obtained as well. To show that the equation admits a unique solution, it suffices to show that the application of

$$F(x, y) = \begin{pmatrix} A & -D \\ -B & I \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

is one-to-one for any diagonal matrix D whose elements are ± 1 and 0. To do so, we prove only that the Null $F = \{0\}$. For that, let $\left(u^T, v^T\right)^T \in \mathbb{R}^{2n}$, we have

$$\begin{aligned} F(u, v) \quad = \quad 0 & \Rightarrow \begin{pmatrix} A & D \\ -B & I \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ & \Rightarrow \begin{cases} Au - Dv = 0, \\ v = Bu, \end{cases} \\ & \Rightarrow \begin{cases} (A - DB)u = 0, \\ v = Bu. \end{cases} \end{aligned}$$

Based on Lemma 2.1, the matrix $(A - DB)$ is non singular for any diagonal matrix D whose elements are ± 1 and 0, then $u = 0$, and consequently, $v = 0$. Thus, Null $F = \{0\}$

and so F is one-to-one. Hence, the NGAVE (1) is uniquely solvable for any b. This completes the proof.

Then it is clear that the NGAVE (1) is uniquely solvable for any b if the matrix of coefficients $(A-DB)$ is non singular for any diagonal matrix D whose elements are ± 1 or 0.

3 Picard's Fixed Point Method of NGAVE

In this section, we provide Picard's fixed point iteration method for computing an approximated solution of the uniquely solvable NGAVE. The principal feature of the method is the use of the following equivalent scheme for NGAVE (1):

$$x_{k+1} = A^{-1}|Bx_k| + A^{-1}b, k = 0, 1, 2, \cdots$$

to find an approximated solution. The details of Picard's iterative algorithm for solving the NGAVE (1) are described in Figure 1.

Algorithm 3.1

Input
An accuracy parameter $\epsilon > 0$;
an initial starting point $x_0 \in \mathbb{R}^n$;
two matrices A and B and a vector b;
set k:=0;
  while $\|Ax_k - |Bx_k| - b\| > \epsilon$ **do**
  begin
  compute x_{k+1} from the linear system $x_{k+1} = A^{-1}(|Bx_k| + b)$;
    $k := k + 1$;
  end;
end.

Figure 1: Picard's algorithm for the NGAVE.

The convergence of Picard's fixed point scheme is based on the Banach fixed point theorem (see [8]).

Theorem 3.1 *Let A be a non singular matrix and if*

$$\left\|A^{-1}\right\| \|B\| < 1,$$

then the sequence $\{x_k\}$ converges to the unique solution $x^\star$ of the NGAVE (1) for any arbitrary $x_0 \in \mathbb{R}^n$. In this case, the error bound is given by

$$\|x_{k+1} - x^\star\| \leq \frac{\left\|A^{-1}\right\| \|B\|}{1 - \|A^{-1}\| \|B\|} \|x_{k+1} - x_k\|, \; k = 0, 1, 2, \cdots$$

Moreover, the sequence $\{x_k\}$ converges to the unique solution $x^\star$ as follows:

$$\|x_{k+1} - x^\star\| \leq \left\|A^{-1}\right\| \|B\| \|x_k - x^\star\|, \; k = 0, 1, 2, \cdots$$

Proof. The proof is similar to the one given in [2].

4 The Proposed Generalized Newton Method for the NGAVE

In this section, we propose a generalized Newton method for solving the NGAVE (1) and we show its convergence.

Let the function $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be defined as

$$F(x) = Ax + |Bx| - b. \tag{6}$$

The generalized Jacobian $\partial F(x)$ of $F(x)$ is given by

$$\partial F(x) = A - D(Bx)B, \tag{7}$$

where $D\left(Bx\right) := diag\left(sign\left(Bx\right)\right),\ x \in \mathbb{R}^n$. The generalized Newton method for finding a zero of the equation $F(x) = 0$ after some simplifications consists then of the following iteration:

$$\left(A - D(Bx_k)B\right) x_{k+1} = b, k = 0, 1, \cdots$$

The details of the algorithm for solving the NGAVE (1) are described in Figure 2.

Algorithm 4.1

Input
An accuracy parameter $\epsilon > 0$;
an initial starting point $x_0 \in \mathbb{R}^n$;
two matrices A and B and a vector b;
set k:=0;
  while $\|Ax_k - |Bx_k| - b\| > \epsilon$ **do**
  begin
  compute x_k from the linear system $\left(A - D\left(Bx_k\right)B\right) x_{k+1} = b$;
   $k := k + 1$;
  end;
end.

Figure 2: A generalized Newton algorithm for the NGAVE.

Next, following Achache [1, 10], we will study the global convergence of the generalized Newton method, first, we give the following lemma.

Lemma 4.1 *For all $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$, we obtain the following results:*

$$\left\||x| - |y|\right\| \leq \|x - y\|.$$

Proof. For a detailed proof, see Lemma 5 [10].

Lemma 4.2 *Suppose that $\left\|\left(A - DB\right)^{-1}\right\| \leq \frac{1}{2\|B\|}$, when $\|B\| \neq 0$, for any diagonal matrix D with diagonal elements of ± 1or 0. Then the generalized Newton iteration converges linearly from any starting point to a solution x^* of the NGAVE (1).*

Proof. Let x^* be a solution of the NGAVE (1), then $\left(A - D\left(Bx^*\right)B\right)x^* = b$. Note that $|Bx^*| = D\left(Bx^*\right)Bx^*$and $|Bx_k| = D\left(Bx_k\right)Bx_k$. Now, subtracting

$(A - D(Bx^*)B)x^* = b$ from $(A - D(Bx_k)B)x_k = b$, we obtain

$$\begin{aligned} A(x_{k+1} - x^*) &= D(Bx_k)Bx_{k+1} - D(Bx^*)Bx^* = 0 \\ &= D(Bx_k)(x_{k+1} + x_k - x_k) - D(Bx^*)Bx^* \\ &= |Bx_k| - |Bx^*| + D(Bx_k)B(x_{k+1} - x^* + x^* - x_k) \\ &= |Bx_k| - |Bx^*| - D(Bx_k)B(x_k - x^*) + D(Bx_k)B(x_{k+1} - x^*). \end{aligned}$$

Hence

$$(A - D(Bx_k)B)(x_{k+1} - x^*) = |Bx_k| - |Bx^*| - D(Bx_k)B(x_k - x^*).$$

Consequently,

$$(x_{k+1} - x^*) = (A - D(Bx_k)B)^{-1}(|Bx_k| - |Bx^*|) - D(Bx_k)B(x_k - x^*).$$

By Lemma 4.1, we have

$$\|x_{k+1} - x^*\| = \|(A - D(Bx_k)B)\|^{-1}\|B\|\|(x_k - x^*)\| - \|B\|\|(x_k - x^*)\|.$$

Hence,

$$\|x_{k+1} - x^*\| \leq 2\|A - D(Bx_k)B\|^{-1}\|B\|\|(x_k - x^*)\|.$$

So by the condition

$$\left\|(A - DB)^{-1}\right\| \leq \frac{1}{2\|B\|},$$

it follows that $\|x_{k+1} - x^*\| < \|x_k - x^*\|$. Hence the sequence $\{x^k\}$ converges linearly to x^*. This completes the proof.

We are now ready to prove our main result of the global convergence. We quote first the following lemma.

Lemma 4.3 *If assumption* $\left\|A^{-1}\right\| \leq \frac{1}{\|B\|}$, *when* $\|B\| \neq 0$, *holds, then* $(A - DB)$ *is nonsingular and*

$$\left\|(A - DB)^{-1}\right\| \leq \frac{\left\|A^{-1}\right\|}{1 - \|A^{-1}B\|}.$$

Proof. The first part follows directly from Lemma 2.1. For the proof of the second part, we have $(A - DB)^{-1}$ can be written in the from

$$(A - DB)^{-1} = \left(I - A^{-1}DB\right)^{-1}A^{-1}.$$

But since $\left(I - A^{-1}DB\right)^{-1}\left(I - A^{-1}DB\right) = I$, it follows that

$$\left(I - A^{-1}DB\right)^{-1} = I + \left(I - A^{-1}DB\right)^{-1}A^{-1}DB.$$

By introducing an induced matrix norm, we get

$$\left\|\left(I - A^{-1}DB\right)^{-1}\right\| \leq \frac{1}{1 - \|A^{-1}\|\|D\|\|B\|} \leq \frac{1}{1 - \|A^{-1}\|\|B\|}.$$

Because

$$\left\|(A - DB)^{-1}\right\| = \left\|\left(I - A^{-1}DB\right)^{-1}A^{-1}\right\|,$$

it follows that

$$\left\|(A - DB)^{-1}\right\| \leq \frac{\left\|A^{-1}\right\|}{1 - \|A^{-1}\|\|B\|}.$$

This completes the proof.

Proposition 4.1 *Let* $\left\|A^{-1}\right\| < \frac{1}{3\|B\|}$ *and* $D\left(Bx_k\right) \neq 0$ *for all* k*. Then the NGAVE is uniquely solvable for each* $b \in \mathbb{R}^n$ *and the generalized Newton iteration is well-defined and converges globally from any starting point to a solution* x^* *for any solvable NGAVE (1).*

Proof. The unique solvability of NGAVE (1) for any b follows from Theorem 2.1 which requires $\left\|A^{-1}\right\| < \frac{1}{\|B\|}$. Next, by Lemma 4.3, we have

$$\left\|(A - DB)^{-1}\right\| \leq \frac{\left\|A^{-1}\right\|}{1 - \left\|A^{-1}\right\| \|B\|} \leq \frac{\frac{1}{3\|B\|}}{1 - \frac{1}{3}} = \frac{1}{2\|B\|}.$$

Hence, by Lemma 4.2, the sequence generated by the generalized Newton method converges to the unique solution of the NGAVE (1) from any starting point x_0. This completes the proof.

5 Numerical Results

In this section, we give some numerical examples to examine the effectiveness of the generalized Newton method (GN) for solving the NGAVE (1). All programs were implemented in **MATLAB** R2016a on a personal PC with 1.40 GHZ AMD E1-2500 APU Radeon(TM) HD Graphic, 8 GB memory and Windows 10 operating system. The starting point and the unique solution of the NGAVE are denoted, respectively, by x_0 and $x^\star$. In the tables of numerical results, we display the following notations: "Iter" and "CPU" state for the number of iterations and the elapsed times. In addition, to show the performance of our methods in practice, we compare the obtained numerical results with those obtained by Picard's fixed point iteration method. The algorithm is terminated if the current iterations satisfy $RES := \|Ax_k - |Bx_k| - b\| \leq 10^{-8}$.

Example 5.1 Consider the problem of NGAVE, where $A, B \in \mathbb{R}^{10\times 10}$ are given by

$$A = \begin{bmatrix} 50 & 5 & -4 & -4 & -4 & -4 & -4 & -4 & -4 & -4 \\ 5 & 50 & 5 & -4 & -4 & -4 & -4 & -4 & -4 & -4 \\ -1 & 5 & 50 & 5 & -4 & -4 & -4 & -4 & -4 & -4 \\ -1 & -1 & 5 & 50 & 5 & -4 & -4 & -4 & -4 & -4 \\ -1 & -1 & -1 & 5 & 50 & 5 & -4 & -4 & -4 & -4 \\ -1 & -1 & -1 & -1 & 5 & 50 & 5 & -4 & -4 & -4 \\ -1 & -1 & -1 & -1 & -1 & 5 & 50 & 5 & -4 & -4 \\ -1 & -1 & -1 & -1 & -1 & -1 & 5 & 50 & 5 & -4 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & 5 & 50 & 5 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 5 & 50 \end{bmatrix},$$

and

$$B = 1/2 \begin{bmatrix} -51 & -5 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -5 & -51 & -5 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -5 & -51 & -5 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -5 & -51 & -5 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -5 & -51 & -5 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -5 & -51 & -5 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & -5 & -51 & -5 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & -5 & -51 & -5 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & -5 & -51 & -5 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -5 & -51 \end{bmatrix}.$$

Applying Theorem 2.1, we have $\left\|A^{-1}\right\| \|B\| = 0.9166 < 1$, then this problem is uniquely solvable for any b. For this example, we take

$$b = [54, 54, 54, 54, 54, 54, 54, 54, 54, 54]^T.$$

Our starting point for this example is taken as

$$x_0 = [0.5, 0.5, 0.5, 0.5, 0.5, 0.5, 0.5, 0.5, 0.5, 0.5]^T.$$

The exact unique solution of this problem is given by

$$x^* = [3.653, 3.008, 3.154, 2.9412, 2.8306, 2.692, 2.567, 2.4604, 2.2705, 2.5987]^T.$$

The obtained numerical results are stated in Table 1.

Algorithms→	**Picard's Algorithm**	**GN Algorithm**
Iter	5	2
CPU	0.032344	0.014184
RSD	$5.9765e-007$	0

Table 1: Numerical results for Example 5.1.

Example 5.2 The hydrodynamic equations (equilibrium problem) are modeled as the following non-differentiable algebraic equations:

$$Cx + \max(0, x) = c,$$

where $C \in \mathbb{R}^{n\times n}, c \in \mathbb{R}^n$ are given. By using the identity

$$\max(a, b) = \frac{1}{2}(a + b + |a - b|),$$

the hydrodynamic equation can be reformulated as an AVE (2). We have

$$Cx + \frac{1}{2}(x + |x|) = c \Leftrightarrow Ax - |x| - b = 0,$$

where $A = -(2C + I), B = I$ and $b = -2c$.

Consider now a random hydrodynamic equation, where $C \in \mathbb{R}^{n\times n}$ and c are given by

$$C=(c_{ij})=\begin{bmatrix} -25.5 & -2.5 & 0 & \cdots & 0 & 0 \\ -2.5 & -25.5 & -2.5 & \cdots & 0 & 0 \\ 0 & -2.5 & -25.5 & \cdots & 0 & \vdots \\ \vdots & \vdots & \ddots & \ddots & -2.5 & 0 \\ 0 & 0 & 0 & \cdots & -25.5 & -2.5 \\ 0 & 0 & \cdots & 0 & -2.5 & -25.5 \end{bmatrix},$$

and

$$c=[-27,-29.5,...,-29.5,-27]^T.$$

For this example, we have taken two initial points such as

$$x_1^0=[0.5,...,0.5]^T \text{ and } x_2^0=[0.9,...,0.9]^T.$$

The computational results with different size of n are summarized in Table 2.

Algorithms→		**Picard's Algorithm**		**GN Algorithm**	
Size n	x^0	**Iter**	**CPU**	**Iter**	**CPU**
100	x_1^0	11	0.064263	2	0.042966
	x_2^0	12	0.487104	3	0.051011
1000	x_1^0	88	26.133205	3	5.022345
	x_2^0	89	2.149593	3	5.884994
2000	x_1^0	335	54.093104	3	45.347198
	x_2^0	333	54.651662	3	45.064239

Table 2: Numerical results for Example 5.2.

Example 5.3 The matrices A and B are given by

$$A=\begin{bmatrix} 4 & -1 & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 \\ -1 & 4 & \ddots & \ddots & \vdots & 0 & 1 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 4 & -1 & \vdots & \ddots & \ddots & 1 & 0 \\ 0 & \cdots & 0 & -1 & 4 & 0 & \cdots & 0 & 0 & 1 \\ 0.3 & 0 & 0 & \cdots & 0 & 1/5 & -2 & -1/3 & \cdots & 0 \\ 0 & 0.3 & \ddots & \ddots & \vdots & -2 & 1/5 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 & 1/4 & \ddots & \ddots & \ddots & -1/3 \\ \vdots & \ddots & \ddots & 0.3 & 0 & \vdots & \ddots & \ddots & 1/5 & -2 \\ 0 & \cdots & 0 & 0 & 0.3 & 0 & \cdots & 1/4 & -2 & 1/5 \end{bmatrix}.$$

$$B = (1/n)\begin{bmatrix} -2 & -1 & 1/5 & \cdots & 1/5 & -1 & 0 & 0 & \cdots & 0 \\ 0.4 & -2 & \ddots & \ddots & \vdots & 0 & -1 & \ddots & \ddots & \vdots \\ 3 & \ddots & \ddots & \ddots & 1/5 & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & -2 & -1 & \vdots & \ddots & \ddots & -1 & 0 \\ 3 & \cdots & 3 & 0.4 & -2 & 0 & \cdots & 0 & 0 & -1 \\ -12 & 0 & 5 & \cdots & 0 & -2 & -1 & 1/5 & \cdots & 0 \\ 0 & -12 & \ddots & \ddots & \vdots & 0.4 & -2 & \ddots & \ddots & \vdots \\ 1/5 & \ddots & \ddots & \ddots & 5 & 3 & \ddots & \ddots & \ddots & 1/5 \\ \vdots & \ddots & \ddots & -12 & 0 & \vdots & \ddots & \ddots & -2 & -1 \\ 0 & \cdots & 1/5 & 0 & -12 & 3 & \cdots & 3 & 0.4 & -2 \end{bmatrix}.$$

$$b = [5, -2, \ldots, 5, -2]^T .$$

The obtained numerical results for different size of n, are summarized in Table 3.

Algorithms→	**Picard's Algorithm**		**GN Algorithm**	
Size n	**Iter**	**CPU**	**Iter**	**CPU**
4	40	0.025384	2	0.040067
8	23	0.028198	2	0.028242
40	26	0.060005	3	0.032638
80	22	0.113059	2	0.056422
200	25	0.560619	3	0.210737
400	30	5.126521	2	1.211867
1000	$\star$	$\star$	3	21.045136

Table 3: Numerical results for Example 5.3.

6 Conclusion

This paper presents a theoretical analysis and numerical study for solving new general absolute value equations (NGAVE). In the first part, we have presented some weaker sufficient conditions for the unique solvability of the NGAVE. For solving this NGAVE, we applied the generalized Newton method. In particular, the sufficient conditions for the convergence of our algorithm are studied. The obtained numerical results deduced from the testing examples illustrate that the suggested algorithms are efficient and valid to solve the NGAVE problems. Finally, an interesting topic of research in the future is solving the NGAVEs by introducing the Splitting method.

References

[1] M. Achache, On the unique solvability and numerical study of absolute value equations. *J. Numer. Anal. Approx. Theory* **48** (2019) 112–121.

[2] M. Achache and N. Anane. On the unique solvability and Picard's iterative method for absolute value equations. *Bulletin of Transilvania. Series III: Mathematics and Computer Sciences* **1** (63) (2021) 13–26.

[3] M. Achache and N. Hazzam. Solving absolute value equations via linear complementarity and interior-point methods. *Journal of Nonlinear Functional Analysis.* Article. ID **39** (2018) 1–10.

[4] N. Anane, Z. Kebaili and M. Achache. A DC Algorithm for Solving non-Uniquely Solvable Absolute Value Equations. *Nonlinear Dynamics and Systems Theory* **21** (2) (2023) 119—128.

[5] M. Hladick. Bounds for the solution of absolute value equations. *Computational Optimization and Applications* **69** (1) (2018) 243–266.

[6] S. Ketabchi and H. Moosaei. An efficient method for optimal correcting of absolute value equations by minimal changes in the right hand side. *Comput. Math. Appl.* **64** (2012) 1882–1885.

[7] Y. Ke. The new iteration algorithm for absolute value equation. *Applied Mathematics Letters* **99** (2020).

[8] E. Kreyszig. *Introductory Functional Analysis with Applications.* John Wiley Sons. New-York, London, Sydney. 1978.

[9] O. L. Mangasarian and R. R. Meyer. Absolute value equations. *Linear Algebra and its Applications* **419** (2006) 359–367.

[10] O. L. Mangasarian. A generalized Newton method for absolute value equations. *Optimization Letters* **3** (2009) 101–108.

[11] F. Mezzardi. On the solution of general Absolute value equations. *Appl. Math. Lett* **117** (2020).

[12] M. A. Noor, J. Iqbal and E. Al-Said, Residual iterative method for solving absolute value equations. *Abst. Appl. Anal.* 2012, Article. ID (2012) 1–9.

[13] J. Rohn. On unique solvability of the absolute value equations. *Optimization Letters* **4** (2) (2010) 287–292.

[14] J. Rohn. A theorem of the alternatives for the equations $Ax - B|x| = b$. *Linear and Multilinear Algebra* **52** (6)(2004) 421–426.

[15] J. Rohn. A theorem of the alternatives for the equations $Ax - |B||x| = b$. *Optimization Letters* **6** (2012) 585–591.

[16] S. L. Wang and C. X. Li. A note on unique solvability of the absolute value equation. *Optimization Letters* (2019).

[17] S. L. Wu. The unique solution of a class of the new generalized absolute value equation. *Appl. Math. Lett.* **116** (2021).

[18] A. Yu. Aleksandrov. Delay-Independent Stability Conditions for a Class of Nonlinear Mechanical Systems. *Nonlinear Dynamics and Systems Theory* **21** (5) (2021) 447–456.