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Adaptive Control for the Stabilization, Synchronization and Anti-Synchronization of New Chaotic System with a Line Equilibrium

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Abstract: This paper derives new results on the adaptive control, synchronization and anti-synchronization for the new chaotic system with a line equilibrium when the system's parameters are unknown. Firstly, we construct an adaptive controller to stabilize the new chaotic system to its unstable equilibrium at the origin. Then, we construct an adaptive controller to synchronize the new identical chaotic systems with unknown parameters. Finally, the corresponding adaptive controller to realize the anti-synchronization is also constructed for the same new identical chaotic systems. The Lyapunov stability theory and adaptive control theory have been applied to prove all the control, synchronization and anti-synchronization results derived in this paper. Numerical simulations have been presented to illustrate the main results for the new chaotic system with a line equilibrium.

Keywords: *chaos control; adaptive control; synchronization and anti-synchronization; Lyapunov stability theory.*

Mathematics Subject Classification (2010): 34H10, 37N35, 93C95, 93C40, 34D08, 34D06.

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1 Introduction

Over the last decades, there has been a great interest in the study of the chaotic behavior in deterministic systems. There are two leading applications in the chaos theory: chaos control and chaos synchronization. The emergence of these two areas in the study of nonlinear systems is due to the classical chaos control theory by Ott, Grebogi and Yorke [3] and chaos synchronization by Pecora and Carroll [7], followed by several types of synchronization that have been studied, namely generalized synchronization [9], projective synchronization [4], phase synchronization [14], anti-synchronization [8], etc.

Chaos, control and synchronization have received much attention due to their applications in many areas such as secure communication [11], biomedical engineering [2], ecological systems [1] and some other fields.

Lots of literature have studied the control, synchronization and anti-synchronization between two identical systems via classical control techniques, including PC and OGY methods, nonlinear optimal control [5], active control [15], adaptive control [12,13], etc.

In this paper, the first case obtained after a systematic computer search from the Sprott case A with a line equilibrium is chosen to illustrate the proposed techniques. We design state control laws to stabilize the new chaotic system around the unstable periodic solutions or unstable equilibrium points. On the other hand, we use the adaptive control method to synchronize two new identical systems with a line equilibrium. Chaos synchronization investigates the linking of the trajectory of one system to the other system with the same parameter values, the two coupled chaotic systems are called synchronized if the error system converges to zero as time goes to infinity. Similarly, two coupled chaotic systems are called anti-synchronized if the sum system converges to zero as time goes to infinity, we demonstrate the effectiveness and validity of the proposed adaptive stabilization, synchronization and anti-synchronization schemes for the new chaotic system with a line equilibrium.

The paper is organized as follows. In Section 2, a system description is given. In Section 3, we apply the adaptive control for stabilizing the chaotic orbits to the equilibrium point of the system. In Section 4, we use the adaptive control method to synchronize two identical systems with unknown parameters. In Section 5, we also build a new adaptive controller to anti-synchronize the same new identical chaotic systems. Finally, some conclusions are given in Section 6.

2 System Description

We consider a general parametric form of the Sprott case A system [6] with quadratic nonlinearities of the form

$$\begin{cases} \dot{x} = y, \\ \dot{y}_2 = a_1x + a_2yz, \\ \dot{z}_2 = a_3x + a_4y + a_5y^2 + a_7xy + a_8xz + a_9yz. \end{cases} \tag{1}$$

This system has a line equilibrium in $(0, 0, z)$ with no other equilibria.

In 2013, after a systematic computer search, six simple chaotic cases $LE_1 - LE_6$ were founded [10] with only six terms and all these cases were dissipative. The typical example (LE_1) is given by

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -x + yz, \\ \dot{z} = -x - axy - bxz, \end{cases} \tag{2}$$

where $(a, b) \in \mathbb{R}^2$.

For $(x_0, y_0, z_0) = (0., 0.5, 0.5)$, the chaotic attractor is shown in Figure 1.

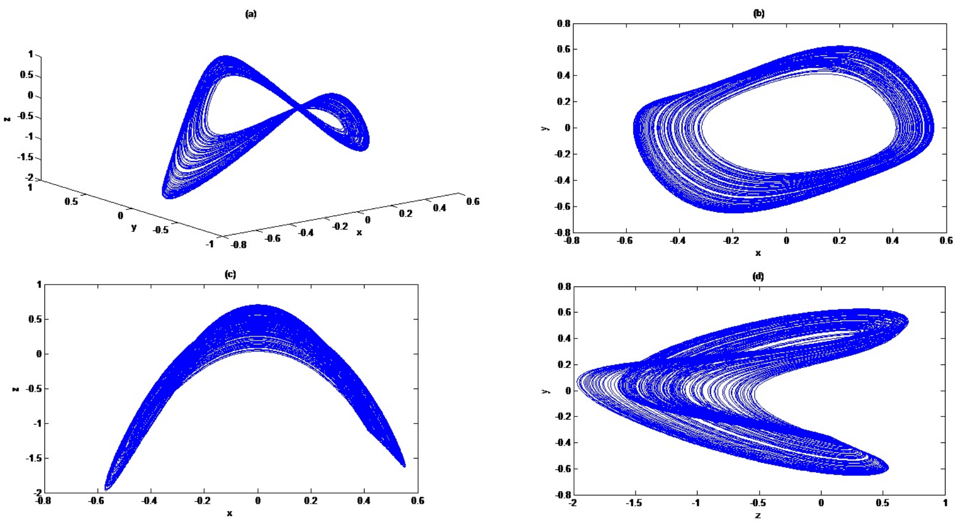

Figure 1: New chaotic system (2) with $a = 15$ and $b = 1$. (a) The strange attractor. (b) Projection on the x-y plane, (c) Projection on the x-z plane and (d) Projection on the z-y plane.

3 Adaptive Control of the New Chaotic System with a Line Equilibrium

3.1 Theoretical results

In this section, we design an adaptive control law for stabilizing the new chaotic system with a line equilibrium when the parameter values are unknown.

We add controllers to the system (2), then the controlled system is given by

$$\begin{cases} \dot{x} = y + u_1, \\ \dot{y} = -x + yz + u_2, \\ \dot{z} = -x - axy - bxz + u_3, \end{cases} \tag{3}$$

where u_1, u_2 and u_3 are the function controllers to be designed using the variables x, y and z.

In order to ensure that the controlled system (3) converges to the zero equilibrium asymptotically, we consider the following adaptive control functions:

$$\begin{cases} u_1(t) = -y - x, \\ u_2(t) = x - yz - y, \\ u_3(t) = x + \hat{a}xy + \hat{b}xz - z, \end{cases} \tag{4}$$

where $\hat{a}$ and $\hat{b}$ are the estimates of the unknown parameters a and b.

Substituting the control law (4) into the controlled system (3), we obtain

$$\begin{cases} \dot{x} = -x, \\ \dot{y} = -y, \\ \dot{z} = (\hat{a} - a)xy + (\hat{b} - b)xz - z. \end{cases} \tag{5}$$

Let us now define the parameter errors as follows:

$$\begin{aligned} e_a &= a - \hat{a}, \\ e_b &= b - \hat{b}. \end{aligned}$$

For the derivation of the update law in order to adjust the parameter estimates $\hat{a}$ and $\hat{b}$, the Lyapunov approach is used.

Consider the quadratic Lyapunov function

$$V = \frac{1}{2}(x^2 + y^2 + z^2 + e_a^2 + e_b^2),$$

which is a positive definite function on $\mathbb{R}^5$.

By deriving V, we get

$$\dot{V} = -x^2 - y^2 + z(-e_a xy - e_b xz - z) + e_a(-\dot{\hat{a}}) + e_b(-\dot{\hat{b}}). \tag{6}$$

In view of Eq.(6), the estimated parameters are updated by the following law:

$$\begin{cases} \dot{\hat{a}} = -zxy + e_a, \\ \dot{\hat{b}} = -z^2 x + e_b. \end{cases} \tag{7}$$

Substituting (7) into (6), we get

$$\dot{V} = -x^2 - y^2 - z^2 - e_a^2 - e_b^2 < 0. \tag{8}$$

Next, we prove the following result

Theorem 3.1 *The new controlled system (3) with unknown parameters is exponentially stabilized by the adaptive control law (4), where the parameter update law is given by (7).*

3.2 Numerical results

For the numerical simulations, the fourth-order Runge-Kutta method with the time step $\Delta t = 0.001$ is used to solve the system (3) with the adaptive control law (4) and parameter update law (7). The parameters of the system (3) are selected as: $a = 15$ and $b = 1$. Suppose that the initial values of the estimated parameters are $\hat{a}(0) = 0$ and $\hat{b}(0) = 4$. The initial states of the controlled system (3) are taken as: $x(0) = 0.5$, $y(0) = -0.5$ and $z(0) = 1.5$. When the adaptive control law (4) and the parameter update law (7) are used, the controlled system converges to the origin exponentially as shown in Figure 2. The time evolution of the parameter estimates is shown in Figure 3.

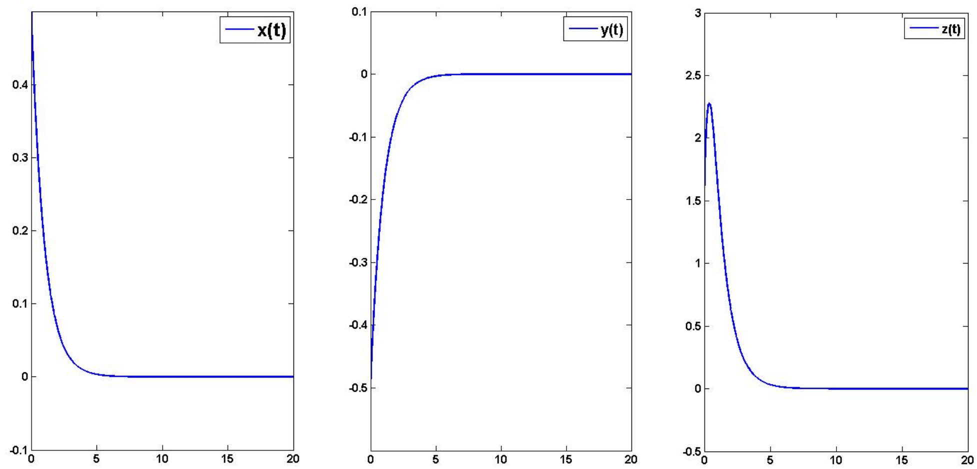

Figure 2: Time responses of the controlled system (3).

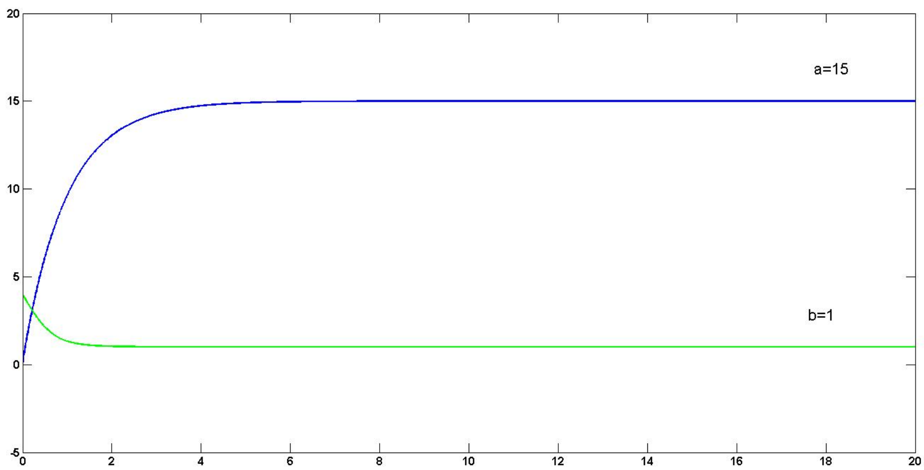

Figure 3: Time evolution of the parameter estimates $\hat{a}(t)$ and $\hat{b}(t)$.

4 Adaptive Synchronization of New Identical Chaotic Systems with a Line Equilibrium

4.1 Theoretical results

In this section, we design a control law for achieving synchronization between two new chaotic systems with two unknown parameters a and b.

The master system is given by

$$\begin{cases} \dot{x}_1 = y_1, \\ \dot{y}_1 = -x_1 + y_1 z_1, \\ \dot{z}_1 = -x_1 - a x_1 y_1 - b x_1 z_1. \end{cases} \tag{9}$$

The slave system is given by

$$\begin{cases} \dot{x}_2 = y_2 + u_1, \\ \dot{y}_2 = -x_2 + y_2 z_2 + u_2, \\ \dot{z}_2 = -x_2 - a x_2 y_2 - b x_2 z_2 + u_3, \end{cases} \tag{10}$$

where $u_i (i = 1, 2, 3)$ are the adaptive control functions to be designed.

It is said that synchronization occurs between master system (9) and slave system (10) if

$$\lim_{t \to \infty} \parallel X_2(t) - X_1(t) \parallel = 0,$$

where

$$X_1 = (x_1, y_1, z_1),$$
$$X_2 = (x_2, y_2, z_2).$$

Now, for our synchronization scheme, let us define error signals between system (9) and system (10) as

$$\begin{cases} e_x(t) = x_2(t) - x_1(t), \\ e_y(t) = y_2(t) - y_1(t), \\ e_z(t) = z_2(t) - z_1(t). \end{cases}$$

The time derivative of the error signal is

$$\begin{cases} \dot{e}_x(t) = e_y + u_1(t), \\ \dot{e}_y(t) = -e_x + y_2 z_2 - y_1 z_1 + u_2(t), \\ \dot{e}_z(t) = -e_x - a(x_2 y_2 - x_1 y_1) - b(x_2 z_2 - x_1 z_1) + u_3(t). \end{cases} \tag{11}$$

New chaotic systems (9) and (10) can be synchronized asymptotically for any different initial conditions with the following adaptive controller:

$$\begin{cases} u_1(t) = -e_y - e_x, \\ u_2(t) = e_x - y_2 z_2 + y_1 z_1 - e_y, \\ u_3(t) = e_x + \hat{a}(x_2 y_2 - x_1 y_1) + \hat{b}(x_2 z_2 - x_1 z_1) - e_z. \end{cases} \tag{12}$$

Let us now define the parameter errors as

$$\begin{aligned} e_a &= a - \hat{a}, \\ e_b &= b - \hat{b}. \end{aligned}$$

Then

$$\begin{cases} \dot{e}_x(t) = -e_x, \\ \dot{e}_y(t) = -e_y, \\ \dot{e}_z(t) = -e_a(x_2 y_2 - x_1 y_1) - e_b(x_2 z_2 - x_1 z_1) - e_z. \end{cases} \tag{13}$$

For the derivation of the update law for adjusting the parameter estimates $\hat{a}$ and $\hat{b}$, let us take the following Lyapunov function candidate:

$$V = \frac{1}{2}(e_x^2 + e_y^2 + e_z^2 + e_a^2 + e_b^2),$$

which is a positive definite function on $\mathbb{R}^5$.

By deriving V, we get

$$\dot{V} = -e_x^2 - e_y^2 + e_z(-e_a(x_2y_2 - x_1y_1) - e_b(x_2z_2 - x_1z_1) - e_z) + e_a(-\dot{\hat{a}}) + e_b(-\dot{\hat{b}}). \quad (14)$$

In view of Eq.(14), the estimated parameter is updated by the following law:

$$\begin{cases} \dot{\hat{a}} = e_z(x_2y_2 - x_1y_1) + e_a, \\ \dot{\hat{b}} = e_z(x_2z_2 - x_1z_1) + e_b. \end{cases} \quad (15)$$

Substituting (15) into (14), we get

$$\dot{V} = -e_x^2 - e_y^2 - e_z^2 - e_a^2 - e_b^2 < 0. \quad (16)$$

Next, we prove the following result

Theorem 4.1 *The new identical chaotic systems (9) and (10) are exponentially synchronized by the adaptive control law (12), where the parameter update law is given by (15).*

4.2 Numerical results

Numerical simulations are performed to verify the efficiency and feasibility of the controller (12). The Runge-Kutta method is employed with the time step $\Delta t = 0.001$. The parameters of the system (9) are selected as $a = 15$ and $b = 1$. Suppose that the initial values of the estimated parameters are: $\hat{a}(0) = 0$ and $\hat{b}(0) = 4$. We take the initial values of the master system (9) as: $x_1(0) = 0$, $y_1(0) = 0.5$ and $z_1(0) = 0.5$. We take the initial values of the slave system (10) as: $x_2(0) = 0.5$, $y_2(0) = -1$ and $z_2(0) = 1$. Figure 4 shows the complete synchronization of the identical systems (9) and (10). Figure 5 shows the time evolution of the parameter estimates $\hat{a}(t)$ and $\hat{b}(t)$.

5 Adaptive Anti-Synchronization of New Identical Chaotic Systems with a Line Equilibrium

5.1 Theoretical results

In this section, we design an adaptive control law for achieving the anti-synchronization of new identical chaotic systems with two unknown parameters.

The master system is given by

$$\begin{cases} \dot{x}_1 = y_1, \\ \dot{y}_1 = -x_1 + y_1z_1, \\ \dot{z}_1 = -x_1 - ax_1y_1 - bx_1z_1. \end{cases} \quad (17)$$

The slave system is given by

$$\begin{cases} \dot{x}_2 = y_2 + u_1, \\ \dot{y}_2 = -x_2 + y_2z_2 + u_2, \\ \dot{z}_2 = -x_2 - ax_2y_2 - bx_2z_2 + u_3, \end{cases} \quad (18)$$

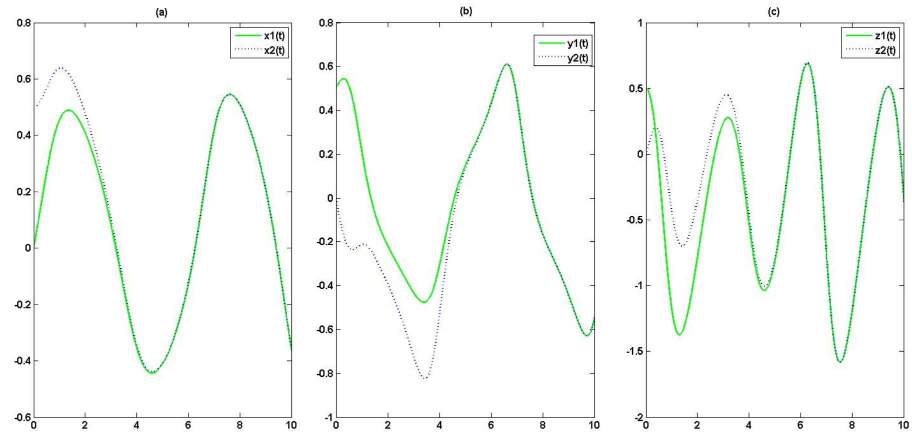

Figure 4: Time series of systems (9) and (10). (a) $x_1(t)$ and $x_2(t)$. (b) $y_1(t)$ and $y_2(t)$. (c) $z_1(t)$ and $z_2(t)$.

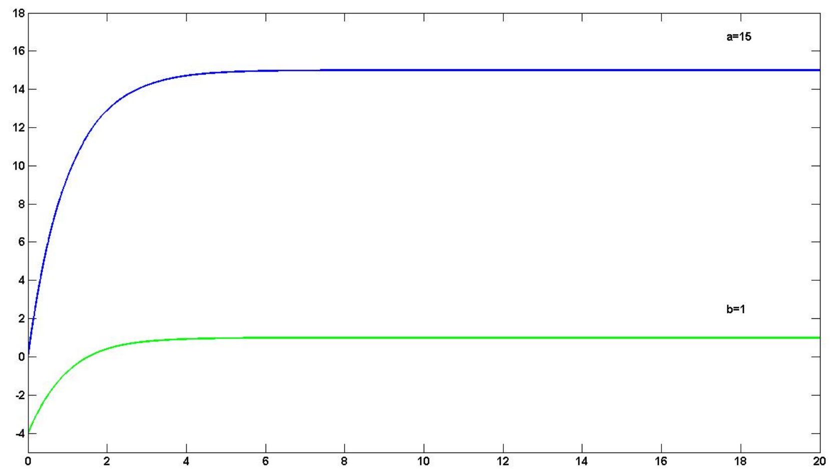

Figure 5: Time evolution of the parameter estimates $\hat{a}(t)$ and $\hat{b}(t)$.

where u_i $(i = 1, 2, 3)$ are the adaptive control fonctions to be designed. The anti-synchronization error between system (17) and system (18) is defined by

$$\lim_{t\to\infty} \parallel X_2(t) + X_1(t) \parallel = 0,$$

where

$$X_1 = (x_1, y_1, z_1)$$
$$X_2 = (x_2, y_2, z_2).$$

Then

$$\begin{cases} e_x(t) = x_2(t) + x_1(t), \\ e_y(t) = y_2(t) + y_1(t), \\ e_z(t) = z_2(t) + z_1(t). \end{cases}$$

The time derivative of the error signal is

$$\begin{cases} \dot{e}_x(t) = e_y + u_1(t), \\ \dot{e}_y(t) = -e_x + y_2 z_2 + y_1 z_1 + u_2(t), \\ \dot{e}_z(t) = -e_x - a(x_2 y_2 + x_1 y_1) - b(x_2 z_2 + x_1 z_1) + u_3(t). \end{cases} \tag{19}$$

New chaotic systems (17) and (18) can be anti-synchronized asymptotically for any different initial conditions with the following adaptive controller:

$$\begin{cases} u_1(t) = -e_y - e_x, \\ u_2(t) = e_x - y_2 z_2 - y_1 z_1 - e_y, \\ u_3(t) = e_x + \hat{a}(x_2 y_2 + x_1 y_1) + \hat{b}(x_2 z_2 + x_1 z_1) - e_z. \end{cases} \tag{20}$$

Let us now define the parameter errors as

$$\begin{aligned} e_a &= a - \hat{a}, \\ e_b &= b - \hat{b}. \end{aligned}$$

Then

$$\begin{cases} \dot{e}_x(t) = -e_x, \\ \dot{e}_y(t) = -e_y, \\ \dot{e}_z(t) = -e_a(x_2 y_2 + x_1 y_1) - e_b(x_2 z_2 + x_1 z_1) - e_z. \end{cases} \tag{21}$$

For the derivation of the update law for adjusting the parameter estimates $\hat{a}$ and $\hat{b}$, let us take the following Lyapunov function candidate:

$$V = \frac{1}{2}(e_x^2 + e_y^2 + e_z^2 + e_a^2 + e_b^2),$$

which is a positive definite function on $\mathbb{R}^5$.

By deriving V, we get

$$\dot{V} = -e_x^2 - e_y^2 - e_z(-e_a(x_2 y_2 + x_1 y_1) - e_b(x_2 z_2 + x_1 z_1) - e_z) + e_a(-\dot{\hat{a}}) + e_b(-\dot{\hat{b}}). \tag{22}$$

In view of Eq.(22), the estimated parameter is updated by the following law:

$$\begin{cases} \dot{\hat{a}} = -e_z(x_2 y_2 + x_1 y_1) + e_a, \\ \dot{\hat{b}} = -e_z(x_2 z_2 + x_1 z_1) + e_b. \end{cases} \tag{23}$$

Substituting (23) into (22), we get

$$\dot{V} = -e_x^2 - e_y^2 - e_z^2 - e_a^2 - e_b^2 < 0. \tag{24}$$

Next, we prove the following result.

Theorem 5.1 *The new identical chaotic systems are exponentially anti-synchronized by the adaptive control law (20), where the parameter update law is given by (23).*

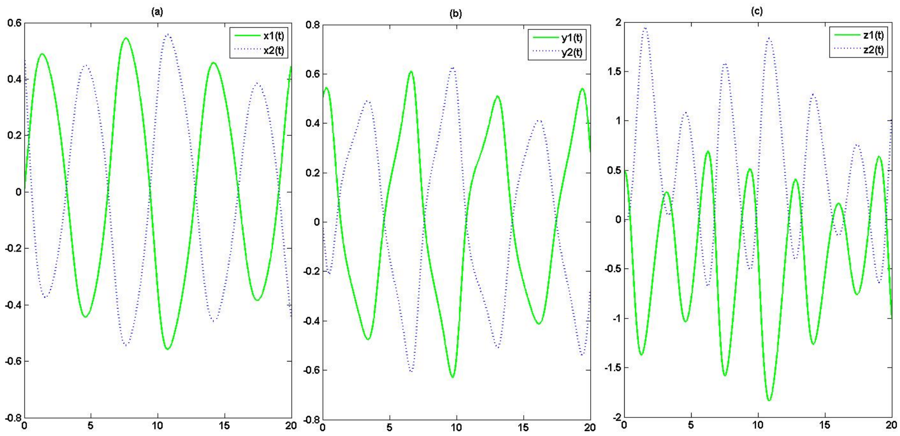

Figure 6: Time series of systems (18) and (19). (a) $x_1(t)$ and $x_2(t)$. (b) $y_1(t)$ and $y_2(t)$. (c) $z_1(t)$ and $z_2(t)$.

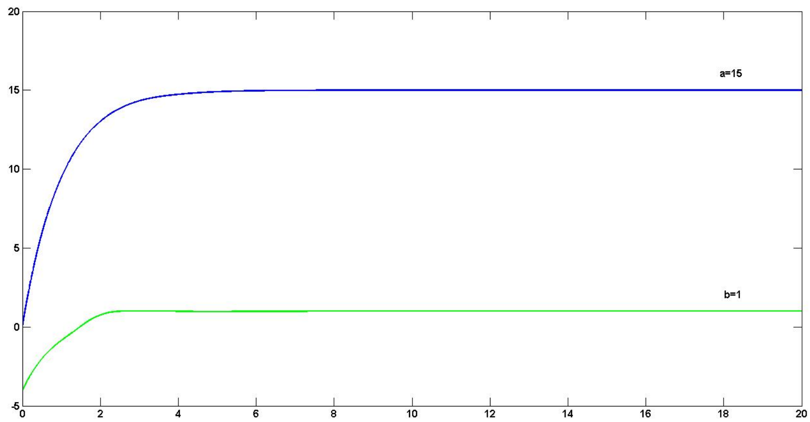

Figure 7: Time evolution of the parameter estimates $\hat{a}(t)$ and $\hat{b}(t)$.

5.2 Numerical results

The Runge-Kutta method is employed with the time step $\Delta t = 0.001$. The parameters of system (17) are selected as: $a = 15$ and $b = 1$. The initial values of the estimated parameters are $\hat{a}(0) = 0$ and $\hat{b}(0) = -4$. We take the initial values of the master system (17) as: $x_1(0) = 0$, $y_1(0) = 0.5$ and $z_1(0) = 0.5$. We take the initial values of the slave system (18) as: $x_2(0) = 0.5$, $y_2(0) = 0$ and $z_2(0) = 0$. Figure 6 shows the anti-synchronization of the identical systems (17) and (18). Figure 7 shows the time evolution of the parameter estimates $\hat{a}(t)$ and $\hat{b}(t)$.

6 Conclusion

In this paper, based on the Lyapunov stability theory and adaptive control theory, new results of stabilization, synchronization and anti-synchronization of the new chaotic system with a line equilibirum obtained from the Sprott case A are demonstrated. First, we designed adaptive control laws to stabilize the new chaotic system with unknown parameters to its unstable equilibrium point at the origin. Then, the synchronization and anti-synchronization of the new identical chaotic systems with unknown parameters are also realized. The proposed adaptive control method is very effective to achieve chaos control, synchronization and anti-synchronization. Numerical simulations are shown to demonstrate the effectiveness of the controllers in this paper.

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