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Existence of Solution for a General Class of Strongly Nonlinear Elliptic Problems

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Abstract: In this paper, we study the existence of solution for a general class of strongly nonlinear elliptic problems associated with the differential inclusion $\beta(u) + A(u)+g(x,u,Du) \ni f$, where A is a Leray-Lions operator from $W_0^{1,p}(\Omega)$ into its dual, β is a maximal monotone mapping such that $0 \in \beta(0)$, while $g(x,s,\xi)$ is a nonlinear term which has a growth condition with respect to ξ and no growth with respect to s but it satisfies a sign condition on s. The right-hand side f is assumed to belong to $L^\infty(\Omega)$.

Keywords: *inclusion problems; Leray-Lions operator; maximal monotone mapping; Sobolev spaces; truncation.*

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1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^N (N \geq 1)$ with sufficiently smooth boundary $\partial\Omega$. Our aim is to show the existence of solutions for the following strongly nonlinear elliptic inclusion:

$$(E,f) \quad \begin{cases} \beta(u) + A(u) + g(x,u,Du) \ni f \text{ in } \mathcal{D}'(\Omega), \\ \\ u \in W_0^{1,p}(\Omega),\ g(x,u,Du) \in L^1(\Omega),\ g(x,u,Du)u \in L^1(\Omega), \end{cases}$$

where A is a Leray-Lions operator from $W_0^{1,p}(\Omega)$ into its dual $W^{-1,p'}(\Omega)$ $(1 < p < \infty)$ defined as $A(u) = -div(a(x,u,Du))$, $f \in L^\infty(\Omega)$, β is a maximal monotone mapping

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such that $0 \in \beta(0)$ and g is a nonlinear lower term having "natural growth" (of order p) with respect to Du, with respect to u, we do not assume any growth restrictions, but we assume the "sign condition" $g(x,s,\xi)s \geq 0$.

The particular instances of the problem (E,f) have been studied for $\beta \equiv 0$, Boccardo, Gallouët and Murat in [7] have proved the existence of at least one solution for the problem. Let us point out that another work in this direction can be found in [6].

For $g \equiv 0$, it is known (cf. [5,10,18,20]) that the problem (E,f) has a solution in the standard sense, the so-called weak solution, that is, a couple $(u,b) \in W_0^{1,p}(\Omega) \times L^1(\Omega)$ such that $b \in \beta(u)$ a.e. in Ω and

$$\int_\Omega b\varphi + \int_\Omega a(x,u,Du) \cdot D\varphi = \int_\Omega f\varphi, \qquad \forall \varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega).$$

In another important work [2], Akdim and Allalou have proved the existence of a renormalized solution for an elliptic problem of diffusion-convection type in the framework of weighted variable exponent Sobolev spaces

$$(E) \begin{cases} \beta(u) - div(a(x,Du) + F(u)) \ni f & \text{in } \Omega, \\ \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

We also refer the reader to [1,3,4,8,12,13,17,19] for more results on problems in this direction. One of the motivations for studying (E,f) comes from applications to electrorheological fluids (see [16] for more details) as an important class of non-Newtonian fluids.

The present paper is organized as follows. In Section 2, we give assumptions and the statement of result. The proof of the theorem is given in Section 3, it consists of the following steps. First, we define approximation equations. We then prove an a priori estimate in $W_0^{1,p}(\Omega)$ for the solutions u_ε of these approximate equations. Finally, we prove that the truncations $T_k(u_\varepsilon)$ are relatively compact in the strong topology of $W_0^{1,p}(\Omega)$, a result which allows us to pass to the limit and obtain the existence result. In the last Section 4, we will present an example for illustrating our abstract result.

2 Assumptions and Main Result

2.1 Assumptions

Let Ω be a bounded domain in $\mathbb{R}^N (N \geq 1)$ with sufficiently smooth boundary $\partial\Omega$ and $1 < p < \infty$ be fixed.

Let A be a nonlinear operator from $W_0^{1,p}(\Omega)$ into its dual $W^{-1,p'}(\Omega)$ $(\frac{1}{p} + \frac{1}{p'} = 1)$ defined by $A(u) = -div(a(x,u,Du))$, where $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is the Carathéodory function satisfying the following assumptions:
(H_1)

$$a(x,s,\xi) \cdot \xi \geq \lambda |\xi|^p, \text{ where } \lambda > 0, \tag{1}$$

$$|a(x,s,\xi)| \leq \alpha(k(x) + |s|^{p-1} + |\xi|^{p-1}), \text{ where } k(x) \in L^{p'}(\Omega),\ k \geq 0,\ \alpha > 0, \tag{2}$$

$$(a(x,s,\xi) - a(x,s,\eta)) \cdot (\xi - \eta) > 0 \text{ for } \xi \neq \eta \in \mathbb{R}^N. \tag{3}$$

Let $g : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ be the Carathéodory function such that
(H_2)

$$g(x, s, \xi)s \geq 0, \tag{4}$$

$$|g(x, s, \xi)| \leq h(|s|)(c(x) + |\xi|^p), \tag{5}$$

where $h : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous increasing function and $c(x)$ is a positive function which is in $L^1(\Omega)$.

Let $\beta : \mathbb{R} \to 2^{\mathbb{R}}$ be a set valued, maximal monotone mapping such that $0 \in \beta(0)$ and consider
(H_3) $f \in L^\infty(\Omega)$.

2.2 Main result

Consider the strongly nonlinear elliptic inclusion problem with the Dirichlet boundary conditions

$$(E, f) \quad \begin{cases} \beta(u) + A(u) + g(x, u, Du) \ni f \text{ in } \mathcal{D}'(\Omega), \\ \\ u \in W_0^{1,p}(\Omega),\ g(x, u, Du) \in L^1(\Omega),\ g(x, u, Du)u \in L^1(\Omega). \end{cases}$$

Definition 2.1 A weak solution to (E, f) is a pair of functions $(u, b) \in W_0^{1,p}(\Omega) \times L^1(\Omega)$ satisfying $b(x) \in \beta(u(x))$ a.e. in Ω, $g(x, u, Du) \in L^1(\Omega)$, $g(x, u, Du)u \in L^1(\Omega)$ and

$$b - div(a(x, u, Du)) + g(x, u, Du) = f \quad \text{in} \quad \mathcal{D}^{'}(\Omega).$$

Our objective is to prove the following existence theorem.

Theorem 2.1 *Under the assumptions $(H_1) - (H_3)$, there exists at least one weak solution of (E, f) in the sense of Definition* 2.1.

3 Proof of Theorem 2.1

Step 1: Approximate problems

From now on, we will use the standard truncation function T_k, $k \geq 0$, defined for all $s \in \mathbb{R}$ by $T_k(s) = max\{-k, min\{s, k\}\}$.

Let $0 < \varepsilon \leq 1$, we introduce the approximate problem

$$(E_\varepsilon, f) \quad \begin{cases} \beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) - div(a(x, u_\varepsilon, Du_\varepsilon)) + g_\varepsilon(x, u_\varepsilon, Du_\varepsilon) = f, \\ \\ u_\varepsilon \in W_0^{1,p}(\Omega), \end{cases}$$

where

$$g_\varepsilon(x, s, \xi) = \frac{g(x, s, \xi)}{1 + \varepsilon|g(x, s, \xi)|}$$

satisfies

$$g_\varepsilon(x, s, \xi)s \geq 0,\ |g_\varepsilon(x, s, \xi)| \leq |g(x, s, \xi)|, \quad |g_\varepsilon(x, s, \xi)| \leq \frac{1}{\varepsilon}$$

and where $\beta_\varepsilon : \mathbb{R} \to \mathbb{R}$ is the Yosida approximation of β. Note that, for any $u \in W_0^{1,p}(\Omega)$, we have

$$\langle \beta_\varepsilon(u), u\rangle \geq 0, \quad |\beta_\varepsilon(u)| \leq \frac{1}{\varepsilon}|u| \quad \text{and} \quad \lim_{\varepsilon\to 0}\beta_\varepsilon(u) = \beta(u).$$

We refer the reader to [9] for more details about the maximal monotone mapping.

Since $|\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))| \leq \dfrac{1}{\varepsilon^2}$ and g_ε is bounded for any fixed $\varepsilon > 0$, there exists at least one solution u_ε of (E_ε, f) (cf. [14], [15]), i.e., for each $0 < \varepsilon \leq 1$ and $f \in W^{-1,p'}(\Omega)$, there exists at least one solution $u_\varepsilon \in W_0^{1,p}(\Omega)$ such that

$$\int_\Omega \beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))\varphi + \int_\Omega a(x,u_\varepsilon,Du_\varepsilon)\cdot D\varphi + \int_\Omega g_\varepsilon(x,u_\varepsilon,Du_\varepsilon)\varphi = \langle f,\varphi\rangle \tag{6}$$

holds for all $\varphi \in W_0^{1,p}(\Omega)$, where $\langle .,.\rangle$ denotes the duality pairing between $W_0^{1,p}(\Omega)$ and $W^{-1,p'}(\Omega)$.

Step 2: A priori estimates

Taking u_ε as a test function in (6), we obtain

$$\int_\Omega \beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))u_\varepsilon + \int_\Omega a(x,u_\varepsilon,Du_\varepsilon)\cdot Du_\varepsilon + \int_\Omega g_\varepsilon(x,u_\varepsilon,Du_\varepsilon)u_\varepsilon = \int_\Omega fu_\varepsilon, \tag{7}$$

as the first term on the left-hand side is nonnegative and since g_ε verifies the sign condition, by (1), we have

$$\lambda||u_\varepsilon||^p_{W_0^{1,p}(\Omega)} \leq C||f||_{L^\infty(\Omega)}||u_\varepsilon||_{W_0^{1,p}(\Omega)},$$

where C is a positive constant coming from the Hölder and Poincaré inequalities, then

$$||u_\varepsilon||_{W_0^{1,p}(\Omega)} \leq C_1. \tag{8}$$

Moreover, from (7) and (8), we infer that

$$0 \leq \int_\Omega g_\varepsilon(x,u_\varepsilon,Du_\varepsilon)u_\varepsilon \leq C_2. \tag{9}$$

For $\delta > 0$, we define $H_\delta^+ : \mathbb{R} \longrightarrow \mathbb{R}$ by

$$H_\delta^+(r) = \begin{cases} 1 & \text{if } r > \delta, \\ \dfrac{r}{\delta} & \text{if } 0 \leq r \leq \delta, \\ 0 & \text{if } r < 0. \end{cases}$$

Clearly, H_δ^+ is an approximation of $sign_0^+$.

We use the test function $\varphi = H_\delta^+(\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) - k)$ in (6), we obtain

$$\int_\Omega \beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))H_\delta^+(\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) - k) + \int_\Omega a(x, u_\varepsilon, Du_\varepsilon) \cdot D(H_\delta^+(\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) - k)) + \int_\Omega g_\varepsilon(x, u_\varepsilon, Du_\varepsilon)H_\delta^+(\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) - k) = \int_\Omega fH_\delta^+(\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) - k).$$

By using (1) and the fact that β_ε is monotone increasing with $\beta_\varepsilon(0) = 0$, we have

$$\int_\Omega a(x, u_\varepsilon, Du_\varepsilon)(H_\delta^+)^{'}(\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) - k)\beta_\varepsilon^{'}(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) \cdot Du_\varepsilon \geq 0.$$

Since g_ε verifies the sign condition, we obtain

$$\int_\Omega g_\varepsilon(x, u_\varepsilon, Du_\varepsilon)H_\delta^+(\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) - k) \geq 0.$$

Consequently, we get

$$\int_\Omega (\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) - k)H_\delta^+(\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) - k) \leq \int_\Omega (f - k)H_\delta^+(\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) - k).$$

Taking $\delta \to 0$ yields

$$\int_\Omega (\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) - k)^+ \leq \int_\Omega (f - k)^+. \tag{10}$$

Similarly, one can show

$$\int_\Omega (\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) + k)^- \leq \int_\Omega (f + k)^-. \tag{11}$$

Combining (10) and (11) gives

$$\int_\Omega (|\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))| - k)^+ \leq \int_\Omega (|f| - k)^+.$$

Choosing $k > ||f||_\infty$, we obtain

$$||\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))||_\infty \leq ||f||_\infty. \tag{12}$$

Step 3: Basic convergence results

By (12), there exists $b \in L^\infty(\Omega)$ such that

$$\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) \overset{*}{\rightharpoonup} b \text{ in } L^\infty(\Omega). \tag{13}$$

Since u_ε remains bounded in $W_0^{1,p}(\Omega)$, we can extract a subsequence, still denoted by u_ε, such that

$$u_\varepsilon \rightharpoonup u \text{ weakly in } W_0^{1,p}(\Omega)$$

and

$$u_\varepsilon \to u \text{ a.e. in } \Omega.$$

We already know that for any fixed $k \in \mathbb{R}^{*+}$,

$$T_k(u_\varepsilon) \rightharpoonup T_k(u) \text{ weakly in } W_0^{1,p}(\Omega).$$

Our objective is to prove that

$$T_k(u_\varepsilon) \to T_k(u) \text{ strongly in } W_0^{1,p}(\Omega).$$

We shall use in (6) the test function

$$v_\varepsilon = \varphi(z_\varepsilon),$$

where

$$z_\varepsilon = T_k(u_\varepsilon) - T_k(u) \text{ and } \varphi(s) = se^{\lambda s^2}.$$

We get

$$\int_\Omega \beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))v_\varepsilon + \int_\Omega a(x, u_\varepsilon, Du_\varepsilon) \cdot Dv_\varepsilon + \int_\Omega g_\varepsilon(x, u_\varepsilon, Du_\varepsilon)v_\varepsilon = \int_\Omega f v_\varepsilon.$$

From now on, we denote by $\eta^1(\varepsilon), \eta^2(\varepsilon), ...$ various sequences of real numbers which converge to zero when ε tends to zero.

Since v_ε converges to zero weakly* in $L^\infty(\Omega)$, we have

$$\int_\Omega f v_\varepsilon \to 0,$$

this implies that

$$\eta^1(\varepsilon) = \int_\Omega \beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))v_\varepsilon + \int_\Omega a(x, u_\varepsilon, Du_\varepsilon) \cdot Dv_\varepsilon + \int_\Omega g_\varepsilon(x, u_\varepsilon, Du_\varepsilon)v_\varepsilon \to 0.$$

Note that

$$\int_\Omega \beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))v_\varepsilon = \int_{\{|u_\varepsilon| \le k\}} \beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))v_\varepsilon + \int_{\{|u_\varepsilon| > k\}} \beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))v_\varepsilon.$$

The fact that the second term on the right-hand side is nonnegative and $\chi_{\{|u_\varepsilon| \le k\}}\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))$ is uniformly bounded, together with the Lebesgue dominated convergence theorem provide that

$$\int_{\{|u_\varepsilon| \le k\}} \beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))v_\varepsilon \to 0.$$

This implies that

$$\int_\Omega a(x, u_\varepsilon, Du_\varepsilon) \cdot Dv_\varepsilon + \int_\Omega g_\varepsilon(x, u_\varepsilon, Du_\varepsilon)v_\varepsilon \le \eta^2(\varepsilon).$$

Using the same arguments as in [7], we obtain

$$0 \le \int_\Omega [a(x, T_k(u_\varepsilon), DT_k(u_\varepsilon)) - a(x, T_k(u_\varepsilon), DT_k(u))] \cdot D(T_k(u_\varepsilon) - T_k(u)) \le \eta^3(\varepsilon).$$

Finally, a result in [8] (see also [11]) implies

$$T_k(u_\varepsilon) \to T_k(u) \text{ strongly in } W_0^{1,p}(\Omega). \tag{14}$$

Step 4: Passing to the limit

In virtue of (14), we have for the subsequence

$$Du_\varepsilon \to Du \text{ a.e. in } \Omega,$$

which with

$$u_\varepsilon \to u \text{ a.e. in } \Omega$$

yields, since $a(x, u_\varepsilon, Du_\varepsilon)$ is bounded in $(L^{p'}(\Omega))^N$,

$$a(x, u_\varepsilon, Du_\varepsilon) \rightharpoonup a(x, u, Du) \text{ weakly in } (L^{p'}(\Omega))^N \tag{15}$$

as well as

$$g_\varepsilon(x, u_\varepsilon, Du_\varepsilon) \to g(x, u, Du) \text{ a.e. in } \Omega. \tag{16}$$

We now use the classical trick in order to prove that $g_\varepsilon(x, u_\varepsilon, Du_\varepsilon)$ is uniformly equi-integrable.

For any measurable subset E of Ω and for any $m \in \mathbb{R}^+$, we have

$$\begin{aligned}\int_E |g_\varepsilon(x, u_\varepsilon, Du_\varepsilon)| &= \int_{E\cap\{|u_\varepsilon|\le m\}} |g_\varepsilon(x, u_\varepsilon, Du_\varepsilon)| + \int_{E\cap\{|u_\varepsilon|>m\}} |g_\varepsilon(x, u_\varepsilon, Du_\varepsilon)| \\ &\le \int_E |g_\varepsilon(x, T_m(u_\varepsilon), DT_m(u_\varepsilon))| + \frac{1}{m}\int_E g_\varepsilon(x, u_\varepsilon, Du_\varepsilon)u_\varepsilon.\end{aligned}$$

Using (5) and (9), we obtain

$$\int_E |g_\varepsilon(x, u_\varepsilon, Du_\varepsilon)| \le h(m)\int_E (c(x) + |DT_m(u_\varepsilon)|^p) + \frac{C_2}{m}.$$

Since the sequence $(DT_m(u_\varepsilon))$ converges strongly in $(L^p(\Omega))^N$, the above inequality implies the equi-integrability of $g_\varepsilon(x, u_\varepsilon, Du_\varepsilon)$.

In view of (16), we thus have

$$g_\varepsilon(x, u_\varepsilon, Du_\varepsilon) \to g(x, u, Du) \text{ strongly in } \; L^1(\Omega). \tag{17}$$

From (13), (15) and (17), we can pass to the limit in (6):

$$\int_\Omega \beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon))\varphi + \int_\Omega a(x, u_\varepsilon, Du_\varepsilon)\cdot D\varphi + \int_\Omega g_\varepsilon(x, u_\varepsilon, Du_\varepsilon)\varphi = \int_\Omega f\varphi,$$

we obtain

$$\int_\Omega b\varphi + \int_\Omega a(x, u, Du)\cdot D\varphi + \int_\Omega g(x, u, Du)\varphi = \int_\Omega f\varphi \text{ for any } \; \varphi \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega).$$

Moreover, since $g_\varepsilon(x, u_\varepsilon, Du_\varepsilon)u_\varepsilon \ge 0$, $g_\varepsilon(x, u_\varepsilon, Du_\varepsilon)u_\varepsilon \to g(x, u, Du)u$ a.e. in Ω and

$$0 \le \int_\Omega g_\varepsilon(x, u_\varepsilon, Du_\varepsilon)u_\varepsilon \le C,$$

by Fatou's lemma, we have

$$g(x,u,Du)u \in L^1(\Omega).$$

Step 5: Subdifferential argument

It remains to prove that $u(x) \in D(\beta(x))$ and $b(x) \in \beta(u(x))$ for almost all $x \in \Omega$. Since β is a maximal monotone graph, there exists a convex, l.s.c and proper function

$$j : \mathbb{R} \to [0,\infty] \text{ such that } \beta(r) = \partial j(r) \text{ for all } r \in \mathbb{R}.$$

According to [9], for $0 < \varepsilon \leq 1$, $j_\varepsilon : \mathbb{R} \to \mathbb{R}$ defined by $j_\varepsilon(r) = \int_0^r \beta_\varepsilon(s)ds$ has the following properties:

i) For any $0 < \varepsilon \leq 1, j_\varepsilon$ is convex and differentiable for all $r \in \mathbb{R}$ so that $j'_\varepsilon(r) = \beta_\varepsilon(r)$ for all $r \in \mathbb{R}$ and any $0 < \varepsilon \leq 1$.

ii) $j_\varepsilon(r) \to j(r)$ for all $r \in \mathbb{R}$ as $\varepsilon \to 0$.

From i), it follows that for any $0 < \varepsilon \leq 1$,

$$j_\varepsilon(r) \geq j_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) + (r - T_{\frac{1}{\varepsilon}}(u_\varepsilon))\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) \tag{18}$$

holds for all $r \in \mathbb{R}$ and almost everywhere in Ω.

Let $E \subset \Omega$ be an arbitrary measurable set and χ_E be its characteristic function. Let $h_l : \mathbb{R} \to \mathbb{R}$ be defined by $h_l(r) = min(1,(l+1-|r|)^+)$ for each $r \in \mathbb{R}$.

We fix $\varepsilon_0 > 0$, multiplying (18) by $h_l(u_\varepsilon)\chi_E$, integrating over Ω and using ii), we obtain

$$j(r)\int_E h_l(u_\varepsilon) \geq \int_E j_{\varepsilon_0}(T_{l+1}(u_\varepsilon))h_l(u_\varepsilon) + (r - T_{l+1}(u_\varepsilon))h_l(u_\varepsilon)\beta_\varepsilon(T_{\frac{1}{\varepsilon}}(u_\varepsilon)) \tag{19}$$

for all $r \in \mathbb{R}$ and all $0 < \varepsilon < min(\varepsilon_0, \frac{1}{l})$. As $\varepsilon \to 0$, taking into account that E is arbitrary, we obtain from (19)

$$j(r)h_l(u) \geq j_{\varepsilon_0}(T_{l+1}(u))h_l(u) + bh_l(u)(r - T_{l+1}(u)) \tag{20}$$

for all $r \in \mathbb{R}$ and almost everywhere in Ω.

Passing to the limit with $l \to \infty$ and then with $\varepsilon_0 \to 0$ in (20) finally yields

$$j(r) \geq j(u(x)) + b(x)(r - u(x)) \tag{21}$$

for all $r \in \mathbb{R}$ and almost everywhere in Ω, hence $u \in D(\beta)$ and $b \in \beta(u)$ for almost everywhere in Ω.

With this last step, the proof of Theorem 2.1 is concluded.

4 Example

Let Ω be a bounded domain of $\mathbb{R}^N (N \geq 1)$. Let us consider the Carathéodory functions

$$a(x,s,\xi) = |\xi|^{p-2}\xi,$$

$$g(x,s,\xi) = \rho s|s|^r|\xi|^p,\ \rho > 0,\ r > 0,$$

and β is the maximal monotone mapping defined by

$$\beta(s) = (s-1)^+ - (s+1)^-.$$

It is easy to show that the Carathéodory function $a(x, s, \xi)$ satisfies the growth condition (2), the coercivity condition (1) and the strict monotonicity condition (3). Also, the Carathéodory function $g(x, s, \xi)$ satisfies the conditions (4) and (5).

Finally, the hypotheses of Theorem 2.1 are satisfied, therefore, for all $f \in L^\infty(\Omega)$, the following problem

$$\begin{cases} \beta(u) - \Delta_p(u) + g(x, u, Du) \ni f \text{ in } \mathcal{D}'(\Omega), \\ \\ u \in W_0^{1,p}(\Omega),\ g(x, u, Du) \in L^1(\Omega),\ g(x, u, Du)u \in L^1(\Omega), \end{cases}$$

has at least one solution.

5 Conclusion

This paper focuses on establishing the existence of solution for a general class of strongly nonlinear elliptic problems associated with the differential inclusion $\beta(u) + A(u) + g(x, u, Du) \ni f$, where A is a Leray-Lions operator from $W_0^{1,p}(\Omega)$ into its dual, β is a maximal monotone mapping such that $0 \in \beta(0)$, while $g(x, s, \xi)$ is a nonlinear term which has a growth condition with respect to ξ and no growth with respect to s but it satisfies a sign condition on s. The right-hand side f is assumed to belong to $L^\infty(\Omega)$. In the future, we aim to expand this study for a measure or L^1 sources.

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