Nonlinear Dynamics and Systems Theory, **24** (4) (2024) 419–430

A Novel Numerical Approach for Solving Nonlinear Volterra Integral Equation with Constant Delay

K. Rouibah [1*], S. Kaouache [1] and A. Bellour [2]

[1] *Laboratory of Mathematics and Their Interactions, Abdelhafid Boussouf Center University, Mila 43000, Algeria.*
[2] *Laboratory of Applied Mathematics and Didactics, Ecole Normale Supérieure de Constantine 25000, Algeria.*

Received: January 14, 2024; Revised: June 6, 2024

Abstract: In this paper, we discuss the application of an iterative collocation method based on the Lagrangian polynomials to the numerical solution of a class of nonlinear Voltera integral equations with constant delay. This application contains, but is not limited to, many important Voltera delay integral equations that arise in physical and biological modeling processes and that are widely used in the analysis of dynamical systems. In addition, the appoximate solution is given in a suitable polynomial spline space by using explicit formulas without resorting to solving any algebric system. The proposed technique is efficient and easy to implement. The error analysis of the proposed numerical method is studied theoreticaly. Finally, illustrative examples are given to demonstrate the efficiency of the proposed method.

Keywords: *nonlinear delay Volterra integral equation; collocation method; iterative method; Lagrange polynomials.*

Mathematics Subject Classification (2010): 45J05, 45G10, 65R20, 70K99.

1 Introduction

In this paper, we study a numerical method for the solution of Volterra integral nonlinear equations with constant delay $\tau > 0$,

$$x(t) = f(t) + \int_0^t k_1(t,s,x(s))ds + \int_0^{t-\tau} k_2(t,s,x(s))ds, t \in I = [0,T], \tag{1}$$

* Corresponding author: mailto:r.khoula@centre-univ-mila.dz

where $x(t) = \Phi(t)$, $t \in [-\tau, 0]$, f, k_1, k_2 and Φ are the functions being sufficiently smooth. Equation (1) is frequently encountered in physical and biological modeling processes [1]. The monograph [2] presents a historical survey of mathematical models in biology, which can be described by Volterra integral equations with constant delays.

Now, some results on the numerical solutions of Volterra integral equations have been investigated [3–10]. Different methodologies have been proposed to approximate the solution of (1). For example, Ali et al. [11] proposed a spectral method for pantograph-type delay integral equations by using the Legendre collocation method. Brunner [2] applied the polynomial collocation method to approximate the solution of (1). Caliò et al. [12, 13] proposed a deficient spline collocation method, Birem et al. [16] developed an algorithm for solving first kind two-dimentional Voltera integral equations by using the collocation method, Horvat [14] used the spline collocation method to find a numerical solution of (1) in the spline space $S^{(d)}_{m+d}(\Pi_N)$ and Rouibah et al. [15] provided a new numerical approach based on the use of continuous collocation Lagrange polynomials for the numerical solution of nonlinear Volterra integral equations.

On the other hand, the Taylor polynomial method for approximating the solution of integral equations has been proposed in the recent years. For example, Bellour and Rawashdeh [17] used the Taylor method to find an approximate solution for first kind integral equations. Darania and Ivaz [18], Maleknejad and Mahmoudi [19], K. Al-Khaled and M.H. Yousef applied the Sumudu decomposition method [20], Sezer and Gülsu [21] applied the Taylor method to certain linear and nonlinear Volterra integral equations.

This paper is concerned with a piecewise polynomial collocation method based on the Lagrangian polynomials. Our goal is to develop an iterative explicit solution to approximate the solution of the Volterra integral equation with a constant delay (1).

The main advantages of the current collocation method are:

1. A more direct and convergent algorithm is introduced to compute the approximation solution and this provides an explicit numerical solution of the equation (1), which is a basic motivation for using an iterative collocation method.

2. In the current method, there is no algebraic system needed to be solved, which makes the proposed algorithm very effective, easy to implement and the calculation cost low.

The rest of the paper is organized as follows. In Section 2, we divide the interval $[0, T]$ into subintervals, and we approximate the proposed solution in each interval by using Lagrange polynomials. Global convergence is established in Section 3, and three numerical examples are provided in Section 4. Finally, conclusion is given in Section 5.

2 Description of the Collocation Method

Let Π_N be a uniform partition of the interval $I = [0, T]$ defined by

$$t^i_n = i\tau + nh, \quad i = 0, ..., r-1, \quad n = 0, ..., N-1, \tag{2}$$

where the stepsize is given by

$$\frac{\tau}{N} = h \text{ and } \tau = \frac{T}{r}.$$

Let $0 \le c_1 < ...... < c_m \le 1$ be the collocation parameters, and $t^i_{n,j} = t^i_n + c_j h$, $j = 1, ..., m, i = 0, ..., r-1, n = 0, ..., N-1$ be the collocation points.

Define the subintervals as $\sigma^i_n = [t^i_n, t^i_{n+1}]$, and $\sigma^i_{N-1} = [t^i_{N-1}, t^i_N]$. Denote by π_m the set of all real polynomials of degree not exceeding m. We define the real polynomial spline space of degree m as follows:

$$S_{m-1}^{(-1)}(I,\Pi_N)=\{u:u_n=u|_{\sigma_n^i}\in\pi_{m-1}, n=0,..,N-1, i=0,...,r-1\}. \tag{3}$$

It is easy to show that for any $y\in C^m([0,T])$,

$$y(t_n^i+sh)=\sum_{j=1}^{m}L_j(s)y(t_{n,j}^i)+\epsilon_n(s),\ \ \epsilon_n(s)=h^m\frac{y^m(\zeta_n)(s)}{m!}\prod_{j=1}^{m}(s-c_j), \tag{4}$$

where $s\in[0,1]$ and $L_j(v)=\prod_{l\neq j}^{m}\dfrac{v-c_l}{c_j-c_l}$ are the Lagrange polynomials associate with the parameters $c_j, j=1,...,m$.

Inserting (4) into (1), we get

$$\begin{aligned} x(t_{n,j}^i)=f(t_{n,j}^i)&+h\sum_{l=0}^{i-1}\sum_{p=0}^{N-1}\sum_{v=1}^{m}b_vk_1(t_{n,j}^i,t_{pv}^l,x(t_{p,v}^l))+h\sum_{p=0}^{n-1}\sum_{v=1}^{m}b_vk_1(t_{n,j}^i,t_{pv}^i,x(t_{pv}^i))\\ &+h\sum_{v=1}^{m}a_{j,v}k_1(t_{n,j}^i,t_{n,v}^i,x(t_{n,v}^i))+h\sum_{l=0}^{i-2}\sum_{p=0}^{N-1}\sum_{v=1}^{m}b_vk_2(t_{n,j}^i,t_{p,v}^l,x(t_{p,v}^l))\\ &+h\sum_{p=0}^{n-1}\sum_{v=1}^{m}b_vk_2(t_{n,j}^i,t_{p,v}^{i-1},x(t_{p,v}^{i-1}))+h\sum_{v=1}^{m}a_{j,v}k_2(t_{n,j}^i,t_{n,v}^{i-1},x(t_{n,v}^{i-1}))\\ &+o(h^m),\end{aligned}$$

where $j=1,...,m, i=0,...,r-1, n=0,...,N-1$, such that

$$a_{j,v}=\int_0^{c_j}L_v(\eta)d\eta$$

and

$$b_v=\int_0^1L_v(\eta)d\eta.$$

It holds for any $u\in S_{m-1}^{-1}(I,\Pi_N)$ that

$$u(t_n^i+sh)=\sum_{j=1}^{m}L_j(s)u(t_{n,j}^i), s\in[0,1]. \tag{5}$$

Now, we approximate the exact solution x by $u\in S_{m-1}^{-1}(I,\Pi_N)$ such that $u(t_{n,j}^i)$ satisfy the following nonlinear system:

$$\begin{aligned} u(t_{n,j}^i)=f(t_{n,j}^i)&+h\sum_{l=0}^{i-1}\sum_{p=0}^{N-1}\sum_{v=1}^{m}b_vk_1(t_{n,j}^i,t_{pv}^l,u(t_{p,v}^l))+h\sum_{p=0}^{n-1}\sum_{v=1}^{m}b_vk_1(t_{n,j}^i,t_{pv}^i,u(t_{pv}^i))\\ &+h\sum_{v=1}^{m}a_{j,v}k_1(t_{n,j}^i,t_{n,v}^i,u(t_{n,v}^i))+h\sum_{l=0}^{i-2}\sum_{p=0}^{N-1}\sum_{v=1}^{m}b_vk_2(t_{n,j}^i,t_{p,v}^l,u(t_{p,v}^l))\\ &+h\sum_{p=0}^{n-1}\sum_{v=1}^{m}b_vk_2(t_{n,j}^i,t_{p,v}^{i-1},u(t_{p,v}^{i-1}))+h\sum_{v=1}^{m}a_{j,v}k_2(t_{n,j}^i,t_{n,v}^{i-1},u(t_{n,v}^{i-1})),\end{aligned}$$

where $v = 1, ..., m$, $n = 0, ..., N-1$, $i = 0, ..., r-1$, and $u(t) = \phi(t) \in [-\tau, 0]$.

According to the nonlinearity of the previous system, we can use an iterative collocation solution $u^q \in S_{m-1}^{-1}(I, \Pi_N), q \in \mathbb{N}$, to approximate the exact solution of (1) such that

$$u^q(t_n^i + sh) = \sum_{j=1}^{m} L_j(s) u^q(t_{n,j}^i), s \in [0,1], \tag{6}$$

where the coefficients $u^q(t_{n,j}^i)$ are given by

$$\begin{aligned} u^q(t_{n,j}^i) &= f(t_{n,j}^i) + h\sum_{l=0}^{i-1}\sum_{p=0}^{N-1}\sum_{v=1}^{m} b_v k_1(t_{n,j}^i, t_{pv}^l, u^q(t_{p,v}^l)) \\ &+ h\sum_{p=0}^{n-1}\sum_{v=1}^{m} b_v k_1(t_{n,j}^i, t_{pv}^i, u^q(t_{pv}^i)) + h\sum_{v=1}^{m} a_{j,v} k_1(t_{n,j}^i, t_{n,v}^i, u^{q-1}(t_{n,v}^i)) \\ &+ h\sum_{l=0}^{i-2}\sum_{p=0}^{N-1}\sum_{v=1}^{m} b_v k_2(t_{n,j}^i, t_{p,v}^l, u^q(t_{p,v}^l)) + h\sum_{p=0}^{n-1}\sum_{v=1}^{m} b_v k_2(t_{n,j}^i, t_{p,v}^{i-1}, u^q(t_{p,v}^{i-1})) \\ &+ h\sum_{v=1}^{m} a_{j,v} k_2(t_{n,j}^i, t_{n,v}^{i-1}, u^q(t_{n,v}^{i-1})), \end{aligned}$$

where the initial values $u^0(t_{n,j}^i) \in J$ (J is a bounded interval). The above formula is explicit and the approximate solution u^q is given without needing to solve any algebraic system.

In the following section, we prove the convergence of the approximate solution u^q to the exact solution x of (1). Moreover, we can show that the order of convergence is m for all $q \geq m$.

3 Convergence Analysis

Here, we assume that the functions k_1 and k_2 satisfy the Lipschitz condition with respect to the third variable; i.e., there exists $L_i \geq 0$ $(i = 1, 2)$ such that

$$|k_i(t, s, y_1) - k_i(t, s, y_2)| \leq L_i |y_1 - y_2|.$$

The following lemmas are very important.

Lemma 3.1 *(Discrete Gronwall-type inequality [2])*

Let $\{k_j\}_{j=0}^n$ be a given non-negative sequence and the sequence $\{\varepsilon_n\}$ satisfy $\varepsilon_0 \leq p_0$ and

$$\varepsilon_n \leq p_0 + \sum_{i=0}^{n-1} k_i \varepsilon_i, \quad n \geq 1,$$

where $p_0 \geq 0$, then ε_n can be bounded by

$$\varepsilon_n \leq p_0 \exp\left(\sum_{j=0}^{n-1} k_j\right), \quad n \geq 1.$$

Lemma 3.2 *(Discrete Gronwall-type inequality [22])*
If $\{f_n\}_{n\geq 0}$, $\{g_n\}_{n\geq 0}$ *and* $\{\varepsilon_n\}_{n\geq 0}$ *are non negative sequences and*

$$\varepsilon_n \leq f_n + \sum_{i=0}^{n-1} g_i \varepsilon_i, \quad n \geq 0,$$

then

$$\varepsilon_n \leq f_n + \sum_{i=0}^{n-1} f_i g_i \exp\left(\sum_{k=0}^{n-1} g_k\right), \quad n \geq 0.$$

The following result gives the existence and uniqueness of the bounded solution for the nonlinear system (2).

Lemma 3.3 *For sufficiently small* h*, the nonlinear system* (2) *has a unique solution* $u \in S_{m-1}^{-1}$*. Moreover, the function* u *is bounded.*

Proof. **1.** The nonlinear system (2) has a unique solution in $\in S_{m-1}^{-1}$. We will use the induction combined with the Banach fixed point theorem.

(i) On the interval $\sigma_0^0 = [t_0^0, t_1^0]$, for $j = 1...m$, where $x(t) = \Phi(t)$ for $t \in [-\tau, 0]$, we have

$$u(t_{0,j}^0) = f(t_{0,j}^0) + h\sum_{v=1}^{m} a_{j,v} k_1(t_{0,j}^0, t_{0,v}^0, u(t_{0,v}^0)) + h\sum_{v=1}^{m} a_{j,v} k_2(t_{0,j}^0, t_{0,v}^0 - \tau, \phi(t_{0,v}^0)).$$

Let :$F_0^0 : R^m \to R^m \quad for \quad j = 1...m$, so

$$F_{0,j}^0(x) = f(t_{0,j}^0) + h\sum_{v=1}^{m} a_{j,v} k_1(t_{0,j}^0, t_{0,v}^0, x_v) + h\sum_{v=1}^{m} a_{j,v} k_2(t_{0,j}^0, t_{0,v}^0 - \tau, \phi(t_{0,v}^0)).$$

From the Banach fixed point theorem, we have

$$\|F_0^0(x) - F_0^0(y)\| \leq hL_1\|x - y\|,$$

which ensures the existence and uniqueness of $u \in \sigma_0^0$ for h being sufficiently small.

(ii) Suppose that u exists and is unique on each interval $\sigma_k^l, l = 0, ..., i-1, k = 0, ..., N-$

1, and we show that u exists and is unique on $\sigma_n^i = [t_n^i, t_{n+1}^i]$, $j = 1, ..., m$. Hence

$$\begin{aligned}
F_{n,j}^i(x) = f(t_{n,j}^i) &+ h\sum_{l=0}^{i-1}\sum_{p=0}^{N-1}\sum_{v=1}^{m} b_v k_1(t_{n,j}^i, t_{pv}^l, u(t_{p,v}^l)) \\
&+ h\sum_{p=0}^{n-1}\sum_{v=1}^{m} b_v k_1(t_{n,j}^i, t_{pv}^i, u(t_{pv}^i)) \\
&+ h\sum_{v=1}^{m} a_{j,v} k_1(t_{n,j}^i, t_{n,v}^i, x_v) \\
&+ h\sum_{l=0}^{i-2}\sum_{p=0}^{N-1}\sum_{v=1}^{m} b_v k_2(t_{n,j}^i, t_{p,v}^l, u(t_{p,v}^l)) \\
&+ h\sum_{p=0}^{n-1}\sum_{v=1}^{m} b_v k_2(t_{n,j}^i, t_{p,v}^{i-1}, u(t_{p,v}^{i-1})) \\
&+ h\sum_{v=1}^{m} a_{j,v} k_2(t_{n,j}^i, t_{n,v}^{i-1}, u(t_{n,v}^{i-1})).
\end{aligned}$$

For all $i = 0....r-1, n = 0...N-1, j = 1...m$, we have

$$\|F_{n,j}^i(x) - F_{n,j}^i(y)\| \le hL_1\|x-y\|,$$

which ensures the existence and uniqueness of $u \in \sigma_n^i$ for h being sufficiently small.

2. The solution u is bounded. We have

$$\begin{aligned}
\left|u(t_{n,j}^i)\right| \le \alpha &+ hL_1\sum_{l=0}^{i-1}\sum_{p=0}^{N-1}\sum_{v=1}^{m} b_v\left|u(t_{p,v}^l)\right| + hL_2\sum_{l=0}^{i-2}\sum_{p=0}^{N-1}\sum_{v=1}^{m} b_v\left|u(t_{p,v}^l)\right| \\
&+ hL_1\sum_{p=0}^{n-1}\sum_{v=1}^{m} b_v\left|u(t_{pv}^i)\right| + hL_2\sum_{p=0}^{n-1}\sum_{v=1}^{m} b_v\left|u(t_{p,v}^{i-1}))\right| \\
&+ hL_1\sum_{v=1}^{m} a_{j,v}\left|u(t_{n,v}^i)\right| + hL_2\sum_{v=1}^{m} a_{j,v}\left|u(t_{n,v}^{i-1}))\right|.
\end{aligned}$$

Let $\alpha = ||f|| + (\tau + T + h)(||k_1|| + ||k_2||)$ and let $y_n^i = max\{u(t_{n,P}^i), p = 1....m\}$. We have

$$\begin{aligned}
y_n^i - hL_1y_n^i &\le \alpha + hL_1\sum_{l=0}^{i-1}\sum_{p=0}^{N-1} y_p^l + hL_2\sum_{l=0}^{i-2}\sum_{p=0}^{N-1} y_p^l \\
&\quad + hL_2\sum_{p=0}^{n-1} y_p^{i-1} + hL_2 y_n^{i-1} + hL_1\sum_{p=0}^{n-1} y_p^i \\
&\le \alpha + h(L_1 + 3L_2)\sum_{l=0}^{i-1}\sum_{p=0}^{N-1} y_p^l + hL_1\sum_{p=0}^{n-1} y_p^i.
\end{aligned}$$

Hence, for all $h \in (0, \frac{1}{2L_1}]$, we get

$$y_n^i \leq 2\alpha + hL_3 \sum_{l=0}^{i-1} \sum_{p=0}^{N-1} y_p^l + hL_4 \sum_{p=0}^{n-1} y_p^i, \tag{7}$$

where $L_3 = 6L_2 + 2L_1$ and $L_4 = 2L_1$. Then, by Lemma 3.1, we obtain

$$\begin{aligned} y_n^i &\leq (2\alpha + hL_3 \sum_{l=0}^{i-1} \sum_{p=0}^{N-1} y_p^l) \exp(\tau L_4) \\ &\leq \alpha_2 + hL_5 \sum_{l=0}^{i-1} \sum_{p=0}^{N-1} y_p^l. \end{aligned}$$

We put $z_n^i = max\{y_n^i, n = 0....N-1\}$, we have $z^i \leq \alpha_2 + hL_5 \sum_{l=0}^{i-1} Nz^l$. Therefore, by Lemma 3.1, we obtain $z^i \leq \alpha_2 \exp(TL_5)$. So $(u(t_{n,j}^i))$ is bounded.

The following result gives the convergence of the approximate solution u to the exact solution x.

Theorem 3.1 *Let f, k_1, k_2 and Φ be m times continuously differentiable on their respective domains. Then for sufficiently small h, the collocation solution u converges to the exact solution x, and the resulting error function $e = x - u$ satisfies $\|e\|_{L^\infty(I)} \leq Ch^m$, where C is a finite constant independent of h.*

Proof. We calculate the error between x and the approximate solution u for $v = 1, 2, ...m$, $n = 0, 1, 2, ..., N-1$ and $i = 0, 1, ..., r-1$. By setting $e = x - u$ as the collocation error, we get

$$\begin{aligned} |e(t_{n,j}^i)| \leq{}& hL_1 \sum_{l=0}^{i-1} \sum_{p=0}^{N-1} \sum_{v=1}^{m} b_v \left|e(t_{p,v}^l)\right| + hL_2 \sum_{l=0}^{i-2} \sum_{p=0}^{N-1} \sum_{v=1}^{m} b_v \left|e(t_{p,v}^l)\right| \\ &+ hL_1 \sum_{p=0}^{n-1} \sum_{v=1}^{m} b_v \left|e(t_{pv}^i)\right| + hL_2 \sum_{p=0}^{n-1} \sum_{v=1}^{m} b_v \left|e(t_{p,v}^{i-1}))\right| \\ &+ hL_1 \sum_{v=1}^{m} a_{j,v} \left|e(t_{n,v}^i)\right| + hL_2 \sum_{v=1}^{m} a_{j,v} \left|e(t_{n,v}^{i-1}))\right|. \end{aligned}$$

Let $e_n^i = max\{e(t_{n,v}^i), v = 1, ..., m\}$, we have

$$e_n^i \leq hL_3 \sum_{l=0}^{i-1} \sum_{p=0}^{N-1} e_p^l + hL_1 \sum_{p=0}^{n-1} e_p^i + o(h^m), \text{ where } L_3 = 3L_2 + L_1.$$

Hence, for all $h \in (0, \frac{1}{2L_1}]$, we get

$$e_n^i \leq 2hL_3 \sum_{l=0}^{i-1} \sum_{p=0}^{N-1} e_p^l + 2hL_1 \sum_{p=0}^{n-1} e_p^i + 2o(h^m). \tag{8}$$

By Lemma 3.1, one can obtain

$$e_n^i \le (2hL_3 \sum_{l=0}^{i-1} \sum_{p=0}^{N-1} e_p^l + 2o(h^m)) \exp(2hL_1 N) \le h\alpha \sum_{l=0}^{i-1} \sum_{p=0}^{N-1} e_p^l + ch^m,$$

where $\alpha = 2hL_3 \exp(2hL_1 N)$.
Let $e^i = max\{e_n^i, n = 0...N-1\}$. So

$$e^i \le \tau\alpha \sum_{l=0}^{i-1} e_p^l + ch^m \le ch^m \exp(T\alpha) \le Ch^m.$$

Thus, the proof is completed by taking $C = c\exp(T\alpha)$. The following result gives the convergence of the iterative solution u^q to the exact solution x.

Theorem 3.2 *For any initial condition $u^0(t_{n,j}^i) \in J$, the sequence $u^q(t_{n,j}^i)$ converges to the exact solution x. Moreover, the following error estimates hold:*

$$\left|u^q(t_{n,j}^i) - x\right| \le (hd)^q \left|(u)^0 - x\right| + Cd^q h^{m+q} + Ch^m, \tag{9}$$

where $d = L_1 \exp(\tau L_1) + rL_1 \exp(\tau L_1)\tau(L_1 + 2L_2) \exp(\tau L_1) \exp(r\tau(L_1 + 2L_2) \exp(\tau L_1))$.

Proof. Let $(e_n^i)^q = max\left|u^{q+1}(t_{p,v}^l) - u(t_{p,v}^l)\right| v = 1....m$. So

$$\begin{aligned}(e_n^i)^{q+1} &\le hL_1 \sum_{l=0}^{i-1} \sum_{p=0}^{N-1} (e_p^l)^{q+1} + hL_1 \sum_{p=0}^{n-1} (e_p^i)^{q+1} + hL_1 (e_n^i)^q \\ &+ hL_2 \sum_{l=0}^{i-2} \sum_{p=0}^{N-1} (e_p^l)^{q+1} + hL_2 \sum_{p=0}^{n-1} (e_p^{i-1})^{q+1} + hL_2 (e_n^{i-1})^{q+1} \\ &\le hL_3 \sum_{l=0}^{i-1} \sum_{p=0}^{N-1} (e_p^l)^{q+1} + hL_1 (e_n^i)^q + hL_1 \sum_{p=0}^{n-1} (e_p^i)^{q+1},\end{aligned}$$

where $L_3 = L_1 + 3L_2$.
Let $e^i = max\{e_n^i, n = 0....N-1\}$. This implies

$$(e_n^i)^{q+1} \le hL_1 (e^i)^q + \tau L_3 \sum_{l=0}^{i-1} (e^l)^{q+1} + hL_1 \sum_{p=0}^{n-1} (e_p^i)^{q+1}.$$

In view of the well-known result on discrete Gronwall inequalities 3.1, we get

$$\begin{aligned}(e_n^i)^{q+1} &\le (hL_1 (e^i)^q + \tau L_3 \sum_{l=0}^{i-1} (e^l)^{q+1}) \exp(\tau L_1) \\ &\le hL_4 (e^i)^q + L_5 \sum_{l=0}^{i-1} (e^l)^{q+1},\end{aligned}$$

where $L_4 = L_1 \exp(\tau L_1)$ and $L_5 = \tau L_3 \exp(\tau L_1)$. By Lemma 3.2, one can obtain

$$(e^i)^{q+1} \le hL_4 (e^i)^q + \sum_{l=0}^{i-1} hL_4 (e^i)^q L_5 \exp(rL_5). \tag{10}$$

Let $(e)^q = max\{(e^i)^q, i = 0, 1, ..., r-1\}$, we have

$$\begin{aligned}(e)^{q+1} &\le hL_4(e)^q + hrL_4(e)^q L_5 \exp(rL_5),\\ &\le hd(e)^q \qquad d = L_4 + rL_4L_5\exp(rL_5),\\ &\le (hd)^q \left|(u)^0 - x\right| + Cd^q h^{m+q},\end{aligned}$$

where $j = 1, ..., m, i = 0, ..., r-1, n = 0, ..., N-1, q \in N^*$. This implies

$$\begin{aligned}\left|u^q(t^i_{n,j}) - x(t^i_{n,j})\right| &\le \left|u^q(t^i_{n,j}) - u(t^i_{n,j})\right| + \left|u(t^i_{n,j}) - x(t^i_{n,j})\right|\\ &\le (hd)^q \left|(u)^0 - x\right| + Cd^q h^{m+q} + Ch^m.\end{aligned}$$

4 Numerical Examples

In order to verify the theoretical results, we present the following examples with $\tau = 0.5$ and $T = 1$. All the exact solutions x are already known. In each example, we calculate the error between x and the iterative collocation solution u^m.

Example 4.1 Consider the nonlinear Volterra delay integral equation

$$x(t) = f(t) + \int_0^t k_1(t, s, x(s))ds + \int_0^{t-\tau} k_2(t, s, x(s))ds, \ t \in [0, 1], \tag{11}$$

where $k_1(t, s, z) = s\sin(t + 2z - s)$, $k_2(t, s, z) = \dfrac{se^{t-z}}{1+t}$ and f is chosen so that the exact solution is $x(t) = 2t + 1$.

The absolute errors for $(m, N) = \{(4, 4), (5, 5), (6, 6), (8, 8)\}$, for $t = 0, 0.2, ..., 1$ are shown in Table 1. From columns $2, 3, 4$, we note that the absolute error reduces as m increases, and from columns $4, 5$, we note that the absolute error reduces as N increases.

t	$m = 4, N = 5$	$m = 5, N = 5$	$m = 7, N = 5$	$m = 7, N = 10$
0.0	0.146×10^{-5}	0.69×10^{-7}	0.25×10^{-7}	0.33×10^{-7}
0.2	0.145×10^{-4}	0.342×10^{-6}	0.68×10^{-7}	0.6×10^{-8}
0.4	0.272×10^{-4}	0.75×10^{-7}	0.34×10^{-7}	0.6×10^{-8}
0.6	0.256×10^{-4}	0.82×10^{-6}	0.27×10^{-7}	0.4×10^{-8}
0.8	0.193×10^{-4}	0.20×10^{-5}	0.59×10^{-7}	0.56×10^{-7}
1.0	0.147×10^{-3}	0.13×10^{-4}	0.39×10^{-6}	0.7×10^{-7}

Table 1: Absolute errors of Example 4.2.

Example 4.2 The given functions in equation (1) are

$$k_1(t, s, z) = 2\cos(t + z - s)s^2,$$

$$k_2(t, s, z) = \frac{s^{tz}}{1+t^2}, \text{ and } x(t) = \sin(t) + 1.$$

The absolute errors for $(m, N) = \{(2, 5), (4, 5), (5, 5), (6, 10)\}$, for $t = 0, 0.2, ..., 1$ are shown in Table 2. From columns $2, 3, 4$, we note that the absolute error reduces as m increases, and from columns $4, 5$, we note that the absolute error reduces as both m and N increase.

t	$m=2, N=5$	$m=4, N=5$	$m=5, N=5$	$m=6, N=10$
0.2	0.123×10^{-3}	0.81×10^{-7}	0.14×10^{-7}	0.1×10^{-8}
0.2	0.472×10^{-3}	0.16×10^{-6}	0.17×10^{-7}	0.2×10^{-8}
0.4	0.794×10^{-3}	0.34×10^{-6}	0.4×10^{-8}	0.41×10^{-7}
0.6	0.232×10^{-2}	0.58×10^{-5}	0.19×10^{-7}	0.12×10^{-7}
0.8	0.813×10^{-2}	0.83×10^{-4}	0.312×10^{-6}	0.52×10^{-7}
1	0.599×10^{-2}	0.22×10^{-4}	0.245×10^{-6}	0.1×10^{-7}

Table 2: Absolute errors of Example 4.3.

Example 4.3 ([23,24]) Consider the following nonlinear Volterra integral equation:

t	Method in [23]		Method in [24]		Our method	
	$N=10$	$N=20$	$N=10$	$N=20$	$N=10$	$N=20$
0.1	1.0×10^{-5}	1.5×10^{-6}	1.2×10^{-5}	2.5×10^{-8}	9.8×10^{-8}	5.5×10^{-9}
0.2	2.4×10^{-5}	3.2×10^{-6}	1.6×10^{-6}	3.4×10^{-7}	1.4×10^{-7}	3.3×10^{-9}
0.3	3.6×10^{-5}	4.7×10^{-6}	2.0×10^{-4}	9.1×10^{-7}	2.0×10^{-7}	1.4×10^{-8}
0.4	4.6×10^{-5}	5.8×10^{-6}	2.0×10^{-5}	1.4×10^{-6}	2.7×10^{-7}	1.6×10^{-8}
0.5	5.2×10^{-5}	6.6×10^{-6}	3.8×10^{-5}	1.8×10^{-6}	3.5×10^{-7}	2.2×10^{-8}
0.6	5.5×10^{-5}	6.9×10^{-6}	5.1×10^{-5}	2.1×10^{-6}	4.3×10^{-7}	2.6×10^{-8}
0.7	5.5×10^{-5}	6.9×10^{-6}	7.2×10^{-5}	1.8×10^{-6}	5.1×10^{-7}	3.4×10^{-8}
0.8	5.2×10^{-5}	6.4×10^{-6}	6.4×10^{-5}	6.4×10^{-6}	5.6×10^{-7}	3.2×10^{-8}
0.9	4.6×10^{-5}	5.7×10^{-6}	1.9×10^{-5}	1.0×10^{-4}	6.0×10^{-7}	4.5×10^{-8}
01	3.9×10^{-5}	4.7×10^{-6}	6.3×10^{-4}	9.2×10^{-4}	4.6×10^{-7}	2.4×10^{-8}

Table 3: Comparison of the absolute errors of Example 4.3

$$x(t) = 1 + (\sin(t))^2 - \int_0^t 3\sin(t-s)(x(s))^2 ds, \ t \in [0,1],$$

which is a particular case of Equation (1), where $k_1(t,s,z) = -3\sin(t-s)z^2$ and $k_2(t,s,z) = 0$. Here $x(t) = \cos(t)$ is the exact solution.

The absolute errors for $N = 10, 20$ and $m = 4$ at $t = 0, 0.1, ..., 1$ are displayed in Table 3. The numerical results of the present method are considerably accurate in comparison with the numerical results obtained in [23, 24]. From Table 3, it is easy to see that the results obtained by the present method are very superior to those obtained by the methods in [23, 24].

5 Conclusion

In this paper, an iterative collocation method based on Lagrangian polynomials has been developed to achieve numerical solution of nonlinear Volterra delay integral equations in a suitable spline space. This method is easy to implement and the coefficients of the approximation solution are determined by using iterative formulas without the need to solve any system of algebraic equations. Numerical examples show that the method is convergent with a good accuracy. In addition, the results in these examples confirm the theoretical results. Moreover, from Tables 1 and 2, we note that the absolute error reduces as N or m increases.

Acknowledgements

This research was supported by the Algerian General Directorate for Scientific Research and Technological Development (DG-RSDT).

References

[1] K. L. Cooke. An epidemic equation with immigration. *Math. Biosci.* **29** (1976) 135–158.

[2] H. Brunner. *Collocation Methods for Volterra Integral and Related Functional Differential Equations*. Cambridge university press, Cambridge, 2004.

[3] H. Brunner. The discretization of neutral functional integro-differential equations by collocation methods. *Journal of analysis and its applications* **18** (1999) 393–406.

[4] H. Brunner. Iterated collocation methods for Volterra integral equations with delay arguments. *Mathematics of Computation* **62** (1994) 581–599.

[5] H. Brunner. Collocation and continuous implicit Runge-Kutta methods for a class of delay Volterra integral equations. *Journal of Computational and Applied Mathematics* **53** (1994) 61–72.

[6] H. Brunner and Y. Yatsenko. Spline collocation methods for nonlinear Volterra integral equations with unknown delay. *Journal of Computational and Applied Mathematics* **71** (1996) 67–81.

[7] Q. Hu. Multilevel correction for discrete collocation solutions of Volterra integral equations with delay arguments. *Applied Numerical Mathematics* **31** (1999) 159–171.

[8] B. Cahlon and L. J. Nachman. Numerical solutions of Volterra integral equations with a solution dependant delay. *Journal of Mathematical Analysis and Applications* **112** (1985) 541–562.

[9] A. Bellour and M. Bousselsal. A Taylor collocation method for solving delay integral equations. *Numer. Algor.* **65** (2014) 843–857.

[10] H. Laib, A. Bellour and A. Boulmerka, Taylor collocation method for high-order neutraldelay Volterra integro-differential equations. *J. Innov. Appl. Math. Comput. Sci.* **2** (1) (2022) 53–77.

[11] I. Ali, H. Brunner and T. Tang. Spectral methods for pantograph-type differential and integral equations with multiple delays. *Front. Math. China* **4** (2009) 49–61.

[12] F. Caliò, E. Marchetti and R. Pavani. About the deficient spline collocation method for particular differential and integral equations with delay. *Rend. Sem. Mat. Univ. Pol. Torino* **61** (2003) 287–300.

[13] F. Caliò, E. Marchetti, R. Pavani and G. Micula. About some Volterra problems solved by a particular spline. *Studia Univ. Babes-Bolyai* **48** (2003) 45–52.

[14] V. Horvat. On collocation methods for Volterra integral equations with delay arguments. *Mathematical Communications* **4** (1999) 93–109.

[15] K. Rouibah, A. Bellour, P. Lima and E. Rawa Shdeh. Iterative continuous collocation method for Solving nonlinear Volterra integral equations. *Kragujevac Journal of Mathematics* **64** (4) (2022) 635–648.

[16] F. Birem, A. Boulmarka, H. Laib and C. Hennous. An algorithm for solving first-kind two dimentional voltera integral equations using collocation method. *Nonlinear Dynamics and Systems Theory* **23** (5) (2023) 475–486.

[17] A. Bellour and E. Rawashdeh. Numerical solution of first kind integral equations by using Taylor polynomials. *Journal of Inequalities and Special Functions* **1** (2) (2010) 23–29.

[18] P. Darania and K. Ivaz. Numerical solution of nonlinear Volterra-Fredholm integro-differential equations. *Computers and Mathematics with Applications* **56** (9) (2008) 2197–2209.

[19] K. Maleknejad and Y. Mahmoudi. Taylor polynomial solution of high-order nonlinear Volterra-Fredholm integro-differential equations. *Applied Mathematics and Computation* **2** (2003) 641–653.

[20] K. Al-Khaled and M.H. Yousef. Sumudu decomposition method for solving higher-order nonlinear Volterra-Fredholm fractional integro-differential equations. *Nonlinear Dynamics and Systems Theory* **19** (3) (2019) 348–361.

[21] M. Sezer and M. Gülsu. Polynomial solution of the most general linear Fredholm-Volterra integro-differential difference equations by means of Taylor collocation method. *Applied Mathematics and Computation* **185** (2007) 646–657.

[22] R.P. Agrwal. *Difference equations and inequalities: Theory, Methods and Applications*. Second edition. Marcel Dekker, Inc. New York, 2000.

[23] N. Ebrahimi and J. Rashidinia. Collocation method for linear and nonlinear Fredholm and Volterra integral equations. *Appl. Math. Comput.* **270** (2015) 156–164.

[24] S.H. Javadi, A. Davari and E. Babolian. Numerical implementation of the Adomian decomposition method for nonlinear Volterra integral equations of the second kind. *Int. J. Comput. Math.* **84** (2007) 75–79.