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A New Numerical Scheme for Solving Time-Fractional Variable-Order Partial Differential Equations

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Abstract: In this work, we study approximations of solutions of fractional differential equations of order α by using an implicit finite difference scheme (IFDS). A discretization and development of the scheme is obtained by using different approaches to fractional derivatives. The implicit finite difference scheme (IFDS) approach is followed in order to derive a simple discretization of the space fractional derivatives. The consistency, stability and convergence of the method are proved. Several examples illustrating the accuracy of the method are given. Moreover, we study the stability and convergence of the implicit finite difference scheme (IFDS) applied to the numerical solution of the fractional differential equations of order α. Two tests for our problem are solved numerically to verify the effectiveness of the proposed numerical scheme.

Keywords: *fractional derivatives; stability; consistence; convergence; numerical scheme.*

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1 Introduction

In viscous boundary problems, there is a viscodynamic operator in the Biot poroviscoelastic theory which may be formulated with a fractional derivative (see [9, 10, 13, 14]). The power law and stretched exponential temporal responses of nonideal capacitors can also be shown to relate to the electro-elastic and electro-visco-elastic models. A fractional-order differential equation is a generalized form of an integer-order differential equation. This one is useful in many areas, e.g., for the depiction of physical, mechanical and biological models of several phenomena in pure mathematics and applied science. Numerical analysis (see for more details [5, 15, 16]) is a very important branch of mathematics in which we analyse and solve sevaral problems which require calculations with different techniques. The important role of numerical methods for fractional calculus is how to apply numerical methods for fractional integrals and fractional derivatives (see [2,4,14]), for example, finite element methods (FEM), finite difference methods, Multi-Grid methods (MGM) for fractional partial and ordinary differential equations. The development of fractional differential equations and their solutions are carried out by using the Simulink Matlab Program (see for more details [1,3,7,17]), which calculates the approximate solutions of fractional differential equations of order α by using the implicit finite difference scheme (IFDS). The structure of this paper is organized as follows. Section 2 includes the basic concepts of the implicit finite difference scheme (IFDS) and we present the steps of discretization and development of the scheme and several techniques of the proposed approach. Section 3 discusses the stability of the approximate scheme presented in Section 2. In Section 4, we prove the convergence of the approximate scheme for the fractional differential equations of order α. In Section 5, we focus on the solvability of the approximate scheme. Finally, Section 6 includes the examples of specific algorithms for a variety of boundary and initial conditions and conclusions are presented in Section 7.

2 Discretization and Development of the Implicit Finite Difference Scheme

In this section, we apply the NSFD to obtain the numerical solution for the linear partial differential equations with time-fractional derivative

$$\begin{cases} \frac{\partial^\alpha u}{\partial t^\alpha} + a(x,t)\frac{\partial^2 u}{\partial x^2}(x,t) + c(x,t)\frac{\partial u}{\partial x} = f(x,t) \;\; 0<x<L, \;\; t>0, \;\; 0<\alpha<1, \\ u(0,t) = q(t), \\ u(L,t) = p(t), \\ u(x,0) = s(x). \end{cases} \tag{1}$$

Let $\phi_1(h)$ and $\phi_2(k)$ be two strictly positive functions and $[0, L]$ be the interval of interest. For the numerical scheme, define $x_i = i\phi_1(h)$, where $i = \overline{0, M}$ and $t_j = j\phi_2(k)$, where $j = \overline{0, N}$. The parameters $\phi_1(h)$ and $\phi_2(k)$ are the space and time steps, respectively. Now, let us assume that

$$u(x_i, t_j) = u_i^j \text{ and } f(x_i, t_j) = f_i^j, \tag{2}$$

and

$$a(x_i, t_j) = a_i^j \text{ and } c(x_i, t_j) = c_i^j,$$

where u_i^j is the numerical approximation of $u(x_i, t_j)$ and f_i^j is the numerical approximation of $f(x_i, t_j)$.

There exist different approaches to fractional derivatives [11]. For simplification, we consider the interval $[0,t]$,

$$\frac{\partial^{\alpha(x,t)}u(x,t)}{\partial t^{\alpha(x,t)}}=\begin{cases}\frac{1}{\Gamma(1-\alpha(x,t))}\int_0^t \frac{u_\xi(x,\xi)}{(t-\xi)^{\alpha(x,t)}}d\xi \text{ if } 0<\alpha<1,\\ u_t(x,t) \text{ if } \alpha(x,t)=1.\end{cases} \tag{3}$$

Initially, as the boundary value problem needs to be discretized to be able to solve (1), it is first necessary to discretize the variable-order time-fractional derivative (3) as follow:

$$\begin{aligned}\frac{\partial^{\alpha(x_i,t_{j+1})}u(x_i,t_{j+1})}{\partial t^{\alpha x_i,t_{j+1})}} &= \frac{1}{\Gamma(1-\alpha(x_i,t_{j+1}))}\int_0^{t_{j+1}}\frac{u_\xi(x_i,\xi)}{(t_{j+1}-\xi)^{\alpha(x_i,t_{j+1})}}d\xi\\ &= \frac{1}{\Gamma(1-\alpha(x_i,t_{j+1}))}\sum_{s=0}^{j}\int_{sk}^{(s+1)k}\frac{u_\xi(x_i,\xi)}{(t_{j+1}-\xi)^{\alpha(x_i,t_{j+1})}}d\xi.\end{aligned}$$

Then we obtain

$$\frac{\partial^{\alpha(x_i,t_{j+1})}u(x_i,t_{j+1})}{\partial t^{\alpha(x_i,t_{j+1})}}=\frac{1}{\Gamma(1-\alpha(x_i,t_{j+1}))}\sum_{s=0}^{j}\int_{s\phi_2(k)}^{(s+1)\phi_2(k)}\left(\frac{\partial u}{\partial \xi}\right)_i^{s+1}\frac{d\xi}{(t_{j+1}-\xi)^{\alpha(x_i,t_{j+1})}}.$$

The first-order spatial derivative can be approximated by the following expression:

$$\left(\frac{\partial u}{\partial \xi}\right)_i^{s+1}=\frac{u_i^{s+1}-u_i^{s-1}}{\phi_2(k)}+\Delta(\phi_2(k)). \tag{4}$$

Adopting the discrete scheme given in (9), we discretize the variable-order time-fractional derivative as

$$\begin{aligned}\frac{\partial^{\alpha(x_i,t_{j+1})}u(x_i,t_{j+1})}{\partial t^{\alpha(x_i,t_{j+1})}} &= \frac{1}{\Gamma(1-\alpha(x_i,t_{j+1}))}\sum_{s=0}^{j}\frac{u_i^{s+1}-u_i^{s-1}}{\phi_2(k)}\int_{(j-s)\phi_2(k)}^{(j-s+1)\phi_2(k)}\frac{d\eta}{\eta^{\alpha(x_i,t_{j+1})}}\\ &= \frac{1}{\Gamma(1-\alpha(x_i,t_{j+1}))}\sum_{n=0}^{j}\frac{u_i^{j-n+1}-u_i^{j-n}}{\phi_2(k)}\int_{n\phi_2(k)}^{(n+1)\phi_2(k)}\frac{dy}{\eta^{\alpha(x_i,t_{j+1})}},\end{aligned}$$

then we get

$$\frac{\partial^{\alpha(x_i,t_{j+1})}u(x_i,t_{j+1})}{\partial t^{\alpha(x_i,t_{j+1})}}=\frac{\phi_2(k)^{-\alpha(x_i,t_{j+1})}}{(1-\alpha(x_i,t_{j+1}))\Gamma(1-\alpha(x_i,t_{j+1}))}\times$$
$$\sum_{n=0}^{j}(u_i^{j-n+1}-u_i^{j-n})[(n+1)^{1-\alpha(x_i,t_{j+1})}-n^{1-\alpha(x_i,t_{j+1})}].$$

Let $\eta=t_{j+1}-\xi$ and having in mind that $\Gamma(1+\alpha)=\alpha\Gamma(\beta)$ and expanding the summation for n = 0, we find

$$\frac{\partial^{\alpha(x_i,t_{j+1})}u(x_i,t_{j+1})}{\partial t^{\alpha(x_i,t_{j+1})}}=\frac{\phi_2(k)^{-\alpha(x_i,t_{j+1})}}{\Gamma(2-\alpha(x_i,t_{j+1}))}\Bigg[u_i^{j+1}-u_i^{j}+\sum_{n=1}^{j}(u_i^{j-n+1}-u_i^{j-n})\times \tag{5}$$
$$[(n+1)^{1-\alpha(x_i,t_{j+1})}-n^{1-\alpha(x_i,t_{j+1})}]\Bigg],$$

where $\delta_i^{j+1}(n) = (n+1)^{1-\alpha(x_i,t_{j+1})} - n^{1-\alpha(x_i,t_{j+1})}$, $\forall j = \overline{0, N-1}$. We will use the central difference approximation of space derivative is as follows:

$$\frac{\partial u(x_i, t_{j+1})}{\partial x} = \frac{u_{i+1}^{j+1} - u_{i-1}^{j+1}}{\phi_1(h)} + \Delta(\phi_1(h)). \tag{6}$$

The second-order spatial derivative can be approximated by the following expression:

$$\frac{\partial^2 u(x_i, t_{j+1})}{\partial x^2} = \frac{u_{i+1}^{j+1} - 2u_i^{j+1} + u_{i-1}^{j+1}}{\phi_1^2(h)} + \Delta(\phi_2^2(h)). \tag{7}$$

Using approximations (5), (6) and (7), the semi-linear diffusion equation (1), we obtain

$$\begin{gathered}\frac{\phi_2(k)^{-\alpha(x_i,t_{j+1})}}{\Gamma(2-\alpha(x_i,t_{j+1}))}\left[u_i^{j+1} - u_i^j + \sum_{n=1}^{j}(u_i^{j-n+1} - u_i^{j-n})\delta_i^{j+1}(n)]\right] + a_i^j \frac{u_{i+1}^{j+1} - 2u_i^{j+1} + u_{i-1}^{j+1}}{\phi_1^2(h)} + \\ + c_i^j \frac{u_{i+1}^{j+1} - u_{i-1}^{j+1}}{\phi_1(h)} = f_i^{j+1}, \ \forall i = \overline{1, M-1} \text{ and } \forall j = \overline{1, N-1},\end{gathered} \tag{8}$$

where

$$r_i^{j+1} = a_i^j \frac{\phi_2(k)^{\alpha(x_i,t_{j+1})}\Gamma(2-\alpha(x_i,t_{j+1}))}{\phi_1^2(h)}, w_i^{j+1} = c_i^j \frac{\phi_2(k)^{\alpha(x_i,t_{j+1})}\Gamma(2-\alpha(x_i,t_{j+1}))}{\phi_1(h)}$$

$$\rho_i^{j+1} = \phi_2(k)^{\alpha(x_i,t_{j+1})}\Gamma(2-\alpha(x_i,t_{j+1})).$$

The initial and boundary conditions are $u_0(x_i) = s_i$, $\forall i = \overline{0, M}$, and $u_0^{j+1} = q^{j+1}$, $u_M^{j+1} = p^{j+1}$, $\forall j = \overline{0, N-1}$. We obtain the following approximate scheme for equation (1):

$$\begin{cases} (r_i^{j+1} - w_i^{j+1})u_{i-1}^{j+1} + (1 - 2r_i^{j+1})u_i^{j+1} + (r_i^{j+1} + w_i^{j+1})u_{i+1}^{j+1} \\ \qquad = u_i^j - \sum_{n=1}^{j}(u_i^{j-n+1} \\ u_i^{j-n})\delta_i^{j+1}(n) + \rho_i^{j+1} f_i^{j+1}, \ \forall i = \overline{1, M-1} \text{ and } \ \forall j = \overline{1, N-1}, \\ u_0^{j+1} = q^{j+1}, \ u_M^{j+1} = p^{j+1}, \ \forall j = \overline{0, N-1}, \\ u_i^0 = s_i, \ \forall i = \overline{0, M}. \end{cases} \tag{9}$$

The coefficients $\delta_i^{j+1}(n)$ $(j = 0, .., N-1; i = 0, .., M)$ satisfy the following properties:

- **P1:** $\delta_i^{j+1}(0) = 1$.
- **P2:** $0 < \delta_i^{j+1}(0) < 1$.

3 Stability of the Approximate Scheme

In this section, we use the method of Fourier analysis to discuss the stability of the approximate scheme (9). Consider the following equation:

$$\begin{cases} (r_i^{j+1} - w_i^{j+1})u_{i-1}^{j+1} + (1 - 2r_i^{j+1})u_i^{j+1} + (r_i^{j+1} + w_i^{j+1})u_{i+1}^{j+1} = \\ u_i^j - \sum_{n=1}^{j}(u_i^{j-n+1} - u_i^{j-n})\delta_i^{j+1}(n) + \rho_i^{j+1} f_i^{j+1}, \\ \forall i = \overline{1, M-1} \text{ and } \forall j = \overline{1, N-1}. \end{cases} \tag{10}$$

Now, we define the following function:

$$u^j(x) = \begin{cases} u_i^j, & \text{if } x_{i-\frac{1}{2}} < x < x_{i+\frac{1}{2}}, \ \forall i = \overline{1, M-1}, \\ 0 & \text{otherwise,} \end{cases}$$

$u^j(x)$ has the Fourier series expansion

$$u^j(x) = \sum_{p=-\infty}^{+\infty} \xi_j(p) e^{\frac{2\pi p}{L} x}, \ \forall j = \overline{0, N},$$

where

$$\xi_j(p) = \frac{1}{L} \int_0^L u^j(x) e^{-\frac{2\pi p}{L} x} dx.$$

Assume that the solution of the equation (9) has the form

$$\nu_i^j = \xi_j e^{\mu\theta h i}, \tag{11}$$

where $\theta = \frac{2\pi p}{L}, \mu^2 = -1$. Now, replacing (11) in equation (10), we have

$$\begin{aligned} &\xi_{j+1}(r_i^{j+1}(e^{\mu\theta h} + e^{\mu\theta h}) + w_i^{j+1}(e^{\mu\theta h} - e^{-\mu\theta h}) + 1 - 2r_i^{j+1}) \\ = \ &\xi_j - \sum_{n=1}^{j} (\xi_{j-n+1} - \xi_{j-n}) \delta_i^{j+1}(n), \end{aligned} \tag{12}$$

then we get

$$\xi_{j+1}(1 - 4r_i^{j+1} \sin^2(\frac{\theta h}{2}) + 2\mu w_i^{j+1} \sin(\theta h)) = \xi_j - \sum_{n=1}^{j} (\xi_{j-n+1} - \xi_{j-n}) \delta_i^{j+1}(n). \tag{13}$$

One can see that equation (13) can be rewritten as

$$\xi_{j+1} = \frac{\xi_j - \sum_{n=1}^{j} (\xi_{j-n+1} - \xi_{j-n}) \delta_i^{j+1}(n)}{1 - 4r_i^{j+1} \sin^2(\frac{\theta h}{2}) + 2\mu w_i^{j+1} \sin(\theta h)}, \tag{14}$$

then we have the following result.

Theorem 3.1 *The implicit finite difference scheme* (9) *is unconditionally stable for* $0 < \beta < 1$ *if*

$$\exists C > 0 \qquad \|u^j\|_2 = |\xi_j| \leq C\|u^0\|_2 = C|\xi_0|, \ \ j = \overline{1, N}.$$

Proof. We use proof by recurrence for $j = 1$, in view of (14),

$$|\xi_1| = \left| \frac{\xi_0}{1 - 4r_i^1 \sin^2(\frac{\theta h}{2}) + 2\mu w_i^1 \sin(\theta h)} \right| =$$

$$\frac{|\xi_0|}{\sqrt{16[(r_i^1)^2 - (w_i^1)^2] \sin^4(\frac{\theta h}{2}) + 8[2(w_i^1)^2 - r_i^1] \sin^2(\frac{\theta h}{2}) + 1}} = C_i^0 |\xi_0| \leq C|\xi_0|$$

such that $C = \max_{0 \leq i \leq M} C_i^0$. We assume that the following statement is true:

$$|\xi_j| \leq C|\xi_0|, \ j = 1, 2, ..., N, \tag{15}$$

and we prove that the following statement is true:

$$|\xi_{j+1}| \leq C|\xi_0|, \quad j = \overline{0, N-1}, \tag{16}$$

we have

$$|\xi_{j+1}| = \left| \frac{\xi_0 - \sum_{n=1}^{j}(\xi_{j-n+1} - \xi_{j-n})\delta_i^{j+1}(n)}{1 - 4r_i^{j+1}\sin^2(\frac{\theta h}{2}) + 2\mu w_i^{j+1}\sin(\theta h)} \right| =$$

$$\frac{|\xi_0 - \sum_{n=1}^{j}(\xi_{j-n+1} - \xi_{j-n})\delta_i^{j+1}(n)|}{\sqrt{16[(r_i^{j+1})^2 - (w_i^{j+1})^2]\sin^4(\frac{\theta h}{2}) + 8[2(w_i^{j+1})^2 - r_i^{j+1}]\sin^2(\frac{\theta h}{2}) + 1}}$$

$$= C_i^j|\xi_0 - \sum_{n=1}^{j}(\xi_{j-n+1} - \xi_{j-n})\delta_i^{j+1}(n)|$$

$$\leq C^j|\xi_0| + \sum_{n=1}^{j}(|\xi_{j-n+1}| + |\xi_{j-n})||\delta_i^{j+1}(n)| \leq C^j(2N-1)|\xi_0| \leq C|\xi_0|$$

such that

$$C_i^j = \frac{1}{\sqrt{16[(r_i^{j+1})^2 - (w_i^{j+1})^2]\sin^4(\frac{\theta h}{2}) + 8[2(w_i^{j+1})^2 - r_i^{j+1}]\sin^2(\frac{\theta h}{2}) + 1}},$$

$$\forall i = \overline{0, M} \text{ and } \forall j = \overline{0, N-1},$$

$$C^j = \max_{0 \leq i \leq M} C_i^j, \quad C = (2N-1)\max_{0 \leq i \leq M} C^j, \quad \forall j = \overline{0, N-1}.$$

So we find $|\xi_{j+1}| \leq C|\xi_0|, j = 0, 1, .., N-1$. The approximate scheme (9) is unconditionally stable, which concludes the proof of Theorem 3.1.

4 Convergence of the Approximate Scheme

In this section, we use the method of Fourier analysis to discuss the convergence of the approximation error

$$e_i^j = u(x_i, t_j) - u_i^j. \tag{17}$$

Replacing (17) in equation (10), we obtain

$$\begin{aligned} &(r_i^{j+1} - w_i^{j+1})e_{i-1}^{j+1} + (1 - 2r_i^{j+1})e_i^{j+1} + (r_i^{j+1} + w_i^{j+1})e_{i+1}^{j+1} \\ = \quad & e_i^j - \sum_{n=1}^{j}(e_i^{j-n} - e_i^{j-n-1})\delta_i^{j+1}(n) + \epsilon_i^j \end{aligned} \tag{18}$$

for all $i = \overline{1, M-1}$ and $j = \overline{1, N-1}$. Then the error e_i^j takes the following form:

$$\epsilon_i^j = \phi_2(k)^{\alpha(x_i, t_{j+1})}\Gamma(2 - \alpha(x_i, t_{j+1}))[\Delta(\phi_2((k)) + \Delta(\phi_1(h))].$$

Now, we define the grid function $e^j(x)$ by

$$e^j(x) = \begin{cases} e_i^j, & \text{if } x_{i-\frac{1}{2}} < x < x_{i+\frac{1}{2}}, \quad \forall i = \overline{1, M-1} \text{ and } \forall j = \overline{1, N-1}, \\ 0, & \text{otherwise}, \end{cases}$$

and

$$\epsilon^j(x) = \begin{cases} \epsilon_i^j, & \text{if } x_{i-\frac{1}{2}} < x < x_{i+\frac{1}{2}}, \ \ \forall i = \overline{1, M-1} \text{ and } \forall j = \overline{1, N-1}, \\ 0, & \text{otherwise.} \end{cases}$$

Then $e^n(x)$ and $\epsilon^n(x)$ have the Fourier series expansions

$$e^j(x) = \sum_{p=-\infty}^{+\infty} \gamma_j(p) e^{\frac{2\pi m}{L}x}, \ \ \epsilon^j(x) = \sum_{p=-\infty}^{+\infty} \lambda_j(p) e^{\frac{2\pi p}{L}x}, \ \ \forall j = \overline{0, N}, \tag{19}$$

and

$$\gamma_j(p) = \frac{1}{L}\int_0^L e^j(x) e^{-\frac{2\pi p}{L}x} dx, , \ \ \lambda_j(p) = \frac{1}{L}\int_0^L \epsilon^j(x) e^{-\frac{2\pi p}{L}x} dx, \tag{20}$$

then

$$\int_0^L |e^j(x)|^2 dx = \sum_{p=-\infty}^{+\infty} |\gamma_j(p)|^2 \ \text{ and } \ \int_0^L |\epsilon^j(x)|^2 dx = \sum_{p=-\infty}^{+\infty} |\gamma_j(p)|^2, \ \ \forall j = \overline{0, N}.$$

Now, we take the norm in the two previous relations, then we get

$$\|e^j\|_2^2 = \sum_{i=1}^{M-1} |\phi_1(h) e_i^j|^2 = \sum_{p=-\infty}^{+\infty} |\gamma_j(p)|^2, \ \ \forall j = \overline{0, N}, \tag{21}$$

and

$$\|\epsilon^j\|_2^2 = \sum_{i=1}^{M-1} |h\epsilon_i^j|^2 = \sum_{m=-\infty}^{+\infty} |\lambda_j(p)|^2, \ \ \forall j = \overline{0, N}. \tag{22}$$

Now, we suppose

$$e_i^j = \gamma_j e^{\mu\tau h i} \ \text{ and } \ r_i^j = \lambda_j e^{\mu\tau h i}. \tag{23}$$

Replacing (23) in equation (18), we find

$$\begin{aligned} & \gamma_{j+1}(r_i^{j+1}(e^{\mu\theta h} + e^{\mu\theta h}) + w_i^{j+1}(e^{\mu\theta h} - e^{-\mu\theta h}) + 1 - 2r_i^{j+1}) \\ = \ & \gamma_j - \sum_{n=1}^{j} (\gamma_{j-n+1} - \gamma_{j-n}) \delta_i^{j+1}(n) + \lambda_j. \end{aligned} \tag{24}$$

Equation (24) can be rewritten as

$$\gamma_{j+1} = \frac{\gamma_j - \sum_{n=1}^{j} (\gamma_{j-n+1} - \gamma_{j-n}) \delta_i^{j+1}(n) + \lambda_j}{1 - 4r_i^{j+1} \sin^2(\frac{\theta h}{2}) + 2\mu w_i^{j+1} \sin(\theta h)}. \tag{25}$$

Theorem 4.1 *The implicit finite difference scheme* (9) *is convergent for* $0 < \alpha < 1$ *if*

$$\|e^j\|_2 = |\gamma_j| \le C^*(\phi_1(h) + \phi_2(k)), \ \ \forall j = \overline{1, N},$$

such that

$$\phi_1(h) + \phi_2(k) \longrightarrow 0 \ \textit{ when } \ (h, k) \rightarrow (0, 0).$$

Proof. We use the proof by recurrence for $j = 1$, we have

$$\begin{aligned}
|\gamma_1| &= \left| \frac{\gamma_0 + \lambda_0}{1 - 4r_i^{j+1}\sin^2(\frac{\theta h}{2}) + 2\mu w_i^{j+1}\sin(\theta h)} \right| \\
&= \frac{|\gamma_0 + \lambda_0|}{|1 - 4r_i^{j+1}\sin^2(\frac{\theta h}{2}) + 2\mu w_i^{j+1}\sin(\theta h)|} \\
&= \frac{|\gamma_0 + \lambda_0|}{\sqrt{16[(r_i^1)^2 - (w_i^1)^2]\sin^4(\frac{\theta h}{2}) + 8[2(w_i^1)^2 - r_i^1]\sin^2(\frac{\theta h}{2}) + 1}} \\
&\leq \frac{|\gamma_0| + |\lambda_0|}{\sqrt{16[(r_i^1)^2 - (w_i^1)^2]\sin^4(\frac{\theta h}{2}) + 8[2(w_i^1)^2 - r_i^1]\sin^2(\frac{\theta h}{2}) + 1}} \\
&= \frac{|\lambda_0|}{\sqrt{16[(r_i^1)^2 - (w_i^1)^2]\sin^4(\frac{\theta h}{2}) + 8[2(w_i^1)^2 - r_i^1]\sin^2(\frac{\theta h}{2}) + 1}},
\end{aligned}$$

we have $\gamma_0 = e_i^0 = u(x_i, 0) - u_i^0 = 0$. By the convergence of the series on the right-hand side of (22), there is a positive constant C_1 such that

$$\exists C_1 > 0 : \ \ |\epsilon_i^0| \leq C_1(\phi_1(h) + \phi_2(k)), \ \ \forall i = \overline{0, M},$$

then we have

$$\exists C_1 > 0 : \ \ \|\epsilon^0\|_2 = |\lambda_0| \leq C1\sqrt{L}(\phi_1(h) + \phi_2(k)).$$

Subsequently, we obtain

$$\begin{aligned}
|\gamma_1| &\leq \frac{C_1\sqrt{L}}{\sqrt{16[(r_i^1)^2 - (w_i^1)^2]\sin^4(\frac{\theta h}{2}) + 8[2(w_i^1)^2 - r_i^1]\sin^2(\frac{\theta h}{2}) + 1}}(\phi_1(h) + \phi_2(k)) \\
&\leq C^*(\phi_1(h) + \phi_2(k)).
\end{aligned}$$

Let us write

$$C^* = \max_{0 \leq i \leq M} \frac{C_1\sqrt{L}}{\sqrt{16[(r_i^1)^2 - (w_i^1)^2]\sin^4(\frac{\theta h}{2}) + 8[2(w_i^1)^2 - r_i^1]\sin^2(\frac{\theta h}{2}) + 1}}.$$

We assume that the following statement is true:

$$|\gamma_j| \leq C^*(\phi_1(h) + \phi_2(k)), \ \ \forall j = \overline{1, N}, \tag{26}$$

and we prove that the following statement is true:

$$|\gamma_{j+1}| \leq C^*(\phi_1(h) + \phi_2(k)), \ \ \forall j = \overline{1, N}, \tag{27}$$

then we have

$$\begin{aligned}
|\gamma_{j+1}| &= \left| \frac{\gamma_j - \sum_{n=1}^{j}(\gamma_{j-n+1} - \gamma_{j-n})\delta_i^{j+1}(n) + \lambda_j}{1 - 4r_i^{j+1}\sin^2(\frac{\theta h}{2}) + 2\mu w_i^{j+1}\sin(\theta h)} \right| \\
&= \frac{|\gamma_j - \sum_{n=1}^{j}(\gamma_{j-n+1} - \gamma_{j-n})\delta_i^{j+1}(n) + \lambda_j|}{|1 - 4r_i^{j+1}\sin^2(\frac{\theta h}{2}) + 2\mu w_i^{j+1}\sin(\theta h)|} \\
&= \frac{|\gamma_j - \sum_{n=1}^{j}(\gamma_{j-n+1} - \gamma_{j-n})\delta_i^{j+1}(n) + \lambda_j|}{\sqrt{16[(r_i^{j+1})^2 - (w_i^{j+1})^2]\sin^4(\frac{\theta h}{2}) + 8[2(w_i^{j+1})^2 - r_i^{j+1}]\sin^2(\frac{\theta h}{2}) + 1}} \\
&\leq \frac{|\gamma_j| + \sum_{n=1}^{j}(|\gamma_{j-n+1}| + |\gamma_{j-n}|)|\delta_i^{j+1}(n)| + |\lambda_j|}{\sqrt{16[(r_i^{j+1})^2 - (w_i^{j+1})^2]\sin^4(\frac{\theta h}{2}) + 8[2(w_i^{j+1})^2 - r_i^{j+1}]\sin^2(\frac{\theta h}{2}) + 1}} \\
&\leq \frac{C^*(2N-1)(\phi_1(h) + \phi_2(k)) + |\lambda_j|}{\sqrt{16[(r_i^{j+1})^2 - (w_i^{j+1})^2]\sin^4(\frac{\theta h}{2}) + 8[2(w_i^{j+1})^2 - r_i^{j+1}]\sin^2(\frac{\theta h}{2}) + 1}}.
\end{aligned}$$

By the convergence of the series on the right-hand side of (22), there is a positive constant C_1 such that

$$\exists C_1 > 0 : \ \|\epsilon_i^j| \leq C_1(\phi_2(k) + \phi_1(h)), \ \ \forall i = \overline{0, M} \ \text{ and } \ \forall j = \overline{0, N},$$

we obtain

$$\exists C_1 > 0 : \ \ \|\epsilon^j\|_2 = |\lambda_j| \leq C_1\sqrt{L}(\phi_2(k) + \phi_1(h)), \ \ \forall j = \overline{0, N}.$$

So

$$\begin{aligned}
|\gamma_{j+1}| &\leq C_i^j \left((2N-1)C^* + C_1\sqrt{L} \right) (\phi_2(k) + \phi_1(h)) \\
&\leq C \left(C^* + \frac{C_1\sqrt{L}}{2N-1} \right) (\phi_2(k) + \phi_1(h)) \\
&= C^*(\phi_2(k) + \phi_1(h)),
\end{aligned}$$

then we have

$$C_i^j = \frac{1}{\sqrt{16[(r_i^{j+1})^2 - (w_i^{j+1})^2]\sin^4(\frac{\theta h}{2}) + 8[2(w_i^{j+1})^2 - r_i^{j+1}]\sin^2(\frac{\theta h}{2}) + 1}}$$
$$\forall i = \overline{0, M} \ \text{ and } \ \forall j = \overline{0, N-1},$$

where

$$C^j = \max_{0 < i < M} C_i^j, \ \text{ and } \ C = (2N-1)\max_{0 < i < M} C^j, \ \ \forall j = \overline{0, N-1},$$

such that the constant L is given by

$$L = \left(\frac{C^*(1-C)(2N-1)}{C_1 C} \right)^2.$$

So we find

$$|\gamma_j| \leq C^*(\phi_1(h) + \phi_2(k)), \forall j = \overline{1, N}. \tag{28}$$

Therefore the implicit finite difference scheme (9) is convergent, which concludes the proof of Theorem 4.1.

5 Solvability of the Approximate Scheme

We have the second result on the solvability of the approximate scheme which is given by Theorem 4.1.

Theorem 5.1 *The approximate scheme* (9) *is uniquely solvable.*

It can be seen that the corresponding homogeneous linear algebraic equations for the approximate scheme

$$\frac{\phi_2(k)^{-\alpha(x_i,t_{j+1})}}{\Gamma(2-\alpha(x_i,t_{j+1}))}\left[u_i^{j+1}-u_i^j+\sum_{n=1}^{j}(u_i^{j-n+1}-u_i^{j-n})\delta_i^{j+1}(n)]\right]+a_i^j\frac{u_{i+1}^{j+1}-2u_i^{j+1}+u_{i-1}^{j+1}}{\phi_3(h)}$$
$$+c_i^j\frac{u_{i+1}^{j+1}-u_{i-1}^{j+1}}{\phi_1(h)}=f_i^{j+1},$$
$$\forall i=\overline{1,M-1} \text{ and } \forall j=\overline{1,N-1},$$

where

$$r_i^{j+1}=a_i^j\frac{\phi_2(k)^{\alpha(x_i,t_{j+1})}\Gamma(2-\alpha(x_i,t_{j+1}))}{\phi_1^2(h)},w_i^{j+1}=c_i^j\frac{\phi_2(k)^{\alpha(x_i,t_{j+1})}\Gamma(2-\alpha(x_i,t_{j+1}))}{\phi_1(h)}$$
$$\rho_i^{j+1}=\phi_2(k)^{\alpha(x_i,t_{j+1})}\Gamma(2-\alpha(x_i,t_{j+1})),$$

are

$$\begin{cases}(r_i^{j+1}-w_i^{j+1})u_{i-1}^{j+1}+(1-2r_i^{j+1})u_i^{j+1}+(r_i^{j+1}+w_i^{j+1})u_{i+1}^{j+1}\\ \qquad =u_i^j-\sum_{n=1}^{j}(u_{i+1}^{j-n+1}\\ u_i^{j-n})\delta_i^{j+1}(n)+\rho_i^{j+1}f_i^{j+1},\ \forall i=\overline{1,M-1} \text{ and } \forall j=\overline{1,N-1},\\ u_0^{j+1}=q^{j+1},\ u_M^{j+1}=p^{j+1},\ \forall j=\overline{0,N-1},\\ u_i^0=s_i,\ \forall i=\overline{0,M}.\end{cases} \tag{29}$$

Proof. Similar to the proof of Theorem 3.1, we can also verify the solutions of the equations (25) satisfy $\|u^j\|_2 \le C\|u^0\|,\ \forall j=\overline{1,N}$, we have $u^0=0$, so we get $u^j=0,\ \forall j=\overline{1,N}$. This indicates that the equations (29) have only zero solutions, the approximate scheme (9) is uniquely solvable, which concludes the proof of Theorem 5.1.

6 Examples and Numerical Experiments

Here, we present different numerical experiments to support the theoretical and numerical analyses of the previous sections. The following variable-order time-fractional diffusion equation is considered:

$$\begin{cases}\frac{\partial^\alpha u}{\partial t^\alpha}+\frac{\partial^2 u}{\partial x^2}(x,t)+\frac{\partial u}{\partial x}=f(x,t)\ \ 0<x<L,\ \ t>0,\ \ 0<\alpha<1,\\ u(0,t)=0,\ \ u(1,t)=0,\\ u(x,0)=10x^2(1-x),\end{cases} \tag{30}$$

where N=M=N_x=50, 100, 150, 200, and we have tested our numerical approximations by two values of $\alpha(x,t)$. First, we have $\alpha(x,t)=\frac{\sin(x.t)^2}{4}$. But, in the second part,

$\alpha(x,t) = \frac{\sqrt{(cos(x.t)^2+\sqrt{x.t}}}{4}$ and the function $f(x,t)$ are used to test our approaches, where the function $f(x,t)$ is given by

$$f(x,t) = 20x^2(1-x)\left[\frac{t^{2-\alpha(x,t)}}{\Gamma(3-\alpha(x,t))} + \frac{t^{1-\alpha(x,t)}}{\Gamma(2-\alpha(x,t))}\right] + 10(-3x^2-4x+2)(1+t)^2.$$

The exact solution is given by

$$u(x,t) = 10x^2(1-x)(t+1)^2.$$

By the implicit finite-difference scheme discretization method, the derivatives can be approximated as follows.

First, we choose the dominator functions in the following form:

$$\phi_1(h) = h \text{ and } \phi_2(k) = 2(e^k - 1).$$

Let now $[0,1]$ be the interval of inerest we discretise the domain first. We define

$$x_i = i\phi_1(h) \text{ where } i = \overline{0, N_x} \text{ and } t_j = j\phi_2(k) \text{ where } j = \overline{0, N_x},$$

where $\phi_2(k)$ represents the time step size and $\phi_1(h)$ represents the space step length. Let us assume that

$$f(x_i,t_{j+1}) = f_i^{j+1} = 20x_i^2(1-x_i)\left[\frac{t_{j+1}^{2-\alpha(x_i,t_{j+1})}}{\Gamma(3-\alpha(x_i,t_{j+1}))} + \frac{t_{j+1}^{1-\alpha(x_i,t_{j+1})}}{\Gamma(2-\alpha(x_i,t_{j+1}))}\right] + \tag{31}$$
$$+10(-3x_i^2-4x_i+2)(1+t_{j+1})^2,$$
$$q^{j+1} = 0, \tag{32}$$
$$p^{j+1} = 0, \tag{33}$$

where u_i^j is the numerical approximation of $u(x_i,t_j)$ *and* f_i^j is the numerical approximation of $f(x_i,t_j)$.
We obtain the following approximate scheme for equation (30):

$$\begin{cases} (r_i^{j+1} - w_i^{j+1})u_{i-1}^{j+1} + (1-2r_i^{j+1})u_i^{j+1} + (r_i^{j+1} + w_i^{j+1})u_{i+1}^{j+1} \\ \quad = u_i^j - \sum_{n=1}^{j}(u_{i+1}^{j-n+1} \\ u_i^{j-n-1})\delta_i^{j+1}(n) + \rho_i^{j+1}f_i^{j+1}, \ \forall i = \overline{1, N_x-1} \text{ and } \forall j = \overline{1, N_x-1}, \\ u_0^{j+1} = q^{j+1}, \ u_{N_x}^{j+1} = p^{j+1}, \ \forall j = \overline{0, N_x-1}, \\ u_i^0 = s_i, \ \forall i = \overline{0, N_x}, \end{cases} \tag{34}$$

where

$$r_i^{j+1} = a_i^j \frac{\phi_2(k)^{\alpha(x_i,t_{j+1})}\Gamma(2-\alpha(x_i,t_{j+1}))}{\phi_1^2(h)}, w_i^{j+1} = c_i^j \frac{\phi_2(k)^{\alpha(x_i,t_{j+1})}\Gamma(2-\alpha(x_i,t_{j+1}))}{\phi_1(h)}$$
$$\rho_i^{j+1} = \phi_2(k)^{\alpha(x_i,t_{j+1})}\Gamma(2-\alpha(x_i,t_{j+1})), \tag{35}$$

and

$$\delta_i^{j+1}(n) = (n+1)^{1-\alpha(x_i,t_{j+1})} - n^{1-\alpha(x_i,t_{j+1})}, \ \forall j = \overline{0, N_x-1}.$$

In this example, we have tested the numerical solution when N=M=N_x=50, 100, 150, 200 and $\alpha(x,t) = \frac{\sin(x.t)^2}{4}$. But, in the second part of our test, $\alpha(x,t) = \frac{\sqrt{(cos(x.t)^2+\sqrt{x.t}}}{4}$ when N=M=N_x=50, 100, 150, 200. Finally, various numerical tests are presented in both one dimension and for general meshes to illustrate the capacity of the schemes and compare theoretical and experimental results.

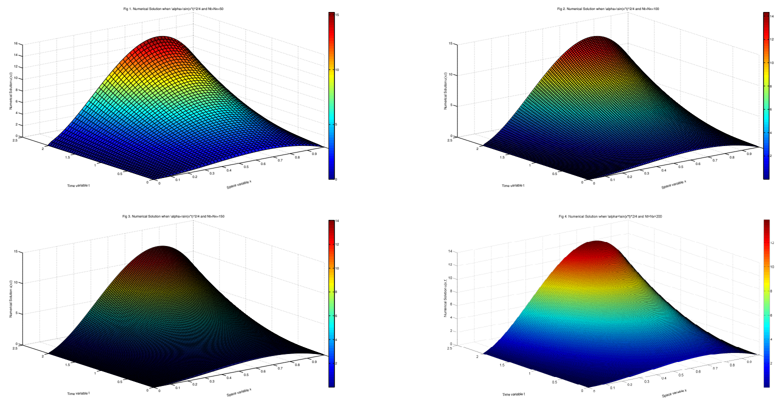

Figure 1: Numerical solution when N=M=N_x=50,100,150,200 and $\alpha(x,t) = \frac{\sin(x.t)^2}{4}$.

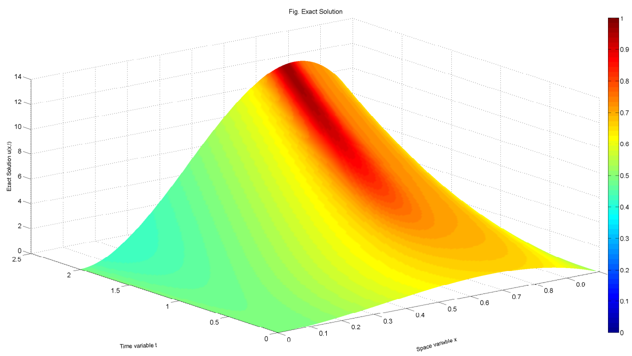

Figure 2: Exact solution when $\alpha(x,t) = \frac{\sin(x.t)^2}{4}$.

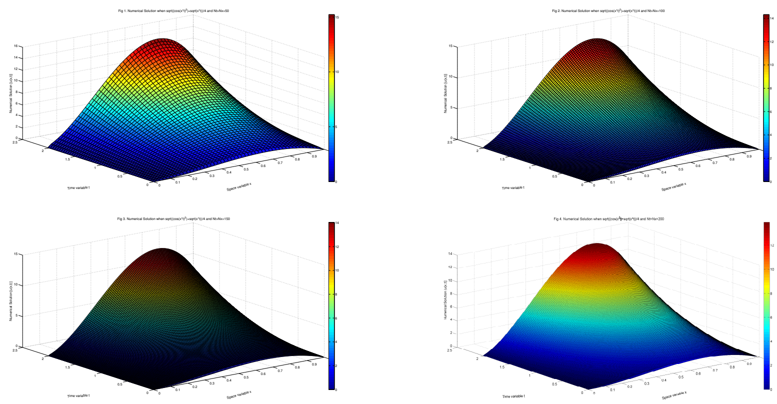

Figure 3: Numerical solution when N=M=N_x=50,100,150,200 and $\alpha(x,t) = \frac{\sqrt{cos(x.t)^2+\sqrt{x.t}}}{4}$.

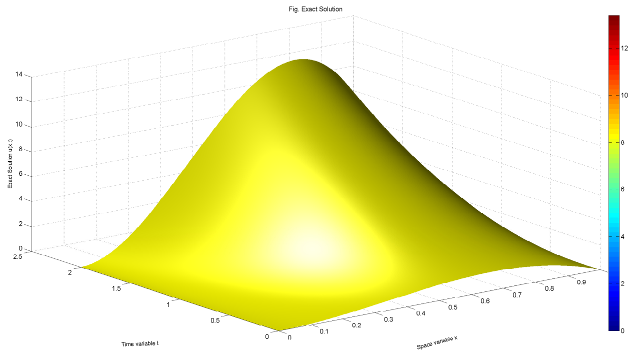

Figure 4: Exact solution when $\alpha(x,t) = \frac{\sqrt{cos(x.t)^2+\sqrt{x.t}}}{4}$.

7 Conclusion and Perspectives

In this work, we have discussed implicit discretization techniques by using an approach to forward fractional derivatives and the first-order spatial derivative can be approximated by the forward operator. These are particularly useful whenever the implicit scheme requires a time step. As in the implicit schemes, the values at interior grid points at new time levels cannot be obtained before computing the values at boundaries, an iteration procedure is employed to handle the nonlocal boundary condition. However, the discussion indicates that it is always important to explore the various algorithms when trying to solve a numerical problem and there is a vast literature available to do this. The numerical test applied to these methods gives acceptable results and suggests convergence to the exact solution when h goes to zero, see [6, 8, 12, 18].

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