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Analysis and Existence of Rumor Spreading with Campaign and Punishment Control

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Abstract: Rumors are pieces of unverified information that are spread from individual to individual. The spread of rumors that are not controlled can impact human life in many ways, one of which is that people start to worry about information that is not necessarily true. As such, efforts have been made to curb the spread of gossip through campaigns and punishment to reduce the number of gossip spreaders. This paper proposes the analysis and existence of a rumor-spreading model with campaigns and punishment control. The positivity, uniqueness, and existence of optimal control in the problem are analyzed and proven. In this paper, positivity and uniqueness are employed to validate the control and the unique solution of the rumor spreading model. Furthermore, based on the positivity and uniqueness of the rumor spreading model with control, the existence of optimal control in the model can be solved.

Keywords: *rumors; optimal control; positivity; uniqueness.*

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1 Introduction

The rapid development of technology has made things easily accessible to many people. Nowadays, humans do everything digitally, one example is obtaining or disseminating information through social media platforms such as Twitter and Instagram. Based on the Katadata Insight Center (KIC) survey, 76% of Indonesians get information from social media [1]. The information obtained or disseminated through this medium can be in the form of rumors. Rumors are still being confirmed and can negatively impact if the information is false [2]. According to the results of a digital literacy survey conducted by the Katadata Insight Center (KIC), this ease of dissemination of information has a huge impact on the community. In the survey, almost 60 % of Indonesians obtained information through rumors or hoaxes when accessing social media [3]. The spread of rumors that are out of control can have an impact on human life, one of which is that people are starting to worry about information that is not necessarily true. Special handling is certainly needed to control the spread of rumors. For example, punishment can be imposed on those who spread rumors, or some kind of campaign can be carried out among people who are vulnerable to information. Handling this kind of problem is very important because rumors can damage people's trust in social media and are difficult to control because they spread quickly.

The concept of optimization can assist in decision-making on achieving a goal and one of the alternatives in determining the optimal control of a problem [4–8]. Modeling of the spread of rumors has been studied by several researchers, such as Daley and Kendall [9], discussing the model of spreading rumors which has similarities with the construction of epidemic models of disease spread. The compartments used are Susceptible (a population that does not yet know rumors), Infected (a population that spreads rumors), and Recovered (a population that does not spread rumors anymore). This model does not consider the time it takes for a rumor to be received or even spread. Mathematical modeling of rumor spread has been further developed by several people such as Dahl et al. [10] in 2016. Efforts to control the spread of rumors entail the addition of control variables to the rumor-spreading model formed. As such, the value of the control variable is immediately given. In 2023, the model introduced by Dhar et al. [10] was used by Jain et al. [11] for control by adding the assumption that there is a delay for thinkers to spread the news. In the research, the implementation of delay for thinkers to spread the news has primary challenge in the attempt to control the spread of rumors. The model constructed by Jain is certainly valid. It has been shown in the model that the constructed model satisfies the positively invariant set property so that the model has a positive, bounded, and unique solution.

Time delays may be a useful tool in controlling the spread of rumor, but the truth is that people's confidence in the information they get is still very strong. This highlights the complexity of human behavior within digital networks and the desire for a more comprehensive strategy to manage the spread of rumors in our rapidly changing digital era. Therefore, based on the description above and several previous studies on the rumor spread mode, in this paper, a model for spreading rumors is constructed with control in the form of campaign and punishment. The campaign is carried out in an effort to impart knowledge to individual who obtaines the rumor, and who is exposed, about the importance of information validation and the dangers of rumor propagation. This control mechanism aims to reduce the number of individuals exposed to rumors, and who subsequently spread them, thereby decreasing the count of infected individuals.

However, punishment control is carried out to provide a deterrent effect on individuals who spread rumors so that the individual will not spread rumors or can be said to be cured. Throughout this paper, we propose and construct a nonlinear dynamic model for rumor spreading, expanding on the previous models by Jain et al. [11]. In this paper, the positivity, uniqueness, and existence of optimal control over the problem are analyzed and proven. Positivity and uniqueness are employed to validate the control and unique solution of the rumor spreading model. Furthermore, based on the positivity and uniqueness of the rumor spreading model with control, the existence of optimal control in the model can be solved.

2 Mathematical Model

Rumors are pieces of unverified information passed from individual to individual. Information spreads on social networks quickly due to their increasing popularity. Individuals disseminate information in seconds without verifying its validity. Therefore, the spread of rumors among social media users can cause unnecessary panic [12]. Before modern communication gadgets such as smartphones existed, rumors were spread by word of mouth, so the spread of information was limited by the length of time it required. Therefore, with the development of technology today, rumors can spread faster than before [13].

Rumor-spreading problems can be formulated as a mathematical model. For example, Jain et al. [11] have described the optimal control of rumor spreading model on homogeneous social network with the consideration of influence of delay on thinkers [11] and Liu et al. [14] have described and analyzed a model of spreading rumors on heterogeneous network. In addition, Liu et al. [15] describe the rumor spreading model in complex social networks with hesitating mechanism. In this paper, the entire population is grouped into four compartments: Susceptible (S), individuals who are not aware of rumors but are vulnerable to knowing rumors; Exposed (E), individuals who know about rumors and have the possibility to spread them; Infected (I), individuals who spread rumors; and Recovered (R), individuals who have stopped spreading rumors. N denotes the total individual population and satisfies $N(t) = S(t) + E(t) + I(t) + R(t)$. The relationship between populations in the rumor-spreading model can be expressed in the following mathematical model:

$$\begin{aligned} \frac{dS}{dt} &= \theta N - \alpha\beta SI - \mu S, \\ \frac{dE}{dt} &= \alpha\beta SI - \sigma E + \gamma mI - \mu E, \\ \frac{dI}{dt} &= \sigma hE - \gamma I - \mu I, \\ \frac{dR}{dt} &= \sigma(1-h)E + \gamma(1-m)I - \mu, R \end{aligned} \tag{1}$$

with

θ : Susceptible individual growth rate,

μ : Natural death rate,

α : The rate of change from *Susceptible* individuals to *Exposed* individuals,

β : The rate of interaction between *Susceptible* individuals and *Infected* individuals,

σ : The rate of individuals leaving the exposed class (*Exposed*),

γ : The rate of individuals leaving the class of Infected (*Infected*),

m : Proportion of Infected individuals to Exposed individuals,

h : Proportion of Exposed individuals to Infected individuals,

and the initial conditions

$$S(0) = S_0, \;\; E(0) = E_0, \;\; I(0) = I_0, \;\; R(0) = R_0.$$

The spread of rumors is getting more and more worrying day by day. There is a lot of fake news or information that is conveyed interestingly and continuously, causing many people to believe it. A rumor is a public communication obtained from a person's hypothesis about an incident with accusations based on non-existent evidence. Rumors from reliable sources tend to be believed by many people [16]. Therefore, action must be taken to reduce the spread of these rumors. This paper proposes two controllers: a campaign for individuals who knew there were rumors and punishment for those who spread rumors. We assume that everyone concerned shares a very high level of confidence in the information they are given. From the assumption, the campaign is carried out in an effort to impart knowledge to individual who obtaines the rumor, and is exposed, about the importance of information validation and the dangers of rumor propagation. This control mechanism aims to reduce the number of individuals exposed to rumors and who subsequently spread them, thereby decreasing the count of infected individuals. However, punishment control is carried out to provide a deterrent effect on individuals who spread rumors so that the individual will not spread rumors or can be said to be cured.

The mathematical model of controlled spread of rumors can be expressed as follows:

$$\begin{aligned} \frac{dS}{dt} &= \theta N - \alpha\beta SI - \mu S, \\ \frac{dE}{dt} &= \alpha\beta SI - \sigma E + \gamma m I - \mu E - u_1 E, \\ \frac{dI}{dt} &= \sigma h E - \gamma I - \mu I - u_2 I, \\ \frac{dR}{dt} &= \sigma (1-h) E + \gamma (1-m) I - \mu R + u_1 E + u_2 I, \end{aligned} \tag{2}$$

with u_1 and u_2 representing an effort to prevent the spread of rumors through a campaign to individuals who know there are rumors and efforts to impose sanctions on individuals who spread rumors, respectively. The objective function can be defined as follows:

$$J(u_1, u_2) = \min \int_{t_0}^{t_f} \left[a_1 I + \frac{1}{2} \left(b_1 u_1^2(t) + b_2 u_2^2(t) \right) \right] dt. \tag{3}$$

3 Positivity and Uniqueness

A model must be valid with existing conditions. This means that in this case, the solution given must be positive. This indicates that the population of each individual in model (2) always exists. Thus, if the initial population of S, E, I and R is positive, then the population at time t must also have a positive value.

We introduce the concept of a positive invariant set to solve the problem. This set is based on the initial solution and the solution of the system. More details can be given as in the following definition.

Definition 3.1 (Positively invariant set). Given $\dot{\boldsymbol{x}} = \boldsymbol{f}(t, \boldsymbol{x})$ is a dynamic system, with the initial conditions $\boldsymbol{x}_0 = \boldsymbol{x}(t_0)$. Given Ω is a subset of $\mathbb{R}^n$. Then Ω is said to be a positive invariant set if for $\boldsymbol{x}_0 \in \Omega$, it implies $\boldsymbol{x}(t, x_0) \in \Omega$ for every $t \geq t_0$.

Now, for $N(t) = S(t) + E(t) + I(t) + R(t)$, we suppose $\boldsymbol{x}(t) = (S(t), E(t), I(t), R(t))$, and the initial condition $\boldsymbol{x}(t_0) = (S(t_0), E(t_0), I(t_0), R(t_0))$, define $\omega = \max\{1, e^{(\theta-\mu)t_f}\}$ and set

$$\Omega_{\boldsymbol{x}(t_0)} := \{\boldsymbol{x}(t) \mid t_0 \leq t \leq t_f, \boldsymbol{x}(t) \geq \mathbf{0},\ N(t) \leq N(0) \cdot \omega\}. \tag{4}$$

The fact that (4) is a positive invariant set can be proven in the following theorem.

Theorem 3.1 *Let* $\omega = \max\{1, e^{(\theta-\mu)t_f}\}$ *and*

$$\Omega_{\boldsymbol{x}(t_0)} := \{\boldsymbol{x}(t) \mid t_0 \leq t \leq t_f, \boldsymbol{x}(t) \geq \mathbf{0},\ N(t) \leq N(0) \cdot \omega\}$$

be the subset all solutions of model (2) *with the initial condition* $\boldsymbol{x}(t_0)$. *Then* $\Omega_{\boldsymbol{x}(t_0)}$ *is a positively invariant set.*

Proof. We can divide the proof into several cases as follows.

1. First, we prove that the total population N is bounded. Based on the dynamic equation (2), note that

$$\begin{aligned} & \frac{dS}{dt} + \frac{dE}{dt} + \frac{dI}{dt} + \frac{dR}{dt} = \theta N - \mu(S + E + I + R), \\ (\Rightarrow) \quad & \frac{dN}{dt} = \theta N - \mu N, \\ (\Rightarrow) \quad & \frac{dN}{dt} = (\theta - \mu)N, \end{aligned}$$

   with the solution $N(t) = N(0)e^{(\theta-\mu)t}$. From this, it is clear that if $\theta - \mu \leq 0$, then $0 < N(t) \leq N(0)$ and if $0 < \theta - \mu$, then $0 < N(t) \leq N(0)\, e^{(\theta-\mu)t_f}$. Then it follows that $N(t) \leq N(0) \cdot \omega$ for every time $0 \leq t \leq t_f$. Thus, this proves that the total population N is bounded at all times.

2. We know that N is positive and bounded for every time t. Then we can use it to prove S is positive. From the dynamic equation (2), we assume that there is $t \in (0, t_f]$ and $S(t) \leq 0$. Suppose $S_* = \{t \in (0, t_f] \mid S(t) \leq 0\}$ and take $t^* = \inf S_*$. It is clear that $t^* \neq 0$ so $S(t) > 0, \forall t \in [0, t^*)$. Then we know that I is a continuous function on $[0, t_f]$, so there is $M \in \mathbb{R}$ and $I(t) < M$ for every $t \in [0, t_f]$. As a result,

$$\begin{aligned} & \frac{dS}{dt} = \theta N - \alpha\beta S I - \mu S, \\ (\Rightarrow) \quad & \frac{dS}{dt} > -\alpha\beta M S - \mu S, \qquad \forall t \in [0, t_f). \end{aligned}$$

   We can solve the above inequality and obtain that

$$S(t^*) > S(0)\, e^{-(\alpha\beta M + \mu)t^*}. \tag{5}$$

   From equation (5), we obtain that $S(t^*) > S(0)\, e^{-(\alpha\beta M+\mu)t^*} > 0$. However, this contradicts the statement $S(t^*) \leq 0$, so from this, it follows that $S(t) > 0$ for every time $t \in [0, t_f]$.

3. By evaluating equation (2), we assume that there is $t \in (0, t_f]$ and $E(t) \leq 0$ or $I(t) \leq 0$. First, assume that $E(t) \leq 0$. Suppose $E_* = \{t \in (0, t_f] \mid E(t) \leq 0\}$ and take $t^* = inf E_*$. It can be easily seen that $t^* \neq 0$, so $E(t) > 0, \forall t \in [0, t^*)$ and

$$\begin{aligned} & \tfrac{dI}{dt} && = && \sigma h E - \gamma I - \mu I - u_2 I, \\ (\Rightarrow) \quad & \tfrac{dI}{dt} && > && -\gamma I - \mu I - u_2 I, \qquad \forall t \in [0, t_f), \\ (\Rightarrow) \quad & \tfrac{dI}{dt} + \gamma I + \mu I + u_2 I && > && 0. \end{aligned}$$

Assuming that there is $t \in (0, t^*)$ and $I(t) \leq 0$, suppose $I_* = \{t \in (0, t^*) \mid I(t) \leq 0\}$ and take $t_I^* = inf I_*$. It can be easily seen that $t_I^* \neq 0$, so $I(t) > 0, \forall t \in [0, t_I^*)$ and

$$\frac{dI}{dt} + \gamma I + \mu I + I \geq \frac{dI}{dt} + \gamma I + \mu I + u_2 I > 0, \qquad \forall t \in [0, t_I^*).$$

By solving the above inequality, it is obtained that

$$I(t_I^*) > I(0)\, e^{-(\gamma+\mu+1)t_I^*}.$$

From this, we obtain that $I(t_I^*) > I(0)\, e^{-(\gamma+\mu+1)t_I^*} > 0$. However, this contradicts the statement $I(t_I^*) \leq 0$, and we obtain that $I(t) > 0$ for every time $t \in [0, t^*)$. From this, we obtain

$$\begin{aligned} & \tfrac{dE}{dt} && = && \alpha\beta S I - \sigma E + \gamma m I - \mu E - u_1 E, \\ (\Rightarrow) \quad & \tfrac{dE}{dt} && > && -\sigma E - \mu E - u_1 E, \ \forall t \in [0, t^*), \\ (\Rightarrow) \quad & \tfrac{dE}{dt} + \sigma E + \mu E + u_1 E && > && 0, \\ (\Rightarrow) \quad & \tfrac{dE}{dt} + \sigma E + \mu E + E > \tfrac{dE}{dt} + \sigma + \mu + u_1 E && > && 0. \end{aligned}$$

Furthermore, by solving the above problems, it is obtained that

$$E(t^*) > E(0)\, e^{-(\sigma+\mu+1)t^*}$$

and $E(t^*) > E(0)\, e^{-(\sigma+\mu+1)t^*} > 0$. However, this contradicts the statement $E(t^*) \leq 0$, and we obtain $E(t) > 0$ for every time $t \in [0, t_f]$. Furthermore, in the same way, for the assumption that there is t such that $I(t) \leq 0$, the assumption given is also wrong. So it follows that $I(t) > 0$ for every time $t \in [0, t_f]$.

4. From the dynamic model system (2), we assume that there is $t \in (0, t_f]$ and $R(t) \leq 0$. Suppose $R_* = \{t \in (0, t_f] \mid R(t) \leq 0\}$ and we take $t^* = \inf R_*$. It can be easily seen that $t^* \neq 0$, so $R(t) > 0, \forall t \in [0, t^*)$ and

$$\begin{aligned} & \tfrac{dR}{dt} && = && \sigma(1-h)E + \gamma(1-m)I - \mu R + u_1 E + u_2 I, \\ (\Rightarrow) \quad & \tfrac{dR}{dt} && > && -\mu R, \qquad \forall t \in [0, t^*), \\ (\Rightarrow) \quad & \tfrac{dR}{dt} + \mu R && > && 0. \end{aligned}$$

From this, we obtain that $R(t^*) > R(0)\, e^{-\mu t^*} > 0$. However, this contradicts the statement $R(t^*) \leq 0$, so it follows that $R(t) > 0$ for every time $t \in [0, t_f]$.

From the above proof, we obtain that M is a *positively invariant set*; as such, the dynamic model system (2) is valid. We have proven that the set defined in (4) is a positively invariant set. This results in the fact that if the initial conditions in the rumor spreading model (2) with control are positive, then the solution of the model is positive for each time interval. However, this model does not guarantee that a single solution is given for an initial condition; so, we have to show whether the rumor spread model (2) with control has a single solution or not.

Now, to guarantee that the model (2) is existent and unique, we can use the concept of the Lipschitz condition in model (2) [17]. For that, we prove that model (2) satisfies the Lipschitz condition as given in the following theorem.

Theorem 3.2 *The rumor-spreading model with control* (2) *that satisfies a given initial condition* $S(t_0), E(t_0), I(t_0), R(t_0) > 0$ *has a unique solution.*

Proof. Let $X = (S, E, I, R)$ and

$$\varphi(X) = \begin{bmatrix} \frac{dS}{dt} \\ \frac{dE}{dt} \\ \frac{dI}{dt} \\ \frac{dR}{dt} \end{bmatrix}.$$

We can write the rumor-spreading model with control (2) as

$$\varphi(X) = \begin{bmatrix} \theta N - \alpha\beta SI - \mu S \\ \alpha\beta SI - \sigma E + \gamma m I - \mu E - u_1 E \\ \sigma h E - \gamma I - \mu I - u_2 I \\ \sigma(1-h)E + \gamma(1-m)I - \mu R + u_1 E + u_2 I \end{bmatrix}$$

and we show that $\varphi(X)$ has a unique solution with the initial condition $(S(0), E(0), I(0), R(0)) > 0$. Note that

$$\varphi(X_1) - \varphi(X_2) = \begin{bmatrix} \theta(N_1 - N_2) - \alpha\beta(S_1 I_1 - S_2 I_2) - \mu(S_1 - S_2) \\ \alpha\beta(S_1 I_1 - S_2 I_2) - \sigma(E_1 - E_2) + \gamma m(I_1 - I_2) - \mu(E_1 - E_2) - u_1(E_1 - E_2) \\ \sigma h(E_1 - E_2) - \gamma(I_1 - I_2) - \mu(I_1 - I_2) - u_2(I_1 - I_2) \\ (\sigma(1-h) + u_1)(E_1 - E_2) + (\gamma(1-m) + u_2)(I_1 - I_2) - \mu(R_1 - R_2) \end{bmatrix}. \quad (6)$$

We know that S and I represent the positive continuous function at $[0, t_f]$ so that S and I are the bounded functions at $[0, t_f]$. Since N is bounded for every initial condition $(S(0), E(0), I(0), R(0))$, there exist M and K such that

$$-M(S_1(t) - S_2(t)) \le -\alpha\beta(S_1(t) I_1(t) - S_2(t) I_2(t)) \le M(S_1(t) - S_2(t))$$

and

$$-K(I_1(t) - I_2(t)) \le \alpha\beta(S_1(t) I_1(t) - S_2(t) I_2(t)) \le K(I_1(t) - I_2(t))$$

for every solution X_1 and X_2 with the initial condition being $X_1(0)$ and $X_2(0)$. Then, from (6), we have

$$-A(X_1 - X_2) \leq \varphi(X_1) - \varphi(X_2) \leq A(X_1 - X_2)$$

with

$$A = \begin{bmatrix} \theta + M - \mu & \theta & \theta & 0 \\ 0 & -\sigma - \mu - u_1 & K + \gamma m & 0 \\ 0 & \sigma h & -\gamma - \mu - u_2 & 0 \\ 0 & -h + u_1 & \sigma + \gamma - m + u_2 & -\mu \end{bmatrix}.$$

We easily show that

$$\| \varphi(X_1) - \varphi(X_2) \|_2 \leq \| A(X_1 - X_2) \|_2 \leq \| A \| \, \| X_1 - X_2 \|_2$$

with $\| X \|_2$ being the Euclidean norm in $\mathbb{R}^4$ and $\| A \| = \sqrt{tr(A^T A)}$; so, we obtain that the function $\varphi(X)$ is uniformly Lipschitz continuous. From this,

$$X(t) = X(t_0) + \int_{t_0}^{t} \varphi(X) \ dt$$

so that X has a unique solution for the initial condition $S(t_0), E(t_0), I(t_0), R(t_0) > 0$.

4 The Existence of Optimal Control

We have shown in Section 3 that the rumor-spreading model with controls (2) has a single and positive solution. From this, we will then determine whether or not the rumor-spreading model with controls (2) has optimal control. Using a result by Fleming and Rishel [18], we can prove the existence of the optimal control checking the following points:

1. The set U defined as

$$U = \{(u_1, u_2) \mid 0 \leq u_1(t), u_2(t) \leq 1, \forall t \in [0, t_f]\}$$

   is a nonempty set.
   This can be seen from Theorem (3.1) and Theorem (3.2), where every control $u \in U$ has a unique and positive solution and is not an empty set.

2. The set U, which is defined as

$$U = \{(u_1, u_2) \mid 0 \leq u_1(t), u_2(t) \leq 1, \forall t \in [0, t_f]\},$$

   is a closed convex set.
   To show that U is a closed convex set, let $v_1 = (u_{11}, u_{21}), v_2 = (u_{12}, u_{22}) \in U$. It can be easily seen that $0 \leq u_{11}(t), u_{21}(t), u_{12}(t), u_{22}(t) \leq 1$ for $t \in [0, t_f]$; so, with every $\lambda \in [0, 1]$, we obtain

$$0 \leq \lambda u_{11}(t) + (1 - \lambda) u_{21}(t) \leq 1$$

   and

$$0 \leq \lambda u_{12}(t) + (1 - \lambda) u_{22}(t) \leq 1$$

for $t \in [0, t_f]$. From this, we have

$$(1-\lambda)\, v_1 + \lambda v_2 = \begin{bmatrix} \lambda u_{11}\,(t) + (1-\lambda)\, u_{21}(t) \\ \lambda u_{12}\,(t) + (1-\lambda)\, u_{22}(t) \end{bmatrix} \in U.$$

As such, we obtain that U is a convex set. To show that U is a closed set, it is enough to show that for every convergent sequence $(u_n) = (u_{1n}, u_{2n}) \subseteq U$, one has $\lim_{n\to\infty} u_n \ \in U$. This statement is equivalent to u_{1n} and u_{2n} is a convergent sequence with $(x, y) = (\lim_{n\to\infty} u_{1n}\ , \lim_{n\to\infty} u_{2n}\ ) \in U$. Now we define

$$\parallel u - v \parallel := \sup\{|u\,(t) - v\,(t)\,|\mid t \in [0, 1]\}\ .$$

Then we know that u_{1n} and u_{2n} is a convergent sequence such that for every $\varepsilon > 0$, there exists $K(\varepsilon) \in \mathbb{N}$ that satisfies

$$\parallel u_{1n} - x \parallel < \varepsilon$$

and

$$\parallel u_{2n} - y \parallel < \varepsilon$$

for every $n \geq K(\varepsilon)$. From this, we obtain

$$\parallel u_{1n} - x \parallel < \varepsilon,$$

$$\begin{array}{ll} (\Rightarrow) & |u_{1n}(t) - x(t)| < \sup\{|u_{1n}(t) - x(t)| \mid t \in [0, t_f]\} < \varepsilon, \\ (\Rightarrow) & -\varepsilon < x(1) - u_{1n}(t) < \varepsilon, \\ (\Rightarrow) & -\varepsilon \leq u_{1n}(t) -\ \varepsilon < x(1) < \varepsilon + u_{1n}(t) \leq \varepsilon + 1, \\ (\Rightarrow) & -\varepsilon < x(t) < 1 + \varepsilon, \end{array}$$

and

$$\parallel u_{2n} - y \parallel < \varepsilon,$$

$$\begin{array}{ll} (\Rightarrow) & |u_{2n}(t) - y(t)| < \sup\{|u_{2n}(t) - y(t)| \mid t \in [0, t_f]\} < \varepsilon, \\ (\Rightarrow) & -\varepsilon < y(1) - u_{2n}(t) < \varepsilon, \\ (\Rightarrow) & -\varepsilon \leq u_{2n}(t) -\ \varepsilon < y(1) < \varepsilon + u_{2n}(t) \leq \varepsilon + 1, \\ (\Rightarrow) & -\varepsilon < y(t) < 1 + \varepsilon. \end{array}$$

Since it is satisfied for every $\varepsilon > 0$, we obtain $0 \leq x(t), y(t) \leq 1$ and $(x, y) \in U$. Consequently, we show that U is a closed set. Furthermore, it can be proven that U is a closed convex set.

3. The function $\mathbb{J} : U \to \mathbb{R}$, which is defined as

$$\mathbb{J}\,(u_1, u_2) = a_1 I + \frac{1}{2}\left(b_1 u_1^2\,(t) + b_2 u_2^2\,(t)\right) dt,$$

is a bounded convex function on U.
To prove that $\mathbb{J}$ is a bounded convex function, first, note that I is a continuous function at $[0, t_f]$, so that there exist $I_{\min}$ and $I_{\max}$ such that $I_min \leq I(t) \leq I_{max}$ and we know $0 \leq u_1(t) \leq 1$ and $0 \leq u_1(t) \leq 1$ so that

$$a_1 I_{min} \leq \mathbb{J}\,((u_1, u_2)) \leq a_1 I_{max} + \frac{1}{2}(b_1 + b_2).$$

From this, we can say that $\mathbb{J}$ is a bounded function. Then, suppose

$$g(u_1, u_2) = \frac{1}{2}b_1u_1^2 + \frac{1}{2}b_2u_2^2$$

and the Hessian matrix of g is

$$H = \begin{bmatrix} \dfrac{\partial^2 g}{\partial u_1^2} & \dfrac{\partial^2 g}{\partial u_1 \partial u_2} \\ \dfrac{\partial^2 g}{\partial u_2 \partial u_1} & \dfrac{\partial^2 g}{\partial u_2^2} \end{bmatrix} = \begin{bmatrix} 2b_1 & 0 \\ 0 & 2b_2 \end{bmatrix}.$$

We can prove that H is semidefinite positive so g is a convex function. From this, we obtain that g and a_1I are convex functions on U. Also,

$$\mathbb{J}\left(u_1, u_2\right) = a_1I + \frac{1}{2}\left(b_1u_1^2\left(t\right) + b_2u_2^2\left(t\right)\right)dt$$

is a convex function on U. Then, we have established that $\mathbb{J}$ is a bounded convex function.

4. We know that $S\left(t\right), E\left(t\right), I\left(t\right)$ and R(t) is a continuous function on $[0, t_f]$ so $S\left(t\right), E\left(t\right), I\left(t\right)$ and R(t) is a bounded function with

$$0 < S\left(t\right) \leq S_{max},\ 0 < E\left(t\right) \leq E_{max}, 0 < I\left(t\right) \leq I_{max}, 0 < R\left(t\right) \leq R_{max}$$

for every $t \in [0, t_f]$. From this, we obtain

(a) $\frac{dS}{dt} = \theta N - \alpha\beta SI - \mu S < \theta(S_{max} + E_{max} + I_{max} + R_{max})$,

(b) $\frac{dE}{dt} = \alpha\beta SI - \sigma E + \gamma mI - \mu E - u_1E < \ \alpha\beta S_{max}I_{max} + \gamma mI_{max}$,

(c) $\frac{dI}{dt} = \sigma hE - \gamma I - \mu I - u_2I < \sigma hE_{max}$,

(d) $\frac{dR}{dt} = \sigma\left(1-h\right)E + \gamma\left(1-m\right)I - \mu R + u_1E + u_2I < \left(\sigma\left(1-h\right)+1\right)E_{max} + \left(\gamma\left(1-m\right)+1\right)I_{max}$,

so that the right-hand side equation of the rumor-spreading model with control (2) is bounded. We also know that the control system (2) can be represented as

$$\frac{dS}{dt} = f_1\left(S, E, I, R\right) + u_1g_1\left(S, E, I, R\right) + u_2h_1\left(S, E, I, R\right),$$

$$\frac{dE}{dt} = f_2\left(S, E, I, R\right) + u_1g_2\left(S, E, I, R\right) + u_2h_2\left(S, E, I, R\right),$$

$$\frac{dI}{dt} = f_3\left(S, E, I, R\right) + u_1g_3\left(S, E, I, R\right) + u_2h_3\left(S, E, I, R\right),$$

$$\frac{dR}{dt} = f_4\left(S, E, I, R\right) + u_1g_4\left(S, E, I, R\right) + u_2h_4\left(S, E, I, R\right).$$

From this, we can say that the control system can be represented as the linear function below u_1 and u_2. So we obtain that the control system is linearly bounded below u_1 and u_2.

5 Conclusion

The nonlinear dynamic model of rumor spreading with control constructed in this paper is based on the very high level of public vulnerability in receiving information. Therefore, the model in this paper applies two controls, a campaign (u_1) for the Exposed population and a punishment (u_2) for the Infected population. By analysis, the constructed model is valid and has a unique solution. This is shown based on the concept of a positively invariant set and reviewing that the model is Lipschitz. The positivity of the resulting solution indicates that the population in the model, namely, Susceptible, Exposed, Infected and Recovered, always exists. The expectation is that although the rumor spreaders (Infected) and the susceptible (Exposed) exist at all times, at least an optimal control is needed to suppress the rate of rumor spread and increase the population of rumor-free people (Recovered). This is in accordance with the existence concept of optimal control as proven in this paper.

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