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On the Synchronization of a Novel Chaotic System with Two Control Methods

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Abstract: This paper reports a new chaotic system and its synchronization via active and adaptive control methods. The novel system is presented and its chaoticity is confirmed using the Lyapunov exponents tool. Furthermore, it is demonstrated that the new system possesses the property of co-existing attractors. Moreover, two control methods are employed: active control and adaptive control. By designing appropriate controllers and estimation laws based on the stability theory of integer-order systems, we achieve synchronization between chaotic systems. Finally, numerical simulations are implemented to demonstrate the effectiveness and flexibility of the synchronization controllers and the estimation laws for the two methods.

Keywords: *chaotic system; strange attractor; Lyapunov exponent; Lyapunov stability theory; adaptive control; synchronization.*

Mathematics Subject Classification (2010): 34D08, 34C28, 37B55, 37B25, 37D45, 70K20, 93D05, 93D21.

1 Introduction

The chaos theory deals with the dynamical behavior of nonlinear dynamical systems which are highly sensitive to initial conditions and system parameters. Recently, chaos theory has achieved great development and has been successfully applied in many fields such as electronic engineering [1], computer science [2], communication systems [3, 4], medical image processing [1,2], complex networks [5], chemical engineering [6], investigation of HIV virus [28] and economic models [7]. By now, numerous chaotic systems with different types of attractors have been proposed and studied, for example those with

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self-excited attractors [8,9], hidden attractors [10,11,12], coexisting attractors [13,14], infinitely many shifted attractors [15,16], multi scroll attractors [17,18], memristor attractors [19,20] and chaotic systems with fractional orders [11,21]. In the past few decades, chaos synchronization has received great attention owing to its applications in designing secure communication system. Synchronization of chaotic systems is a topic of interest in the field of nonlinear dynamics. It involves the design of control strategies that can make two or more chaotic systems behave identically or follow a certain pattern over time. Some of the methods used for chaotic synchronization include active control, adaptive control, fuzzy sliding mode control, sliding mode controller [22-25], backstepping neural network method [26], observer-based synchronization [27] and so on. This paper is organized as follows. Section 2 describes the new chaotic system, its phase plots and equilibrium points. Section 3 describes the dynamic analysis of the new chaotic system including multistability and coexisting attractors. Moreover, the synchronization of the chaotic system with two control methods is realized by designing appropriate active and adaptive controllers and estimation laws using the stability theory of integer-order systems in Section 4. In Section 5, numerical simulations are implemented to demonstrate the effectiveness and flexibility of the synchronization controllers and the estimation laws for the two methods. The conclusion is given in the last section.

2 Description and Analysis of System

The new chaotic system has three positive parameters, cosine and nonlinear terms described by

$$\begin{cases} \dot{x_1} = a(-x_1 cos(x_2) + x_2), \\ \dot{x_2} = -bx_1 - x_3, \\ \dot{x_3} = -x_1(1 - x_2^2) + cx_3, \end{cases} \tag{1}$$

where $X = (x_1, x_2, x_3)^T$ are the states and a, b, c are the positive constants.

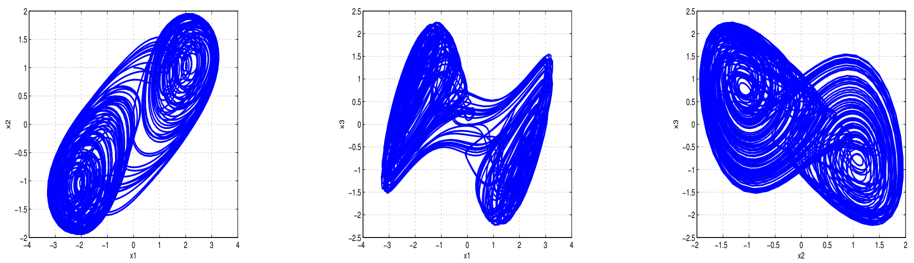

Figure 1: Strange Attractor of New Hyperchaotic System.

2.1 Equilibrium points

Putting equations of the system (1) equal to zero, i.e.,

$$\begin{cases} a(-x_1 cos(x_2) + x_2) = 0, \\ -bx_1 - x_3 = 0, \\ -x_1(1 - x_2^2) + cx_3 = 0, \end{cases} \tag{2}$$

then we can verify the following:

– If $1+bc \leq 0$, the system (1) has only one equilibrium point $p_0=(0,0,0)$,

– If $1+bc>0$ and $1+bc \neq \left(k\frac{\pi}{2}\right)^2$, $k \in \mathbb{Z}^*$, the system (1) has three equilibrium points:

$$\begin{cases} P_0=(0,0,0), \\ P_1=(\frac{\sqrt{1+bc}}{cos(\sqrt{1+bc})}, \sqrt{1+bc}, -\frac{b\sqrt{1+bc}}{cos(\sqrt{1+bc})}), \\ P_2=(-\frac{\sqrt{1+bc}}{cos(\sqrt{1+bc})}, -\sqrt{1+bc}, \frac{b\sqrt{1+bc}}{cos(\sqrt{1+bc})}). \end{cases} \tag{3}$$

In order to check the stability of the equilibrium points, we derive the Jacobian matrix at a point $p(x_1,x_2,x_3)$ of the system (1).

$$J(p)=\begin{pmatrix} -acos(x_2) & ax_1sin(x_2)+a & 0 \\ -b & 0 & -1 \\ -1+x_2^2 & 2x_1x_2 & c \end{pmatrix}. \tag{4}$$

Therefore, the characteristic equation of J is obtained as

$$\lambda^3+\eta_1\lambda^2+\eta_2\lambda+\eta_3=0. \tag{5}$$

Based on the Routh-Hurwitz theorem, p is a stable point if the following conditions are satisfied:

$$\begin{cases} \eta_1>0, \eta_2>0, \\ \eta_1\eta_2>\eta_3. \end{cases} \tag{6}$$

When $a=1.2, b=c=0.46$, system (1) has the following equilibria:

$$\begin{cases} p_0=(0,0,0), \\ p_1=(1.1007, 1.1007, 0.5063), \\ p_2=(-1.1007,-1.1007, 0.5063). \end{cases} \tag{7}$$

The corresponding eigenvalues and their stability are given in the following table.

Equilibrium points	Eigenvalues	stablity
p_0	$\lambda_{1,2}=-0.83620 \pm 0.92742i, \lambda_3=0.93240$	unstable foci point
$p_{1,2}$	$\lambda_{1,2}=0.15272 \pm 1.8336i, \lambda_3=-0.38900$	unstable foci points

3 Dynamic Analysis of the New Chaotic System (1)

3.1 Lyapunouv-exponent and Kaplan-York dimension

In this section, we will prove that the new system (1) is chaotic by calculating the Lyapunov exponents using the Wolf-Swift algorithm [29]. Simulation in Matlab gives us the following Lyapunov exponents:

$$L_1=0.153564, L_2=0, L_3=-0.378558. \tag{8}$$

Also, the Kaplan-York dimension of system (1) is given by

$$D_L=2+\frac{1}{|L_3|}\sum_{i=1}^{2}L_i=2.40565. \tag{9}$$

Since L_1, L_2 are positive Lyapunov exponents and $\sum_{i=1}^{3}L_i<0$, the novel system (1) is a chaotic dissipative system with fractal dimension.

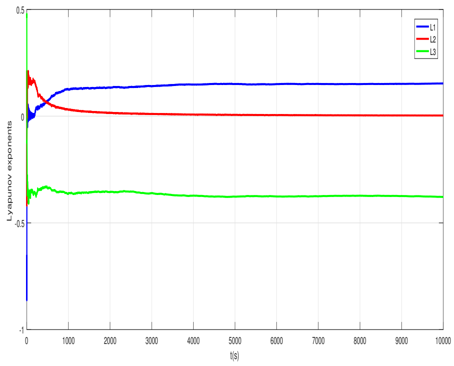

Figure 2: Lyapunouv exponent spectrum.

3.2 Multistability in the new 3-D chaotic sytem (1)

Multistability in chaotic systems refers to a phenomenon where a system can exhibit multiple stable states, or attractors, under certain conditions. This means that the system can evolve into different, distinct chaotic behaviors depending on its initial conditions or external parameters. In order to study the coexistence attractors and other characteristics of the system better, it is necessary to give some disturbance to the initial conditions under the condition of keeping the system parameters constant. The proposed 3D chaotic system (1) has symmetrical attractors with respect to the origine (equilibrum point $(0, 0, 0)$ because it is invariant under the coordinate transformation

$$(x_1, x_2, x_3) \longrightarrow (-x_1, -x_2, -x_3). \tag{10}$$

Fig.3 shows the dynamic behavior with coexistence chaotic attractors with different initial conditions.

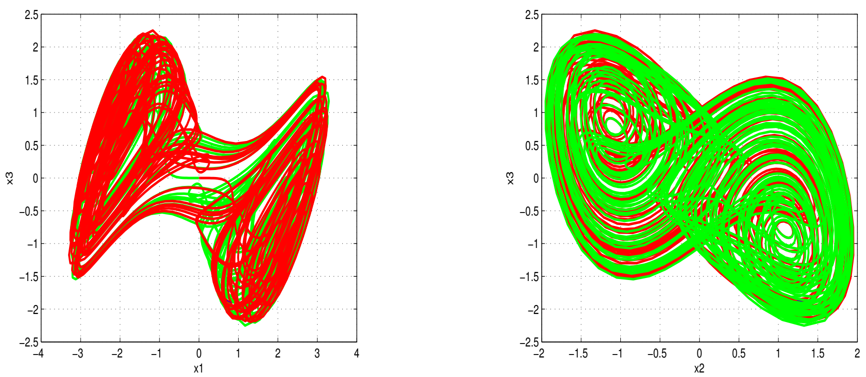

Figure 3: Coexistence of two attractors with different initial values and parameters.

4 Identical Synchronization with Two Method

The synchronization error state is define by

$$e(t) = Y(t) - X(t),$$

where $Y(t) = (y_1, y_2, y_3)$ *and* $X(t) = (x_1, x_2, x_3), e(t) = (e_1, e_2, e_3)$. The error dynamics is obtained as

$$\dot{e}(t) = \dot{Y}(t) - \dot{X}(t).$$

The drive system is described by

$$\begin{cases} \dot{x_1} = a(-x_1 cos(x_2) + x_2), \\ \dot{x_2} = -bx_1 - x_3, \\ \dot{x_3} = -x_1(1 - x_2^2) + cx_3. \end{cases} \tag{11}$$

The response system is

$$\begin{cases} \dot{y_1} = a(-y_1 cos(y_2) + y_2) + u_1, \\ \dot{y_2} = -by_1 - y_3 + u_2, \\ \dot{y_3} = -y_1(1 - y_2^2) + cy_3 + u_3, \end{cases} \tag{12}$$

where $U(t) = (u_1, u_2, u_3)^T$ is the controller.
The synchronization error of system (11) and (12) is described as follows:

$$\begin{cases} \dot{e_1} = ae_2 + a(x_1 cos(x_2) - y_1 cos(y_2)) + u_1, \\ \dot{e_2} = -be_1 - e_3 + u_2, \\ \dot{e_3} = -e_1 + ce_3 + y_1 y_2^2 - x_1 x_2^2 + u_3. \end{cases} \tag{13}$$

4.1 Active Control Method

The controller system is constructed as follows:

$$\begin{cases} u_1 = -ae_2 - a(x_1 cos(x_2) - y_1 cos(y_2)) - \dot{e_1}, \\ u_2 = be_1 + e_3 - \dot{e_2}, \\ u_3 = e_1 - ce_3 - y_1 y_2^2 + x_1 x_2^2 - \dot{e_3}, \end{cases}$$

where

$$\begin{cases} \dot{e_1} = -k_1 e_1, \\ \dot{e_2} = -k_2 e_2, \\ \dot{e_3} = -k_3 e_3, \end{cases}$$

with $k_i > 0, i = 1, 2, 3,$ being the constant gains. The controller system is

$$\begin{cases} u_1 = ae_2 - a(x_1 cos(x_2) - y_1 cos(y_2)) - k_1 e_1, \\ u_2 = be_1 + e_3 - k_2 e_2, \\ u_3 = e_1 - ce_3 - y_1 y_2^2 + x_1 x_2^2 - k_3 e_3. \end{cases} \tag{14}$$

Substituting (14) into (13), we can obtain

$$\begin{cases} \dot{e_1} = -k_1 e_1, \\ \dot{e_2} = -k_2 e_2, \\ \dot{e_3} = -k_3 e_3. \end{cases} \tag{15}$$

It is clear from (15) that the eigenvalues of the Jacobian matrix of the error system are negative, thus the zero solution of the errors systems is asymptotically stable, i.e., $\lim_{t\to\infty} |e(t)| = 0$. Hence the synchronization using active control between systems (11) and (12) is achieved.

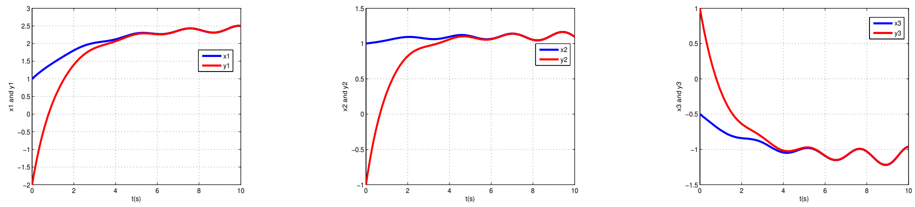

Figure 4: Synchronization between $x_i, y_i, i = 1, 2, 3$, using active control.

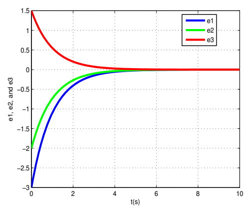

Figure 5: The time-history of the synchronization errors $e_1(t), e_2(t), e_3(t)$ using active control.

4.2 Adaptive control method

We choose system (11) as the driver system and (12) as the response system; similarly, the synchronization error is defined as $\dot{e}(t) = \dot{Y}(t) - \dot{X}(t)$, then system (13) is the synchronization error system. The controller system is constructed as follows:

$$\begin{cases} u_1 = -\hat{a}e_2 - \hat{a}(x_1 cos(x_2) - y_1 cos(y_2)) - \dot{e_1}, \\ u_2 = \hat{b}e_1 + e_3 - \dot{e_2}, \\ u_3 = e_1 - \hat{c}e_3 - y_1 y_2^2 + x_1 x_2^2 - \dot{e_3}. \end{cases} \tag{16}$$

Substituting (16) into (13), we obtain the error dynamics

$$\begin{cases} \dot{e_1} = (a - \hat{a})e_2 + (a - \hat{a})(x_1cos(x_2) - y_1cos(y_2)) - k_1e_1, \\ \dot{e_2} = (b - \hat{b})e_1 - e_3 - k_2e_2, \\ \dot{e_3} = -e_1 + (c - \hat{c})e_3 + y_1y_2^2 - x_1x_2^2 - k_3e_3. \end{cases} \tag{17}$$

Let us now define the parameter estimation error as $e_a = a - \hat{a}, e_b = b - \hat{b}, e_c = c - \hat{c}$. The error dynamics simplifies to

$$\begin{cases} \dot{e_1} = e_ae_2 + e_a(x_1cos(x_2) - y_1cos(y_2)) - k_1e_1, \\ \dot{e_2} = -e_be_1 - k_2e_2, \\ \dot{e_3} = e_ce_3 - k_3e_3. \end{cases} \tag{18}$$

Now, consider the Lyapunov stability function V given by

$$V = \frac{1}{2}(e_1^2 + e_2^2 + e_3^2 + e_a^2 + e_b^2 + e_c^2), \tag{19}$$

then

$$\dot{V} = (e_1\dot{e_1} + e_2\dot{e_2} + \dot{e_3}e_3 + \dot{e_a}e_a + \dot{e_b}e_b + \dot{e_c}e_c) \tag{20}$$

with

$$\dot{e_a} = -\dot{\hat{a}}, \dot{e_b} = -\dot{\hat{b}}, \dot{e_c} = -\dot{\hat{c}}. \tag{21}$$

In view of Eq. (20), the parameter update law can be chosen as

$$\begin{cases} \dot{\hat{a}} = e_1e_2 + e_1(x_1cos(x_2) - y_1cos(y_2)) + k_4e_a, \\ \dot{\hat{b}} = -e_1e_2 + k_5e_b, \\ \dot{\hat{c}} = e_ce_3 + k_ce_c. \end{cases} \tag{22}$$

Consequently,

$$\dot{V} = -k_1e_1^2 - k_2e_2^2 - k_3e_3^2 - k_4e_a^2 - k_5e_b^2 - k_6e_c^2 < 0, \tag{23}$$

which is a negative definite function. Using the Lyapunov stability theory [30], we conclude that the closed-loop system (18) is globally asymptotically stable for all initial values of the error signals e_1, e_2, e_3 by using the adaptive controller (16) and the update parameter law (22). It proves that the adaptive synchronization between the drive and response system is achieved.

5 Numerical Simulation

In order to verify our results, we applied the fourth order Runge-Kutta method in Matlab with the step size $h = 10^{-6}$ to solve the systems of differential equations (11), (12) and (13) for the active contol method with the initial conditions $X_0(1, 1, -0.5), Y_0(-2, -1, 1), e_0(-3, -2, 1.5)$ and the systems of differential equations (11), (12), (17) and (22) for the adaptive control method with the initial conditions $X_0(0.1, 0.2, -0.5), Y_0(-0.5, -0.2, 0.1), e_0(-0.6, -0.4, 0.6), (a_0, b_0, c_0) = (1, 0.5, 0.5), k_1 = k_2 = k_3 = 1$. The parameter values of systems are taken as in the chaotic case $a = 1.2, b = 0.46, c = 0.46$. In Figs.4, 6, the synchronization between the states x_i and $y_i, i = 1, 3$, is depicted. In Figs.5, 7, the time-history of the synchronization errors $e_1(t), e_2(t), e_3(t)$ is depicted for the two used control methods, respectively.

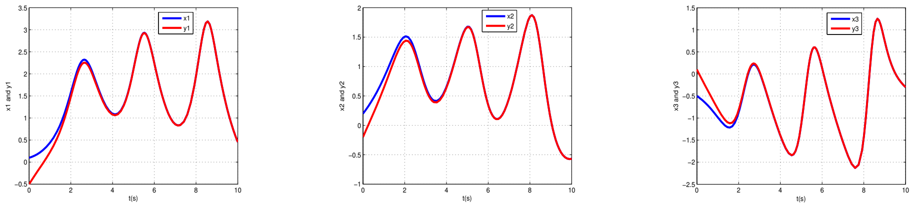

Figure 6: Synchronization between $x_i, y_i, i = 1, 2, 3$, using adaptive control.

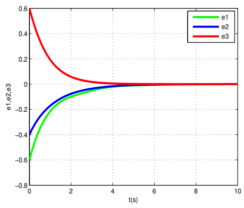

Figure 7: The time-history of the synchronization errors $e_1(t), e_2(t), e_3(t)$ using adaptive control.

6 Conclusion

This paper introduces a new chaotic system and proposes two control methods, active control and adaptive control, for synchronization using this system. We confirm the chaotic nature of the system using the Lyapunov exponents and demonstrate the coexistence of attractors. For active control, we design a controller based on the stability theory of integer-order systems to achieve synchronization. For adaptive control, we develop an estimation law to estimate the unknown parameters of the system and achieve synchronization. Numerical simulations are conducted to demonstrate the effectiveness and flexibility of both control methods.

The main important points in this work are:

- A novel chaotic system is presented and its chaoticity is confirmed using the Lyapunouv exponents.
- The synchronization between identical 3-D chaotic systems using two different control methods is achieved by designing appropriate active and adaptive controllers and estimation laws using the stability theory of integer-order systems.

The results achieved through the study of the novel chaotic system and the proposed control methods can be applied to various fields that involve chaotic systems, for example secure communication systems and encryption and decryption algorithms, Biological modeling and simulation of chaos-based robotics, making further research of the system particularly relevant. As such, they will be given due consideration in future work.

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