Nonlinear Dynamics and Systems Theory, **24** (5) (2024) 431–441

On Localization of Spectrum of an Integro-Differential Convection-Diffusion-Reaction Operator

O. Bahi [1*], A. Khellaf [1] and H. Guebbai [2]

[1] *Ecole Nationale Polytechniques de Constantine, Algeria.*
[2] *University 8 May 1945, Guelma, Algeria. Laboratoire de Mathematiques Applqiuees et Modelisation (LMAM).*

Received: April 10, 2024; Revised: October 1, 2024

Abstract: This paper explores the spectral properties of a non-self-adjoint integral-differential operator defined on an unbounded domain. The operator is governed by the Dirichlet-type conditions. We utilize the pseudo-spectral theory to demonstrate that the operator's spectrum is localized in the real numbers.

Keywords: *non-self-adjoint operators; unbounded operators; spectral analysis; integral-differential operators.*

Mathematics Subject Classification (2010): 47Axx, 58Jxx, 35Gxx.

1 Introduction

Non-self-adjoint and unbounded operators are fundamental in numerous branches of physics and chemistry, where phenomena like convection, diffusion, and reactions are widespread, see [1–3] and references therein. In this study, we focus on the spectral analysis of a non-self-adjoint integral-differential operator of convection-diffusion-reaction type, defined on an unbounded domain and subject to the Dirichlet-type conditions. The operator under consideration, denoted as L, is defined by the expression

$$L\xi = -\Delta\xi + \begin{pmatrix} -y \\ -x \end{pmatrix} \cdot \nabla\xi + (x^2+y^2)\xi + \int_\Gamma k(x,y,z,t)\xi(z,t)dzdt.$$

Convection equations can be considered as dynamic systems [4–6], where the state of the system evolves over time. They describe the transport of a quantity under the effect of a velocity field and can be analysed using the theory of dynamical systems and

* Corresponding author: mailto:bahi.oussama@enpc.com

semi-groups of operators. This approach makes it possible to study the stability and asymptotic behaviour of the solutions. The spectral analysis of the associated operator provides information about the propagation of the initial perturbations.

This study is distinguished by the unbounded and non-self-adjoint characteristics of the operator, which render it a subject of great interest within this field of research [7, 8]. The primary contributions of this study lie in utilizing the pseudo-spectral theory, see [9, 10], to demonstrate that the spectrum of the operator L is localized in $\mathbb{R}$. This innovative approach provides a promising alternative to the traditional spectral theory, with potential implications across various application domains. Our methodology is based mainly on the pseudo-spectral theory, splitting the spatial domain into finite-dimensional domains, then returning to the limit and recovering all the spectral properties. This technique was used in [11, 12].

Nevertheless, despite the notable advancements made, this study is subject to certain limitations, particularly with regard to the assumptions made about the integral operator within the integral-differential operator. These assumptions may prove challenging to verify in practice, although their relevance remains compelling.

The structure of this paper is designed to provide a comprehensive understanding of the problem under study. We begin by defining the theoretical framework in Section 2, then proceed to examine the restriction of the operator L to a bounded domain to localize its spectrum in Section 3. Next, in Section 4, we explore the relationship between the operator L and its restriction using the pseudo-spectral theory.

2 General Framework

Let Γ be an open unbounded set in $\mathbb{R}^2$ defined as follows:

$$\Gamma = \left\{(x, y) \in \mathbb{R}^2 : x > 0 \text{ and } -x < y < x\right\},$$

with its boundary denoted by $\partial\Gamma$. We define the space $L^2(\Gamma)$, the Hilbert space of complex-valued (classes of) functions defined almost everywhere on Γ, provided with their usual inner product $\langle\cdot,\cdot\rangle$. Let L be the integro-differential operator defined on $L^2(\Gamma)$ by

$$L\xi = -\Delta\xi + \begin{pmatrix} -y \\ -x \end{pmatrix} .\nabla\xi + (x^2 + y^2)\xi + \int_\Gamma k(x, y, z, t)\xi(z, t)dzdt,$$

where k is a real-valued function defined on $\Gamma \times \Gamma$, satisfying

$$(H) \left\| \begin{array}{l} i)\ \forall(x, y), (z, t) \in \Gamma : |k(x, y, z, t)| \le k_1(x, y)k_2(z, t), \\ \\ ii)\ k_1 \in L_\infty(\Gamma) \text{ and } k_2 \in L^2(\Gamma), \\ \\ iii)\ \forall(x, y), (z, t) \in \Gamma, \quad e^{xy}k(x, y, z, t) = e^{zt}k(z, t, x, y). \end{array} \right.$$

The operator T is given as follows:

$$T\xi = -\Delta\xi + \begin{pmatrix} -y \\ -x \end{pmatrix} \cdot \nabla\xi + (x^2 + y^2),$$

where this operator falls into the category of convection-diffusion operators, see [13]. Additionally, the operator $\mathcal{K}$ is defined as follows:

$$\forall\xi \in L^2(\Gamma), \forall(x, y) \in \Gamma, \quad \mathcal{K}\xi(x, y) = \int_\Gamma k(x, y, z, t)\xi(z, t)\, dz\, dt$$

representing the integral operator associated with the reaction term [14]. The sesquilinear form q is defined as follows:

$$q(\xi_1,\xi_2)=\int_\Gamma \nabla\xi_1.\overline{\nabla\xi_2}dxdy+\int_\Gamma \begin{pmatrix} -y \\ -x \end{pmatrix}.\nabla\xi_1\bar{\xi_2}dxdy+\int_\Gamma (x^2+y^2)\xi_1\bar{\xi_2}dxdy+\langle\mathcal{K}\xi_1,\xi_2\rangle,$$

and $\mathcal{Q}$ the quadratic form associated with q, is

$$\mathcal{Q}(\xi)=\int_\Gamma |\nabla\xi|^2dxdy+\int_\Gamma \begin{pmatrix} -y \\ -x \end{pmatrix}.\nabla\xi\bar{\xi}dxdy+\int_\Gamma (x^2+y^2)|\xi|^2dxdy+\langle\mathcal{K}\xi,\xi\rangle.$$

Using the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}\left|\int_\Gamma \begin{pmatrix} -y\bar{\xi} \\ -x\bar{\xi} \end{pmatrix}.\nabla\xi dxdy\right| &\le \int_\Gamma \left|\begin{pmatrix} -y\bar{\xi} \\ -x\bar{\xi} \end{pmatrix}\right|.|\nabla\xi|dxdy \\ &\le \frac{1}{2}\left(\|\nabla\xi\|^2_{L^2(\Gamma)}+\int_\Gamma (x^2+y^2)|\xi|^2dxdy\right).\end{aligned}$$

Hence, q is a sectorial form defined on the linear space

$$V=H^1_0(\Gamma)\cap\left\{\xi\in L^2(\Gamma):\int_\Gamma (x^2+y^2)|\xi|^2dxdy<+\infty\right\}$$

and L is the operator associated with q and its domain is

$$D(L)=H^2(\Gamma)\cap H^1_0(\Gamma)\cap\left\{\xi\in L^2(\Gamma):\int_\Gamma (x^2+y^2)|\xi|^2dxdy<+\infty\right\}.$$

Consider the eigenvalue problem, which represents the main problem addressed in this paper:

$$(P)\begin{cases} \text{Find } (\lambda,\xi)\in(\mathbb{C},D(L)\backslash\{0\}): \\ L\xi=\lambda\xi \text{ on } \Gamma, \\ \xi=0 \text{ on } \partial\Gamma. \end{cases}$$

We define the decreasing family $\{\Gamma_\eta\}_{0<\eta<1}$ of open bounded sets of $\mathbb{R}^2$ as

$$\Gamma_\eta=\left\{(x,y)\in\mathbb{R}^2:\eta<x<\eta^{-1}\text{and}-(1-\eta)(x-\eta)<y<(1-\eta)(x-\eta)\right\},$$

this family converges to Γ when η tends to 0. For all $\eta\in]0,1[$, we define on $L^2(\Gamma_\eta)$ the sesquilinear form q_η by

$$\begin{aligned} q_\eta(\xi_1,\xi_2)=&\textstyle\int_{\Gamma_\eta}\nabla\xi_1.\overline{\nabla\xi_2}dxdy+\int_{\Gamma_\eta}\begin{pmatrix} -y \\ -x \end{pmatrix}.\nabla\xi_1\bar{\xi_2}dxdy \\ &+\textstyle\int_{\Gamma_\eta}(x^2+y^2)\xi_1\bar{\xi_2}dxdy+<\mathcal{K}_\eta\xi_1,\xi_2>,\end{aligned}$$

where

$$\mathcal{K}_\eta\xi=\int_{\Gamma_\eta}k_\eta(x,y,z,t)\xi(z,t)dzdt,$$

and k_η is the restriction of k in Γ_η. It is evident that $k_\eta\in L^2(\Gamma_\eta\times\Gamma_\eta)$. To avoid any confusion, $\langle\cdot,\cdot\rangle$ is the usual inner product defined on $L^2(\Gamma_\eta)$. We note also that q_η is a sectorial form defined on $H^1_0(\Gamma_\eta)$ and the operator associated with q_η is L_η , defined on

$$D(L_\eta)=H^2(\Gamma_\eta)\cap H^1_0(\Gamma_\eta).$$

3 Spectrum of L_η

This section will examine the spectrum of the operator L_η. The results are presented in Theorem 1. We begin by defining the inner product on $L^2(\Gamma_\eta)$ by

$$\langle \xi_1, \xi_2 \rangle_\eta = \int_{\Gamma_\eta} e^{xy} \xi_1(x,y) \overline{\xi_2(x,y)} dxdy,$$

where its associated norm is denoted by $\|.\|_\eta$, which is equivalent to the usual norm $\|.\|_{L^2(\Gamma_\eta)}$. Note that the spectrum $sp(L)$ is defined as

$$sp(L) = \left\{ z \in \mathbb{C} : (L - zI)^{-1} \text{ is not bounded operator } \right\},$$

and so, $sp_p(L)$ consists only of the eigenvalues of L. Finally, $sp_{ess}(L) = sp(L) \setminus sp_p(L)$.

Lemma 3.1 *For all $\eta \in]0,1[$, L_η is self-adjoint with respect to $\langle \cdot, \cdot \rangle_\eta$.*

Proof. Let $\xi \in D(L_\eta)$, for all $(x,y) \in \Gamma_\eta$, we define $\tilde{\xi}(x,y) = e^{\frac{xy}{2}} \xi(x,y)$. So, we obtain that

$$\Delta \tilde{\xi} = (\Delta \xi + y \partial_x \xi + x \partial_y \xi + \frac{1}{4}(x^2 + y^2) \xi) e^{\frac{xy}{2}}.$$

Let $\xi_1, \xi_2 \in D(L_\eta)$. By the Green formula and using the above equation, we get

$$\langle T_\eta \xi_1, \xi_2 \rangle_\eta = \int_{\Gamma_\eta} \nabla \tilde{\xi_1} . \overline{\nabla \tilde{\xi_2}} dxdy + \int_{\Gamma_\eta} \frac{5}{4}(x^2 + y^2) \tilde{\xi_1} \overline{\tilde{\xi_2}} dxdy. \tag{1}$$

On the other hand, under the assumption (H), we get

$$\langle \mathcal{K}_\eta \xi_1, \xi_2 \rangle_\eta = \langle \xi_1, \mathcal{K}_\eta \xi_2 \rangle_\eta. \tag{2}$$

Consequently, from (1) and (2), the operator L_η is self-adjoint with respect to $\langle \cdot, \cdot \rangle_\eta$.

□

As a result, $sp(L_\eta)$ is a real value. Because of the impossibility of extending the inner product $\langle \cdot, \cdot \rangle_\eta$ over $L^2(\Gamma)$, it is not possible to ensure that L is self-adjoint.

We define the coefficient $C_{PF} = \frac{d(\Gamma_\eta)}{\sqrt{2}}$, where d is a measure on $\mathbb{R}^2$. The coefficient C_{PF} is known as the Poincaré-Friedrich constant [15]. The following theorem localises the essential and point spectra in the real line for the operators L_η.

Theorem 3.1 *For all $\eta \in]0,1[$, the essential spectrum of L_η, $sp_{ess}(L_\eta)$ is included in $\left] \frac{\eta^2}{2} - \|k_\eta\|_{L^2(\Gamma_\eta \times \Gamma_\eta)}, +\infty \right[$, and the point spectrum of L_η, $sp_p(L_\eta)$ is included in $[C_{PF}^{-2} + \eta^2 - \|k_\eta\|_{L^2(\Gamma_\eta \times \Gamma_\eta)}, +\infty[$.*

Proof. For all $\eta \in]0,1[$ and $\xi \in D(L_\eta)$, we have

$$\begin{aligned} Re(\langle L_\eta \xi, \xi \rangle) &= \frac{1}{2}(\langle L_\eta \xi, \xi \rangle + \overline{\langle L_\eta \xi, \xi \rangle}) = \frac{1}{2}(\langle L_\eta \xi, \xi \rangle + \langle \xi, L_\eta \xi \rangle) \\ &= Re(\langle A_\eta \xi, \xi \rangle) + Re(\langle \mathcal{K}_\eta \xi, \xi \rangle). \end{aligned}$$

So, using the Green formula and integrating by parts, we get

$$\begin{aligned}\langle A_\eta\xi,\xi\rangle &= \int_{\Gamma_\eta}(-\Delta\xi - y\partial_x\xi - x\partial_y\xi + (x^2+y^2)\xi).\bar{\xi}dxdy\\ &= \int_{\Gamma_\eta}\nabla\xi.\nabla\bar{\xi}dxdy - \int_{\partial\Gamma_\eta}\bar{\xi}\frac{\partial\xi}{\partial n}ds + \int_{\Gamma_\eta}y\xi\partial_x\bar{\xi}dxdy\\ &\quad - \int_{\Gamma_\eta}y\partial_x(\xi\bar{\xi})dxdy + \int_{\Gamma_\eta}x\xi\partial_y\bar{\xi}dxdy - \int_{\Gamma_\eta}x\partial_y(\xi\bar{\xi})dxdy\\ &\quad + \int_{\Gamma_\eta}(x^2+y^2)|\xi|^2dxdy,\end{aligned}$$

since $\xi \in H_0^1(\Gamma_\eta)$, this implies $\xi = 0$ a.e on $\partial\Gamma_\eta$, we simplify certain terms as

$$\int_{\partial\Gamma_\eta}\bar{\xi}\frac{\partial\xi}{\partial n}ds = 0,\ \int_{\Gamma_\eta}x\partial_y(\xi\bar{\xi})dxdy = \int_\eta^{\frac{1}{\eta}}[x\xi\bar{\xi}]_{-(1-\eta)(x-\eta)}^{(1-\eta)(x-\eta)}dx = 0,$$

and

$$\int_{\Gamma_\eta}y\partial_x(\xi\bar{\xi})dxdy = \int_{\frac{1-\eta}{\eta}}^{-\frac{1-\eta}{\eta}}[y\xi\bar{\xi}]_{\frac{y}{1-\eta}}^{\frac{1}{\eta}}dy = 0.$$

So,

$$\langle A_\eta\xi,\xi\rangle = \int_{\Gamma_\eta}(|\nabla\xi|^2d - y\xi\overline{\partial_x\xi} - x\xi\overline{\partial_y\xi} + (x^2+y^2)|\xi|^2)dxdy. \tag{3}$$

With the same argument, we find

$$\langle \xi, A_\eta\xi\rangle = \int_{\Gamma_\eta}(|\nabla\xi|^2 + y\xi\overline{\partial_x\xi} + x\xi\overline{\partial_y\xi} + (x^2+y^2)|\xi|^2)dxdy. \tag{4}$$

Thus, by adding (3) to (4) and using the Poincaré inequality, we get

$$Re(\langle A_\eta\xi,\xi\rangle) = \int_{\Gamma_\eta}(|\nabla\xi|^2 + (x^2+y^2)|\xi|^2)dxdy \geq (C_{PF}^{-2} + \eta^2)\,\|\xi\|^2_{L^2(\Gamma_\eta)}.$$

Now, let us estimate the term $Re(\langle\mathcal{K}_\eta\xi,\xi\rangle)$. By applying the Cauchy-Schwarz inequality twice, we obtain

$$\begin{aligned}|Re\,(\langle\mathcal{K}_\eta\xi,\xi\rangle)| &\leq |\langle\mathcal{K}_\eta\xi,\xi\rangle|\\ &\leq \left(\int_{\Gamma_\eta}\left(\int_{\Gamma_\eta}k_\eta(x,y,z,t)\xi(z,t)dzdt\right)^2dxdy\right)^{\frac{1}{2}}\left(\int_{\Gamma_\eta}\xi^2(x,y)dxdy\right)^{\frac{1}{2}}\\ &\leq \left(\int_{\Gamma_\eta}\int_{\Gamma_\eta}k_\eta^2(x,y,z,t)dxdydzdt\right)^{\frac{1}{2}}\left(\int_{\Gamma_\eta}\xi^2(z,t)dzdt\right)^{\frac{1}{2}}\\ &\quad\times\left(\int_{\Gamma_\eta}\xi^2(x,y)dxdy\right)^{\frac{1}{2}}\\ &\leq \|k_\eta\|_{L^2(\Gamma_\eta\times\Gamma_\eta)}\|\xi\|^2_{L^2(\Gamma_\eta)}.\end{aligned}$$

Thus,

$$Re(\langle \mathcal{K}_\eta \xi, \xi \rangle) \geq -\|k_\eta\|_{L^2(\Gamma_\eta \times \Gamma_\eta)} \|\xi\|^2_{L^2(\Gamma_\eta)},$$

then

$$Re(\langle L_\eta \xi, \xi \rangle) \geq (C_{PF}^{-2} + \eta^2 - \|k_\eta\|_{L^2(\Gamma_\eta \times \Gamma_\eta)}) \|\xi\|^2_{L^2(\Gamma_\eta)}. \tag{5}$$

For all $\lambda \in \mathbb{R}$ such that $\lambda < C_{PF}^{-2} + \eta^2 - \|k_\eta\|_{L^2(\Gamma_\eta \times \Gamma_\eta)}$ and $\xi \in D(L_\eta)$, we have the result

$$\|(L_\eta - \lambda I)\xi\|_{L^2(\Gamma_\eta)} \geq (C_{PF}^{-2} + \eta^2 - \|k_\eta\|_{L^2(\Gamma_\eta \times \Gamma_\eta)} - \lambda) \|\xi\|_{L^2(\Gamma_\eta)}. \tag{6}$$

Indeed, for all $\lambda \in \mathbb{R}$,

$$\|(L_\eta - \lambda I)\xi\|^2_{L^2(\Gamma_\eta)} = \|L_\eta \xi\|^2_{L^2(\Gamma_\eta)} - 2\lambda Re(< L_\eta \xi, \xi >) + \lambda^2 \|\xi\|^2_{L^2(\Gamma_\eta)} \geq 0, \tag{7}$$

then $(Re(\langle L_\eta \xi, \xi \rangle))^2 \leq \|\xi\|^2_{L^2(\Gamma_\eta)} \|L_\eta \xi\|^2_{L^2(\Gamma_\eta)}$, which implies that

$$\|L_\eta \xi\|^2_{L^2(\Gamma_\eta)} \geq \frac{(Re(\langle L_\eta \xi, \xi \rangle))^2}{\|\xi\|^2_{L^2(\Gamma_\eta)}}, \ \xi \in D(L_\eta) \setminus \{0\}.$$

Injecting the last inequality in (7), we get

$$\|(L_\eta - \lambda I)\xi\|^2_{L^2(\Gamma_\eta)} \geq (\frac{(Re(\langle L_\eta \xi, \xi \rangle))}{\|\xi\|^2_{L^2(\Gamma_\eta)}} - \lambda)^2 \|\xi\|^2_{L^2(\Gamma_\eta)},$$

by (5), for all $\lambda \in \mathbb{R}$ such that $\lambda < C_{PF}^{-2} + \eta^2 - \|k_\eta\|_{L^2(\Gamma_\eta \times \Gamma_\eta)}$, we find

$$\|(L_\eta - \lambda I)\xi\|^2_{L^2(\Gamma_\eta)} \geq (C_{PF}^{-2} + \eta^2 - \|k_\eta\|_{L^2(\Gamma_\eta \times \Gamma_\eta)} - \lambda)^2 \|\xi\|_{L^2(\Gamma_\eta)},$$

we conclude (6).

The operator $L_\eta - \lambda I$ is injective for $\lambda < C_{PF}^{-2} + \eta^2 - \|k_\eta\|_{L^2(\Gamma_\eta \times \Gamma_\eta)}$, which means that the point spectrum of L_η is included in $[C_{PF}^{-2} + \eta^2 - \|k_\eta\|_{L^2(\Gamma_\eta \times \Gamma_\eta)}, +\infty[$.

Now, we define the problems (P_η) and $(\tilde{P}_\eta)$ as follows:

$$(P_\eta) \begin{cases} \text{for all } g \in L^2(\Gamma_\eta), \text{ find } \xi \in D(L_\eta) \backslash \{0\} : \\ L_\eta \xi - \lambda \xi = g \text{ on } \Gamma_\eta, \\ \xi = 0 \text{ on } \partial \Gamma_\eta \end{cases}$$

and

$$(\tilde{P}_\eta) \begin{cases} \text{find } \xi \in H_0^1(\Gamma_\eta) \backslash \{0\} : \\ a_{\lambda,\eta}(\xi, v) = l(v) \text{ for } v \in H_0^1(\Gamma_\eta), \end{cases}$$

where

$$a_{\lambda,\eta}(\xi, v) = \int_{\Gamma_\eta} \nabla \xi . \overline{\nabla v} dx dy + \int_{\Gamma_\eta} \begin{pmatrix} -y \\ -x \end{pmatrix} . \nabla \xi \bar{v} dx dy + \int_{\Gamma_\eta} (x^2 + y^2 - \lambda) \xi \bar{v} dx dy + \langle \mathcal{K}_\eta \xi, v \rangle,$$

and

$$l(v) = \int_{\Gamma_\eta} g \tilde{v} dx dy.$$

The sesquilinear form $a_{\lambda,\eta}(\cdot,\cdot)$ is continuous and coercive in $H_0^1(\Gamma_\eta)$ for $\lambda < \frac{\eta^2}{2} - \|k_\eta\|_{L^2(\Gamma_\eta\times\Gamma_\eta)}$. Indeed, using the Cauchy-Schawrz inequality

$$\begin{aligned}|a_{\lambda,\eta}(\xi,v)| &\le \|\nabla\xi\|_{L^2(\Gamma_\eta)}\|\nabla v\|_{L^2(\Gamma_\eta)} + \frac{1}{2}\Big(\|\nabla\xi\|_{L^2(\Gamma_\eta)}\|\nabla v\|_{L^2(\Gamma_\eta)} + \\ &\quad \int_{\Gamma_\eta}(x^2+y^2)\xi\bar{v}dxdy\Big) + \int_{\Gamma_\eta}(x^2+y^2-\lambda)\xi\bar{v}dxdy \\ &\quad +\|k_\eta\|_{L^2(\Gamma_\eta\times\Gamma_\eta)}\|\xi\|_{L^2(\Gamma_\eta)}\|v\|_{L^2(\Gamma_\eta)} \\ &\le \frac{3}{2}\|\nabla\xi\|_{L^2(\Gamma_\eta)}\|\nabla v\|_{L^2(\Gamma_\eta)} + C_1\|\xi\|_{L^2(\Gamma_\eta)}\|v\|_{L^2(\Gamma_\eta)} \\ &\quad +\|k_\eta\|_{L^2(\Gamma_\eta\times\Gamma_\eta)}\|\xi\|_{L^2(\Gamma_\eta)}\|v\|_{L^2(\Gamma_\eta)},\end{aligned}$$

where $C_1 = \sup_{\Gamma_\eta}\left\{\frac{3}{2}(x^2+y^2)-\lambda\right\}$, we have

$$|a_{\lambda,\eta}(\xi,v)| \le C(\eta,\lambda,\|k_\eta\|_{L^2(\Gamma_\eta\times\Gamma_\eta)})\|\xi\|_{H^1(\Gamma_\eta)}\|v\|_{H^1(\Gamma_\eta)}$$

with

$$C(\eta,\lambda,\|k_\eta\|_{L^2(\Gamma_\eta\times\Gamma_\eta)}) = \max\{\frac{3}{2}, C_1 + \|k_\eta\|_{L^2(\Gamma_\eta\times\Gamma_\eta)}\}.$$

For the coercivity of $a_{\lambda,\eta}(.,.)$, we have

$$\langle\mathcal{K}_\eta\xi,\xi\rangle \ge -\|k_\eta\|_{L^2(\Gamma_\eta\times\Gamma_\eta)}\|\xi\|^2_{L^2(\Gamma_\eta)}.$$

So,

$$\begin{aligned}a_{\lambda,\eta}(\xi,\xi) &= \int_{\Gamma_\eta}|\nabla\xi|^2dxdy + \int_{\Gamma_\eta}\begin{pmatrix}-y\\-x\end{pmatrix}.\nabla\xi\bar{\xi}dxdy \\ &\quad + \int_{\Gamma_\eta}(x^2+y^2-\lambda)|\xi|^2dxdy + \langle\mathcal{K}_\eta\xi,\xi\rangle \\ &\ge \|\nabla\xi\|^2_{L^2(\Gamma_\eta)} - \frac{1}{2}\left(\|\nabla\xi\|^2_{L^2(\Gamma_\eta)} + \int_{\Gamma_\eta}(x^2+y^2)|\xi|^2dxdy\right) \\ &\quad + \int_{\Gamma_\eta}(x^2+y^2-\lambda)|\xi|^2dxdy - \|k_\eta\|_{L^2(\Gamma_\eta\times\Gamma_\eta)}\|\xi\|^2_{L^2(\Gamma_\eta)} \\ &\ge \frac{1}{2}\|\nabla\xi\|^2_{L^2(\Gamma_\eta)} + \min_{\Gamma_\eta}\left\{\frac{1}{2}(x^2+y^2)-\lambda\right\}\|\xi\|^2_{L^2(\Gamma_\eta)} - \|k_\eta\|_{L^2(\Gamma_\eta\times\Gamma_\eta)}\|\xi\|^2_{L^2(\Gamma_\eta)} \\ &\ge \frac{1}{2}\|\nabla\xi\|^2_{L^2(\Gamma_\eta)} + \left(\frac{\eta^2}{2} - \lambda - \|k_\eta\|_{L^2(\Gamma_\eta\times\Gamma_\eta)}\right)\|\xi\|^2_{L^2(\Gamma_\eta)}.\end{aligned}$$

Then

$$a_{\lambda,\eta}(\xi,\xi) \ge \min\left\{\frac{1}{2}, \frac{\eta^2}{2} - \lambda - \|k_\eta\|_{L^2(\Gamma_\eta\times\Gamma_\eta)}\right\}\|\xi\|^2_{H_0^1(\Gamma_\eta)}.$$

Otherwise, the semilinear form l is continuous in $H_0^1(\Gamma_\eta)$. Therefore, the Lax-Milgram theorem gives that, for all $g \in L^2(\Gamma_\eta)$ and for all λ such that $\lambda < \frac{1}{2}\eta^2 - \|k_\eta\|_{L^2(\Gamma_\eta\times\Gamma_\eta)}$, the problem $(\tilde{P}_\eta)$ has a unique solution $u \in H_0^1(\Gamma_\eta)$ for all $v \in H_0^1(\Gamma_\eta)$. This solution also satisfy the problem (P_η). We conclude that (P_η) has unique solution, for all $\eta \in]0,1[$. this completes the proof.

□

4 Pseudo-Spectra and Spectra of L

In this section, we will establish the relationships concerning the spectrum and the pseudo-spectrum of the operators L and L_η. We need to show an important density lemma. We denote by $\mathcal{D}(\Gamma)$ the space of infinitly differentiable functions defined on Γ with compact support in it, and we denote by $\|\cdot\|_L$ the graph norm of the operator L, which is defined by $\|\cdot\|_L = \|L\cdot\|_{L^2(\Gamma)} + \|\cdot\|_{L^2(\Gamma)}$.

Lemma 4.1 *$\mathcal{D}(\Gamma)$ is dense in $D(L)$ with respect to the graph norm $\|.\|_L$.*

Proof. See [12].

We will show an important set equality, which gives us a relationship between the pseudo-spectrum of L and the pseudo-spectrum of L_η. The definition of the pseudo-spectrum of an unbounded linear operator A on a Hilbert space H, denoted $sp_\epsilon(A)$, is the set of $\lambda \in \mathbb{C}$ for which there exists a vector $x \in H$ with $\|x\| = 1$ such that

$$\|(A - \lambda I)x\| < \epsilon.$$

Formally, this is written as

$$sp_\epsilon(A) = \{\lambda \in \mathbb{C} : \exists x \in H, \|x\| = 1, \|(A - \lambda I)x\| < \epsilon\}.$$

This definition indicates that for each λ in the pseudo-spectrum, there exists a unit vector x such that the action of $A - \lambda I$ on x is very small. In other words, λ is almost an eigenvalue of A in the sense that A acts on x almost like multiplication by λ, see [10].

Theorem 4.1 *For all $\varepsilon > 0$, we have the relation*

$$sp_\varepsilon(L) = \bigcup_{0<\eta<1} sp_\varepsilon(L_\eta).$$

Proof. Let $\lambda \in \bigcup_{0<\eta<1} sp_\varepsilon(L_\eta)$. So, there exists $\eta_1 \in]0, 1[$ such that $\lambda \in sp_\varepsilon(L_{\eta_1})$. But the operator L_{η_1} is self-adjoint, for that, there exists $u \in D(L_{\eta_1})$ with $\|\xi\|_{L^2(\Gamma_{\eta_1})} = 1$ such that

$$\|(L_{\eta_1} - \lambda I)\xi\|_{L^2(\Gamma_{\eta_1})} < \varepsilon.$$

On the other hand, we have $\mathcal{D}(\Gamma_{\eta_1})$ is dense in $D(L_{\eta_1})$ with respect to the graph norm (see Lemma 4.1). Then, there exists a sequence $(\xi_n)_{n\in\mathbb{N}}$ in $\mathcal{D}(\Gamma_{\eta_1})$ such that

$$\lim_{n \longrightarrow +\infty} \|\xi_n - \xi\|_{L_{\eta_1}} = 0,$$

where $\|\xi\|_{L_{\eta_1}} = \|\xi\|_{L^2(\Gamma_{\eta_1})} + \|L_{\eta_1}\xi\|_{L^2(\Gamma_{\eta_1})}$ is the graph norm.

As a result, $((L_{\eta_1} - \lambda I)\xi_n)_{n\in\mathbb{N}}$ converges to $(L_{\eta_1} - \lambda I)\xi$ in $L^2(\Gamma_{\eta_1})$. Now, for all $\theta > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have

$$\left| \frac{\|(L_{\eta_1} - \lambda I)\xi_n\|_{L^2(\Gamma_{\eta_1})}}{\|\xi_n\|_{L^2(\Gamma_{\eta_1})}} - \frac{\|(L_{\eta_1} - \lambda I)\xi\|_{L^2(\Gamma_{\eta_1})}}{\|\xi\|_{L^2(\Gamma_{\eta_1})}} \right| < \theta.$$

Let $\theta = \varepsilon - \|(L_{\eta_1} - \lambda I)\xi\|_{L^2(\Gamma_{\eta_1})} > 0$. Then there exists $n_0 \in \mathbb{N}$ such that

$$\frac{\|(L_{\eta_1} - \lambda I)\xi_{n_0}\|_{L^2(\Gamma_{\eta_1})}}{\|\xi_{n_0}\|_{L^2(\Gamma_{\eta_1})}} < \varepsilon,$$

we find $\xi_{n_0} \in \mathcal{D}(\Gamma_{\eta_1})$. Then we extend ξ_{n_0} by zero to Γ , we denote its extension by $\tilde{\xi}_{n_0}$. So, it is clear that $\tilde{\xi}_{n_0} \in \mathcal{D}(\Gamma) \subset D(L)$ and we have $\|\tilde{\xi}_{n_0}\|_{L^2(\Gamma_{\eta_1})} = \|\xi_{n_0}\|_{L^2(\Gamma_{\eta_1})}$. Let $v_{n_0} = \frac{\xi_{n_0}}{\|\xi_{n_0}\|} \in \mathcal{D}(\Gamma_{\eta_1})$ and its extension by zero to Γ is defined by $\tilde{v}_{n_0} = \frac{\tilde{\xi}_{n_0}}{\|\tilde{\xi}_{n_0}\|} \in \mathcal{D}(\Gamma)$. Then

$$\|(L - \lambda I)\tilde{v}_{n_0}\|_{L^2(\Gamma)} = \|(L_{\eta_1} - \lambda I)v_{n_0}\|_{L^2(\Gamma_{\eta_1})} = \frac{\|(L_{\eta_1} - \lambda I)\xi_{n_0}\|_{L^2(\Gamma_{\eta_1})}}{\|\xi_{n_0}\|_{L^2(\Gamma_{\eta_1})}} < \varepsilon.$$

So, we find that $\lambda \in sp_\varepsilon(L)$.

Reciprocally, let $\lambda \in sp_\varepsilon(L)$. There exists $\xi \in D(L)$ with $\|\xi\|_{L^2(\Gamma)} = 1$ such that

$$\|(L - \lambda I)\xi\|_{L^2(\Gamma)} < \varepsilon.$$

By Lemma 4.1, there exists a sequence $(\xi_n)_{n\in\mathbb{N}}$ in $\mathcal{D}(\Gamma)$ such that $\|\xi_n - \xi\|_L \to 0$. According to the same arguments as above, there exists $n_1 \in \mathbb{N}$ such that

$$\frac{\|(L - \lambda I)\xi_{n_1}\|_{L^2(\Gamma)}}{\|\xi_{n_1}\|_{L^2(\Gamma)}} < \varepsilon, \tag{8}$$

as $\operatorname{supp} \xi_{n_1} \subset \Gamma$, there exists $\eta_0 \in]0, 1[$ such that $\operatorname{supp} \xi_{n_1} \subset \Gamma_{\eta_0}$. We get from (8) that

$$\|(L_{\eta_0} - \lambda I)v_{n_1}\|_{L^2(\Gamma_{\eta_0})} < \varepsilon, \text{ and } v_{n_1} \in D(L_{\eta_0}),$$

where $v_{n_1} = \frac{\xi_{n_1}}{\|\xi_{n_1}\|_{L^2(\Gamma_{\eta_0})}}$. We conclude that $\lambda \in sp_\varepsilon(L_{\eta_0})$. This completes the proof.

□

Remark 4.1 It is clear that the operators $(L_\eta)_{\eta\in]0,1[}$ are normal operators with respect to $\langle\cdot,\cdot\rangle_\eta$.

The following theorems are the main focus of our results. The ε-neighborhood of a set S in $\mathbb{C}$ is denoted by $N_\varepsilon(S)$.

Theorem 4.2 *For all $\varepsilon > 0$, we obtain the relation*

$$\bigcup_{0<\eta<1} sp_\varepsilon(L_\eta) = N_\varepsilon(\bigcup_{0<\eta<1} sp(L_\eta)).$$

Proof. Let $\lambda \in \bigcup_{0<\eta<1} sp_\varepsilon(L_\eta)$. There exists $\eta_1 \in]0, 1[$ such that

$$\lambda \in sp_\varepsilon(L_{\eta_1}) = N_\varepsilon(sp(L_{\eta_1})).$$

So, $\lambda = z + s$, where $s \in sp(L_{\eta_1})$ and $|z| < \varepsilon$. But $s \in \bigcup_{0<\eta<1} sp(L_\eta)$, this implies that

$$\lambda = z + s \in N_\varepsilon(\bigcup_{0<\eta<1} sp(L_\eta)).$$

Reciprocally, let $\lambda \in N_\varepsilon(\bigcup_{0<\eta<1} sp(L_\eta))$. Then $\lambda = z + s$, where $s \in \bigcup_{0<\eta<1} sp(L_\eta)$ and $|z| < \varepsilon$. There is $\eta_2 \in]0, 1[$ such that $\lambda = z + s \in N_\varepsilon(sp(L_{\eta_2})) = sp_\varepsilon(L_{\eta_2})$. Then

$$\lambda \in \bigcup_{0<\eta<1} sp_\varepsilon(L_\eta).$$

□

The next theorem is our main result, characterising the spectrum of the operator L as a union of operators spectrum of L_η; $\eta \in]0,1[$, which allows to determine that the spectrum of the operator L is purely real.

Theorem 4.3 *The spectrum of L is localized in $\mathbb{R}$, where*

$$sp(L) = \bigcup_{0<\eta<1} sp(L_\eta).$$

Proof. We apply Theorem 4.1 and Theorem 4.2, we find

$$sp_\varepsilon(L) = \bigcup_{0<\eta<1} sp_\varepsilon(L_\eta) = N_\varepsilon(\bigcup_{0<\eta<1} sp(L_\eta)).$$

We use the propriety $\bigcap\limits_{0<\varepsilon<1} N_\varepsilon(S) = S$, where S is a set in $\mathbb{C}$. Also, we use the fact that

$$sp(L) = \bigcap_{0<\varepsilon<1} sp_\varepsilon(L).$$

Then we conclude that

$$sp(L) = \bigcap_{0<\varepsilon<1} sp_\varepsilon(L) = \bigcap_{0<\varepsilon<1} N_\varepsilon(\bigcup_{0<\eta<1} sp(L_\eta)) = \bigcup_{0<\eta<1} sp(L_\eta).$$

This completes the proof. □

Acknowledgment

We extend our heartfelt appreciation to the reviewers and editors for their insightful feedback and valuable contributions to improving the quality of this manuscript. Their expertise and dedication greatly enhanced the final outcome of our work.

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