Nonlinear Dynamics and Systems Theory, **24** (5) (2024) 442–459

A Priori Predictions for a Weak Solution to Time-Fractional Nonlinear Reaction-Diffusion Equations Incorporating an Integral Condition

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Received: February 3, 2024; Revised: September 21, 2024

Abstract: Within this paper, we lay out the necessary criteria that ensure a solution's presence and distinctiveness within a functionally weighted Sobolev space. This pertains to a specific group of initial-boundary value problems accompanied by an integral condition, all related to nonlinear partial fractional reaction-diffusion (RD) equations. Our findings are derived through the utilization of a priori estimates in Bouziani fractional spaces. By employing an iterative approach built upon outcomes from the linear counterpart, we successfully validate the existence and uniqueness of a weak generalized solution for the nonlinear conundrum.

Keywords: *fractional partial differential equation; existence; uniqueness; energy inequality.*

Mathematics Subject Classification (2010): 35R11, 35A01, 35A02, 70K75.

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1 Introduction

The nonlinear diffusion equation is a partial differential equation that describes the behavior of a diffusing quantity in a medium where the diffusion rate depends on the concentration or magnitude of the quantity itself, these are suggested as mathematical models of physical problems in many fields such as image processing, heat conduction in composite materials, reaction-diffusion systems, nonlinear diffusion in fluid mechanics and population dynamics [1–3].

Fractional differential equations (FDEs) generalize ordinary and partial differential equations by incorporating fractional derivatives instead of integer-order derivatives. FDEs have attracted considerable attention because they can describe complex phenomena characterized by long-range memory, anomalous diffusion, and fractal-like behavior, fractional differential equations (FDEs) are used to model a wide range of phenomena in various fields such as visco-elasticity, biological, electrical circuits, control systems, and geological systems [4–8]. These are just a few examples of the diverse range of applications where FDEs are treated. Fractional calculus and FDEs provide a powerful mathematical framework to capture complex dynamics, memory effects, and non-local interactions in various systems, as well as a variety of other physical phenomena. Recently, there has been a lot of progress in the study of fractional differential equations [9–13]. This is due to several recent studies in this field, see the monographs of Kilbas et al. [14], Miller and Ross [15], Samko et al. [16], and the papers of Agarwal et al. [17], Anguraj A. and Karthikeyan P. [18], Belmekki et al. [19], Daftardar-Gejji and Jafari [20, 21], Kaufmann and Mboumi [22], Kilbas and Marzan. [23], Yu and Gao [24], Oussaeif [25], and also the general references in Baleanu et al. [26], and the references therein.

However, many phenomena can better be described by integral boundary conditions, which are often used in problems where the system's physical or mathematical characteristics require considering the solution's cumulative behavior over a specific region. They can arise in various fields, including heat transfer, fluid mechanics, quantum mechanics, and population dynamics. Bouziani [24, 27] has extensively studied the topic and generated significant interest in various works. The recent surge in interest in nonlinear fractional reaction-diffusion (RD) equations [28, 29] can be attributed to their ability to exhibit self-organization phenomena and introduce the fractional index as a new parameter in the equation. Moreover, the analysis of these equations from both analytical and numerical perspectives has generated considerable attention. These equations provide a fertile, promising research area, offering rich mathematical insights. Despite efforts to investigate fractional RD equations under specific boundaries and initial conditions, explicit solutions are often elusive. This study delves into a more comprehensive examination of a generalized model for nonlinear time-fractional RD equations.

The aim of this paper is to expand the utilization of the energy inequality method to establish the existence and uniqueness of weak solutions in functionally weighted Sobolev spaces. Specifically, we focus on a class of initial-boundary value problems with a nonlocal condition referred to as the "integral condition" for a broader range of nonlinear partial fractional differential equations. To the best of our knowledge, this particular class of equations has not been previously investigated. Additionally, this work serves as an explanation and complement to our previous paper [24]. Furthermore, this research introduces novel theoretical concepts involving Bouziani fractional spaces.

2 Preliminaries

Let $\Omega = [0;T]$ be a finite interval of the real numbers R and $\Gamma(\cdot)$ denote the gamma function. For any $0 < \alpha < 1$ being a positive integer, the Caputo and Riemann-Liouville derivatives are, respectively, defined as follows:

- The left Caputo fractional derivative of order α is defined respectively by

$$ {}^{C}D_t^{\alpha}\mu(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial \mu(x,s)}{\partial s} \frac{1}{(t-s)^{\alpha}} ds. \tag{1} $$

- The right Caputo fractional derivatives of order α is defined respectively by

$$ {}_{t}^{C}D^{\alpha}\mu(x,t) = \frac{-1}{\Gamma(1-\alpha)} \int_t^T \frac{\partial \mu(x,s)}{\partial s} \frac{1}{(s-t)^{\alpha}} ds. \tag{2} $$

- The left Riemann-Liouville fractional derivative of order α is defined respectively by

$$ {}^{R}D_t^{\alpha}\mu(x,t) = \frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial t} \int_0^t \frac{\mu(x,s)}{(t-s)^{\alpha}} ds. \tag{3} $$

- The right Riemann-Liouville fractional derivative of order α is defined respectively by

$$ {}_{t}^{R}D^{\alpha}\mu(x,t) = \frac{-1}{\Gamma(1-\alpha)} \frac{\partial}{\partial t} \int_t^T \frac{\mu(x,s)}{(s-t)^{\alpha}} ds. \tag{4} $$

Many authors think that Caputo's version is more natural because it makes the handling of homogeneous initial conditions easier. Then the two definitions (1) and (3) are linked by the following relationship, which can be verified by a direct calculation:

$$ {}_{a}^{R}D_t^{\alpha}\mu(x,t) = {}_{a}^{C} D_t^{\alpha}\mu(x,t) + \frac{\mu(x,0)}{\Gamma(1-\alpha)t^{\alpha}}. \tag{5} $$

Definition 2.1 [30, 31] For any real $\theta > 0$ and finite interval $[a,b]$ of the real axis $\mathbb{R}$, we define the semi-norm

$$ |v|^2_{{}^{l}H^{\theta}(\Omega)} = \left\| {}^{R}D_t^{\theta} v \right\|^2_{L^2(\Omega)} $$

and the norm

$$ \|v\|^2_{{}^{l}H^{\theta}(\Omega)} = \|v\|^2_{L^2(\Omega)} + |v|^2_{{}^{l}H^{\theta}(\Omega)}. \tag{6} $$

Next, we define ${}^{l}H^{\theta}(\Omega)$ as the closure of $C_0^{\infty}(\Omega)$ with respect to the norm $\|\cdot\|_{{}^{l}H^{\theta}(\Omega)}$.

Definition 2.2 [30, 31] For any real $\theta > 0$ and finite interval $[a,b]$ of the real axis $\mathbb{R}$, we define the semi-norm

$$ |v|^2_{{}^{r}H^{\theta}(\Omega)} = \left\| {}_{t}^{R}D^{\theta} v \right\|^2_{L^2(\Omega)} $$

and the norm

$$ \|v\|^2_{{}^{r}H^{\theta}(\Omega)} = \|v\|^2_{L^2(\Omega)} + |v|^2_{{}^{r}H^{\theta}(\Omega)}. \tag{7} $$

In what follows, we define ${}^{l}H^{\theta}(\Omega)$ as the closure of $C_0^{\infty}(\Omega)$ with respect to the norm $\|\cdot\|_{{}^{l}H^{\theta}(\Omega)}$.

Definition 2.3 For any real $\theta > 0$ and finite interval $[a,b]$ of the real axis $\mathbb{R}$, we define the semi-norm

$$|v|^2_{{}^{c}H^{\theta}(\Omega)} = \left| \frac{({}^{R}D_t^{\theta}v, {}_t^{R}D^{\theta}v)_{L^2(\Omega)}}{\cos(\theta\pi)} \right|$$

and the norm

$$\|v\|^2_{{}^{c}H^{\theta}(\Omega)} = \|v\|^2_{L^2(\Omega)} + |v|^2_{{}^{c}H^{\theta}(\Omega)}. \tag{8}$$

Lemma 2.1 *[30, 31] For any real $\theta \in \mathbb{R}_+$, if $u \in {}^{l}H^{\theta}(\Omega)$ and $v \in C_0^{\infty}(\Omega)$, then we have*

$$({}^{R}D_t^{\theta}u(t), v(t))_{L^2(\Omega)} = (u(t), {}_t^{R}D^{\theta}v(t))_{L^2(\Omega)}.$$

Lemma 2.2 *[30, 31] For $0 < \theta < 2$, $\theta \neq 1$ and $u \in H_0^{\frac{\theta}{2}}(\Omega)$, then*

$${}^{R}D_t^{\theta}u(t) = {}^{R}D_t^{\frac{\theta}{2}}u\,{}^{R}D_t^{\frac{\theta}{2}}u(t).$$

Lemma 2.3 *[30, 31] For any real $\theta \in \mathbb{R}_+$ and $\theta \neq n + \frac{1}{2}$, the semi norms $|\cdot|_{{}^{l}H^{\theta}(\Omega)}$, $|\cdot|_{{}^{r}H^{\theta}(\Omega)}$, and $|\cdot|_{{}^{c}H^{\theta}(\Omega)}$ are equivalent, then we pose*

$$|\cdot|_{{}^{l}H^{\theta}(\Omega)} \cong |\cdot|_{{}^{r}H^{\theta}(\Omega)} \cong |\cdot|_{{}^{c}H^{\theta}(\Omega)}.$$

Lemma 2.4 *[30, 31] For any real $\theta > 0$, the space ${}^{R}H_0^{\theta}(\Omega)$ with respect to the norm (7) is complete.*

Definition 2.4 We denote by $L_2(0,T,L_2(0,1)) = L_2(Q)$ the space of functions which are square integrable in the Bochner sense with the scalar product

$$(u,w)_{L_2(0,T,L_2(0,1))} = \int_0^T ((u,\cdot),(w,\cdot))_{L_2(0,1)}\,dt. \tag{9}$$

Since the space $L_2(0,1)$ is a Hilbert space, it can be shown that $L_2(0,T,L_2(0,1))$ is a Hilbert space as well. Now, let $C^{\infty}(0,T)$ denote the space of infinitely differentiable functions on $(0,T)$ and $C_0^{\infty}(0,T)$ denote the space of infinitely differentiable functions with compact support in $(0,T)$.

3 Bouziani Functional Spaces

We introduce the function spaces needed in our investigation. Let $L^2(0,1)$ and $L^2(0,T,L^2(0,1))$ be the standard function spaces. Also, we denote by $C_0(0,1)$ the vector space of continuous functions with compact support in (0.1). Since such functions are Lebesgue integrable with respect to dx, we can define it on $C_0(0,1)$. The bilinear form is given by

$$(u.w) = \int_0^1 \Im_x u.\Im_x w dx, \tag{10}$$

where $\Im_x u = \int_0^x u(\zeta,\cdot)d\zeta$ and $\Im_x^* u = \int_x^1 u(\zeta,\cdot)d\zeta$. The previous bilinear form (22) is considered as a scalar product on $C_0(0,1)$ for which $C_0(0,1)$ is not complete.

Definition 3.1 We denote by $B^2(0,1)$ a completion of $C_0(0,1)$ for the scalar product (13), which is denoted by $(\cdot,\cdot)_{B^2(0,1)}$. It is also called the Bouziani space or the space of square-integrable primitive function on $(0,1)$. By the norm of the function u from $B^2(0,1)$, we can understand the non negative number

$$\|u\|_{B^2(0,1)} = \sqrt{(u,u)_{B^2(0,1)}} = \|\Im_x u\|_{L^2(0,1)}.$$

For $u \in L^2(0,1)$, we have the elementary inequality

$$\|u\|^2_{B^2(0,1)} \leq \frac{1}{2}\|u\|^2_{L^2(0,1)}. \tag{11}$$

We denote by $L^2(0,T,B^2(0,1)) = B^2(\Omega)$ the space of functions, which are called the square integrable in the Bochner sense with the scalar product

$$(u.w)_{B^2(\Omega)} = \int_0^1 ((u,\cdot).(w,\cdot))_{B^2(0,1)}dx \tag{12}$$

for which $B^2(\Omega)$ is a Hilbert space.

4 Solvability of Solution of Diffusion Fractional Dirichlet Problems

4.1 Formulation of the problem

In this part, we assume $\Omega = (0;1)$ and $I = (0;T)$ with $0 \prec T \prec +\infty$. Also, we consider the following nonlinear fractional problem:

$$\begin{cases} {}^cD_t^\alpha u(x,t) - \frac{\partial}{\partial x}(a(x,t)\frac{\partial u(x,t)}{\partial x}) + bu(x,t) = f(x,t,u,\frac{\partial u}{\partial x}), & \forall (x,t) \in Q, \\ u(x,0) = 0 \qquad \forall x \in (0,1), \\ \int_\Omega x^k u(x,t)dx = 0 \qquad \forall t \in (0,T) \text{ and } k = \{0,1\}, \end{cases} \tag{P}$$

where $Q = \Omega \times I$ is an open bounded interval of $\mathbb{R}$ and a, b, f are known functions. Also, we suppose the following conditions:

- (A_1) We assume that $0 \prec a_0 \leq a(x,t) \leq a_1$ and $a_2 \leq \frac{\partial^2 a(x,t)}{\partial x^2} \leq a_3$ for all $(x,t) \in Q$.
- (A_2) We assume that the compatibility conditions

$$\int_\Omega x^k u(x,t)dx = 0 \qquad \forall t \in (0,T) \text{ and } k = \{0,1\}$$

are verified.

4.2 The associated linear problem

In this part, we show the existence and uniqueness of the strong solution of the linear problem. The proof is based on an a priori estimate and the density of the set of values of the image of the operator generated by the problem

$$\begin{cases} {}^cD_t^\alpha u(x,t) - \frac{\partial}{\partial x}(a(x,t)\frac{\partial u(x,t)}{\partial x}) + bu(x,t) = f(x,t), \forall (x,t) \in Q, \\ u(x,0) = 0 \qquad \forall x \in (0,1), \\ \int_\Omega x^k u(x,t)dx = 0 \qquad \forall t \in (0,T) \text{ and } k = \{0,1\}, \end{cases} \tag{P_1}$$

whose diffusion problem is given as follows:

$$\pounds u =^{c} D_t^{\alpha} u(x,t) - \frac{\partial}{\partial x}(a(x,t)\frac{\partial u(x,t)}{\partial x}) + bu(x,t) = f(x,t) \tag{13}$$

with the initial condition

$$lu = u(x,0) = 0 \quad \forall x \in [-1,1] \tag{14}$$

and the integral conditions

$$\int_{\Omega} x^k u(x,t) dxdt = 0 \qquad \forall t \in (0,T) \text{ and } k = \{0,1\}, \tag{15}$$

where $f(x,t)$ is a given function and α satisfies the assumption $0 \leq \alpha \leq 1$, for which $(x,t) \in \bar{Q}$.

4.3 A priori estimation

In this part, we aim to establish an a priori bound and prove the existence of a solution to the problems (13)-(15) with $Lu = F$, where $L = (\mathscr{L}, l)$ and $F = f$ is the operator equation corresponding to problems (13)-(15). To obtain a full overview about such an estimation, the reader may refer to [32]. The operator L acts from E to F, which is defined as follows. The Banach space E consists of all functions $u(x,t)$ with the finite norm

$$\|u\|_E^2 = \|\Im_x u\|_{L^2(Q)}^2 + \|u\|_{L^2(Q)}^2 . \tag{16}$$

The Hilbert space F consists of the vector-valued functions $F = f$ with the norm

$$\| \mathscr{F} \|_F^2 = \|f\|_{L^2_{(Q)}}^2 + \int_0^T I^{\alpha} \|f\|_{L^2_{(0,1)}}^2 dt + \|\Im_x \varphi\|_{L^2_{(0,1)}}^2 . \tag{17}$$

4.4 A priori bound

Theorem 4.1 *If the assumption (A_1) is satisfied, then for any function $u \in D(L)$, there exists a positive constant c independent of u such that*

$$\|\Im_x u\|_{L^2(Q)}^2 + \|u\|_{L^2(Q)}^2 \leq k \left(\|f\|_{L^2_{(Q)}}^2 + \int_0^T I^{\alpha} \|f\|_{L^2_{(0,1)}}^2 dt + \|\Im_x \varphi\|_{L^2_{(0,1)}}^2 \right) \tag{18}$$

for which $D(L)$ is the domain of definition of the operator L defined by

$$D(L) = \{u \in L^2(Q) \ / \ \Im_x u \in L^2(Q)\},$$

satisfying conditions (15).

Proof. By taking the scalar product in $L^2(Q)$ on (13) and the operator

$$Mu = \int_x^1 \left(\int_0^{\varsigma} u(\eta,t)\, d\eta \right) d\zeta,$$

where $Q^\tau = \Omega \times (0,T)$, we obtain

$$\begin{aligned}(\pounds u, Mu)_{L^2(Q^\tau)} &= \left({}^c D_t^\alpha u, \int_x^1 \left(\int_0^\varsigma u(\eta,t)\,d\eta\right) d\zeta\right)_{L^2(Q)} \\ &- \left(\frac{\partial}{\partial x}(a(x,t)\frac{\partial u(x,t)}{\partial x}) \int_x^1 \left(\int_0^\varsigma u(\eta,t)\,d\eta\right) d\zeta\right)_{L(Q)} \\ &+ \left(bu(x,t), \int_x^1 \left(\int_0^\varsigma u(\eta,t)\,d\eta\right) d\zeta\right)_{L^2(Q)} = \left(\tilde{f}, u\right)_{L^2(Q)}.\end{aligned} \tag{19}$$

The successive integration by parts of integrals on the right-hand side of (17) yields

$$\begin{aligned}\left({}^R D_t^\alpha u, Mu\right)_{L^2(Q)} &= \int_Q \left({}^c D_t^\alpha u(x,t) \,.\, \int_x^1 \left(\int_0^\varsigma u(\eta,t)\,d\eta\right) d\zeta\right) dx \\ &= \int_{Q^\tau} \left({}^c D_t^\alpha \int_0^x u(\zeta,t)\,d\zeta. \int_0^1 u(\zeta,t)\,d\zeta\right) dxdt \\ &= \int_{Q^\tau} \left({}^c D_t^{\frac{\alpha}{2}} \int_0^x u(\zeta,t)\,d\zeta . {}^c D_t^{\frac{\alpha}{2}} \int_0^x u(\zeta,t)\,d\zeta\right) dxdt \\ &= \left\| {}^c D_t^{\frac{\alpha}{2}} \Im_x u \right\|^2_{L^2(Q^\tau)}\end{aligned} \tag{20}$$

and

$$\begin{aligned}&-\left(\frac{\partial}{\partial x}(a\frac{\partial u}{\partial x}), Mu\right)_{L(Q)} = -\int_{Q^\tau} -\left(\frac{\partial}{\partial x}(a(x,t)\frac{\partial u(x,t)}{\partial x}) \int_x^1 \left(\int_0^\varsigma u(\eta,t)\,d\eta\right) d\zeta\right) dxdt \\ &= \int_{Q^\tau} \left(\left(a(x,t)\frac{\partial u(x,t)}{\partial x}\right)\left(\int_0^x u(\zeta,t)\,d\zeta\right)\right) dxdt \\ &= \int_{Q^\tau} a(x,t)\,(u(x,t))^2\,dxdt - \frac{1}{2}\int_{Q^\tau} \left(\frac{\partial^2 a(x,t)}{\partial x^2}\right)\left(\int_0^x u(\zeta,t)\,d\zeta\right) dxdt \\ &\geq a_0 \|u\|^2_{L^2(Q^\tau)} - \frac{a_3}{2} \|\Im_x u\|^2_{L^2(Q^\tau)}.\end{aligned} \tag{21}$$

Consequently, we have

$$\begin{aligned}(bu, Mu)_{L(Q)} &= \int_{Q^\tau} \left(b(x,t)\,u(x,t) \int_x^1 \left(\int_0^\varsigma u(\eta,t)\,d\eta\right) d\zeta\right) dxdt \\ &\geq b\int_{Q^\tau} \left(u(x,t) \int_x^1 \left(\int_0^\varsigma u(\eta,t)\,d\eta\right) d\zeta\right) dxdt \\ &\geq b\int_{Q^\tau} \left(\int_0^1 u(\zeta,t)\,d\zeta\right)\left(\int_0^1 u(\zeta,t)\,d\zeta\right) dxdt \\ &\geq b\int_{Q^\tau} (\Im_x u(x,t)^2\,dxdt \\ &\geq b\,\|\Im_x u\|^2_{L^2_{(0,1)}}.\end{aligned} \tag{22}$$

Substituting (20), (21) and (22) into (19) gives

$$\left\| {}^c D_t^{\frac{\alpha}{2}} \Im_x u \right\|^2_{L^2(Q^\tau)} + a_0 \|u\|^2_{L^2(Q^\tau)} - \frac{a_3}{2} \|\Im_x u\|^2_{L^2(Q^\tau)} + b\,\|u\|^2_{L^2(Q^\tau)} \leq \left(\tilde{f}, Mu\right). \tag{23}$$

Now, we estimate the last term on the right-hand side of (23) by applying the Cauchy inequality $\left(|ab| \leq \frac{a^2}{2\varepsilon} + \frac{\varepsilon b^2}{2}\right)$ with ε. In other words, we have

$$\begin{aligned}
&\left\| {}^cD_t^{\frac{\alpha}{2}} \Im_x u \right\|^2_{L^2(Q^\tau)} + a_0 \|u\|^2_{L^2(Q^\tau)} - \frac{a_3}{2} \|\Im_x u\|^2_{L^2(Q^\tau)} + b_0 \|u\|^2_{L^2(Q^\tau)} \\
\leq\ & \int_{Q^\tau} f(x,t) \left(\int_x^1 \left(\int_0^\varsigma u(\eta,t)\, d\eta \right) d\zeta \right) dxdt \\
\leq\ & \frac{1}{2\varepsilon} \int_{Q^\tau} \left(\int_0^x u(\zeta,t) d\zeta \right)^2 dxdt + \frac{\varepsilon}{2} \int_{Q^\tau} \left(\int_0^x f(\zeta,t) d\zeta \right)^2 dxdt. \qquad (24)
\end{aligned}$$

By using the Cauchy-Schwartz inequality, we obtain

$$\begin{aligned}
&\frac{1}{2\varepsilon} \int_{Q^\tau} \left(\int_0^x u(\zeta,t) d\zeta \right)^2 dxdt + \frac{\varepsilon}{2} \int_{Q^\tau} \left(\int_0^x f(\zeta,t) d\zeta \right)^2 dxdt \\
\leq\ & \frac{1}{2\varepsilon} \int_{Q^\tau} (\Im_x u(x,t))^2 dxdt + \frac{\varepsilon}{2} \int_{Q^\tau} (\Im_x f(x,t))^2 dxdt \\
\leq\ & \frac{1}{4\varepsilon} \int_{Q^\tau} (u(x,t))^2 dxdt + \frac{\varepsilon}{4} \int_{Q^\tau} (f(x,t))^2 dxdt \\
=\ & \frac{1}{4\varepsilon} \|u\|^2_{L^2(Q^\tau)} + \frac{\varepsilon}{4} \|f\|^2_{L^2(Q^\tau)}. \qquad (25)
\end{aligned}$$

This, consequently, yields

$$\left\| {}^cD_t^{\frac{\alpha}{2}} \Im_x u \right\|^2_{L^2(Q^\tau)} + \left(a_0 + b - \frac{1}{4\varepsilon} \right) \|u\|^2_{L^2_{(Q^\tau)}} \leq \frac{\varepsilon}{4} \|f\|^2_{L^2(Q^\tau)}$$

and

$$\left\| {}^cD_t^{\frac{\alpha}{2}} \Im_x u \right\|^2_{L^2(Q^\tau)} + \|u\|^2_{L^2_{(Q^\tau)}} \leq \frac{\varepsilon}{4 \min\left\{1, \left(a_0 + b - \frac{1}{4\varepsilon}\right)\right\}} \|f\|^2_{L^2_{(Q^\tau)}}.$$

Now, we present

$$C = \frac{\varepsilon}{4 \min\left\{1, \left(a_0 + b - \frac{1}{4\varepsilon}\right)\right\}}$$

as

$$I^\alpha \left({}^cD_t^\alpha u(x,t) \right) = u(x,t) + \varphi(x).$$

This leads to

$$\|\Im_x u\|^2_{L^2(Q^\tau)} + \|u\|^2_{L^2_{(0,1)}} \leq C I^\alpha \|f\|^2_{L^2_{(0,1)}} + \|\Im_x \varphi\|^2_{L^2(0,1)}.$$

Consequently, we have

$$\|\Im_x u\|^2_{L^2(Q^\tau)} \leq C I^\alpha \|f\|^2_{L^2_{(Q^\tau)}} + \|\Im_x \varphi\|^2_{L^2(Q^\tau)}.$$

So, finally, we get

$$\|u\|^2_{L^2_{(Q^\tau)}} \leq C \|f\|^2_{L^2_{(Q^\tau)}},$$

where

$$C_1 = \max\{1, C\}$$

and

$$\|\Im_x u\|^2_{L^2(Q^\tau)} + \|u\|^2_{L^2_{(0,1)}} \le C_1 \left(I^\alpha \|f\|^2_{L^2_{(0,1)}} + \|f\|^2_{L^2_{(0,1)}} + \|\Im_x \varphi\|^2_{L^2(0,1)} \right).$$

With the use of successive integration $(0,T)$, we get

$$\|\Im_x u\|^2_{L^2_{(Q^\tau)}} + \|u\|^2_{L^2_{(Q^\tau)}} \le (C_1 \max\{1,T\}) \left(\int_0^T I^\alpha \|f\|^2_{L^2_{(Q^\tau)}} dt + \|f\|^2_{L^2_{(Q^\tau)}} + \|\Im_x \varphi\|^2_{L^2_{(Q^\tau)}} \right)$$

which implies

$$k = \quad (C_1 \max\{1,T\})^{\frac{1}{2}}$$

for which

$$\|u\|_E \le k \|Lu\|_F . \tag{26}$$

Let $R(L)$ be the range of the operator L. However, since we do not have any information about $R(L)$, except that $R(L) \subset F$, we must extend L so that (26) holds for the extension, and its range is the whole space F. For this purpose, we state the following proposition.

Proposition 4.1 *The operator $L : E \longrightarrow F$ has a closure.*

Proof. Let $(u_n)_{n\in\mathbb{N}} \subset D(L)$ be a sequence where

$$u_n \longrightarrow 0 \qquad \text{in } E$$

and

$$Lu_n \longrightarrow \left(\tilde{f}; 0\right) \qquad \text{in } F \ . \tag{27}$$

Herein, we must prove that

$$f \equiv 0.$$

The convergence of u_n to 0 in E leads to

$$u_n \longrightarrow 0 \qquad \text{in } D'(Q) \ . \tag{28}$$

According to the continuity of the derivation of $D'(Q)$ in $D'(Q)$, the relation (28) involves

$$\mathscr{L} u_n \longrightarrow 0 \qquad \text{in } D'(Q) \ . \tag{29}$$

Moreover, the convergence of $\mathscr{L} u_n$ to f in $L^2(Q)$ gives

$$\mathscr{L} u_n \longrightarrow f \qquad \text{in } D'(Q) \ . \tag{30}$$

As we have the uniqueness of the limit in $D'(Q)$, we conclude from (29) and (30) that $f = 0$. Then, L is closable of this operator with the domain of definition $D(L)$.

Definition 4.1 A solution of the operator equation

$$\bar{L}u = F$$

is called a strong solution to problems (13)-(15). The a priori estimate (18) can be extended to strong solutions, i.e., we have the estimate

$$\|\Im_x u\|^2_{L^2(Q)} + \|u\|^2_{L^2(Q)} \le k \left(\|f\|^2_{L^2_{(Q)}} + \int_0^T I^\alpha \|f\|^2_{L^2_{(0,1)}} dt + \|\Im_x \varphi\|^2_{L^2_{(0,1)}} \right).$$

We deduce from the estimate (18) the subsequent result.

Corollary 4.1 *The range $R(\bar{L})$ of the operator $\bar{L}$ is closed in F and is equal to the closure $\overline{R(L)}$ of $R(L)$, that is, $R(\bar{L}) = \overline{R(L)}$.*

Proof. Let $z \in \overline{R(L)}$. Then, there is a Cauchy sequence $(z_n)_{n\in\mathbb{N}}$ in F constituting of the elements of the set $R(L)$ such that

$$\lim_{n\longrightarrow+\infty} z_n = z.$$

There is then a corresponding sequence $u_n \in D(L)$ such that

$$z_n = Lu_n.$$

With the use of estimate (18), we get

$$\|u_p - u_q\|_E \le C\,\|Lu_p - Lu_q\|_F \to 0,$$

where p, q tend towards infinity. We can deduce that $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in E. So, as E is a Banach space, there exists $u \in E$ such that

$$\lim_{n\longrightarrow+\infty} u_n = u \text{ in } E.$$

By virtue of the definition of $\bar{L}$ ($\lim\limits_{n\longrightarrow+\infty} u_n = u$ in E, if $\lim\limits_{n\longrightarrow+\infty} Lu_n = \lim\limits_{n\longrightarrow+\infty} z_n = z$, then $\lim\limits_{n\longrightarrow+\infty} \bar{L}u_n = z$ as $\bar{L}$ is closed, so $\bar{L}u = z$), the function u satisfies

$$v \in D\left(\bar{L}\right),\ \bar{L}v = z.$$

Then $z \in R(\bar{L})$, and so we have

$$\overline{R(L)} \subset R(\bar{L}).$$

Also, we conclude here that $R(\bar{L})$ is closed because it is Banach (any complete subspace of a metric space (not necessarily complete) is closed). Thus, it remains to show the reverse inclusion. To this aim, it should be noted that either $z \in R(\bar{L})$ and then there exists a Cauchy sequence $(z_n)_{n\in\mathbb{N}}$ in F constituting of the elements of the set $R(\bar{L})$ such that

$$\lim_{n\longrightarrow+\infty} z_n = z,$$

or $z \in R(\bar{L})$ because $R(\bar{L})$ is a closed subset of a completed F and so $R(\bar{L})$ is complete. There is then a corresponding sequence $u_n \in D(\bar{L})$ such that

$$\bar{L}u_n = z_n.$$

As a result, we get from (18) that

$$\|u_p - u_q\|_E \le C\left\|\bar{L}u_p - \bar{L}u_q\right\|_F \to 0,$$

where p and q tend towards infinity. We can then deduce that $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in E, and so as E is a Banach space, there exists $u \in E$ such that

$$\lim_{n\longrightarrow+\infty} u_n = u \ \text{ in } E.$$

Once again, there is a corresponding sequel $(Lu_n)_{n\in\mathbb{N}} \subset R(L)$ such that

$$\bar{L}u_n = Lu_n \text{ on } R(L),\ \forall n \in \mathbb{N}.$$

So, we obtain

$$\lim_{n\longrightarrow+\infty} Lu_n = z.$$

Consequently, we obtain $z \in \overline{R(L)}$, and then we conclude that

$$R(\bar{L}) \subset \overline{R(L)}.$$

4.5 Existence of solution

Theorem 4.2 *Let the assumptions (A_1) be satisfied. Then for all $F = (f, 0) \in F$, there exists a unique strong solution $u = \bar{L}^{-1}\mathscr{F} = \overline{L^{-1}}\mathscr{F}$ of the problem (2)-(4).*

Proof. We have

$$(Lu, W)_F = \int_Q \mathscr{L}u.wdxdt, \tag{31}$$

where

$$W = (w, 0).$$

So, for $w \in L^2(Q)$ and for all $u \in D_0(L) = \{u,\ u \in D(L) : \ell u = 0\}$, we have

$$\int_Q u.wdxdt = 0.$$

By putting $w = u$ and using the same estimate as previously, we obtain

$$\left\| {}^cD_t^{\frac{\alpha}{2}} \Im_x u \right\|^2_{L^2(0,1)} + \|u\|^2_{L^2_{(0,1)}} = 0,$$

which implies

$$\|u\| \leq 0 \Rightarrow u = 0.$$

So, we get $u = w = 0$.

Corollary 4.2 *If for any function $u \in D(L)$, we have the following estimate:*

$$\|u\|_E \leq C\,\|Lu\|_F\,,$$

then the solution of problem (P_1), if it exists, is unique.

Proof. Let u_1 and u_2 be two solutions to problem (P_1), i.e.,

$$\begin{cases} Lu_1 = \mathcal{F} \\ Lu_2 = \mathcal{F} \end{cases} \Longrightarrow Lu_1 - Lu_2 = 0,$$

where L is a linear operator. As a result, we obtain

$$L(u_1 - u_2) = 0. \tag{32}$$

Now, according to (32), we obtain

$$\|u_1 - u_2\|^2_E \leq c\,\|0\|^2_F = 0,$$

which, consequently, gives

$$u_1 = u_2.$$

5 Solvability of the Weak Solution of the Nonlinear Problem

This section is devoted to the proof of the existence and uniqueness of the solution of the nonlinear problem (Pr):

$$\begin{cases} {}^cD_t^\alpha u(x,t) - \frac{\partial}{\partial x}(a(x,t)\frac{\partial u(x,t)}{\partial x}) + bu(x,t) = f(x,t,u,\frac{\partial u}{\partial x}), \forall (x,t) \in Q, \\ u(x,0) = 0 \qquad \forall x \in (-1,1), \\ \int_\Omega x^k u(x,t)dx = 0 \qquad \forall t \in (0,T) \text{ and } k = \{0,1\}, \end{cases} \quad (P_2)$$

for which the function f is Lipchitzian. As a consequence, there is a positive constant k such that

$$\begin{aligned} &\left\| f\left(x,t,u_1,\frac{\partial u_1}{\partial x}\right) - f\left(x,t,u_2,\frac{\partial u_1}{\partial x}\right) \right\|_{L^2(Q)} \\ &\qquad \le k\left(\|u_1 - u_2\|_{L^2(Q)} + \left\| \frac{\partial u_1}{\partial x} - \frac{\partial u_1}{\partial x} \right\|_{L^2(Q)} \right). \end{aligned} \quad (33)$$

Now, we shall prove that (P_2) has a unique weak solution. To this end, we let $u \in \widetilde{C^1}(Q)$ and $u \in C^1(Q)$. Also, we shall compute the integral $\int_Q (f \Im_x^* v)dxdt$. For this purpose, we assume $u, v \in \widetilde{C^1}(Q)$, $\int_Q x^k u(x,t)dx = 0$, and $\int_Q x^k u(x,t)dx = 0$ for all $k = \{0,1\}$. By using the condition on u and v, we have

$$\int_Q ({}^cD_t^\alpha u \cdot \Im_x^* v)dxdt = -\int_Q (v \cdot {}^c D_t^\alpha \Im_x^* u)dxdt,$$

$$-\int_Q (\frac{\partial}{\partial x}(a(x,t)\frac{\partial u}{\partial x}) \cdot \Im_x^* v)dxdt = \int_Q (v \cdot a(x,t)\frac{\partial u}{\partial x})dxdt,$$

and

$$b\int_Q (u \cdot \Im_x^* v)dxdt = -\int_Q (v \cdot \Im_x^* u)dxdt.$$

It then follows, from [24], that

$$A(u,v) = -\int_Q (v \cdot {}^c D_t^\alpha \Im_x^* u)dxdt + \int_Q (v \cdot a(x,t)\frac{\partial u}{\partial x})dxdt - \int_Q (v \cdot \Im_x^* u)dxdt.$$

Definition 5.1 A function u is called a weak solution of problem (P_1) and $u \in L^2(0,T,H^1(0,1))$.

By building a recurring sequence starting with $u^{(0)} = 0$, we can define the sequence $\left(u^{(n)}\right)_{n\in\mathbb{N}}$ as follows: given the element $u^{(n-1)}$, for $n = 1,2,3,\ldots$, we will solve the following problem:

$$\begin{cases} {}^cD_t^\alpha u^{(n)}(x,t) - \frac{\partial}{\partial x}(a(x,t)\frac{\partial u^{(n)}(x,t)}{\partial x}) + b(x,t)u^{(n)}(x,t) = f(x,t,u^{(n-1)},\frac{\partial u^{(n-1)}}{\partial x}), \\ \int_\Omega x^k u^{(n)}(x,t)dx = 0 \qquad \forall t \in (0,T) \text{ and } k = \{0,1\}. \end{cases} \quad (P_3)$$

Theorem 5.1 *According to the study of the previous linear problem and by fixing n, problem (P_3) admits a unique solution $u^{(n)}(x,t)$.*

Now, by supposing

$$z^{(n)}(x,t) = u^{(n+1)}(x,t) - u^{(n)}(x,t),$$

we might get a new problem, which has the form

$$\begin{cases} {}^{c}D_t^{\alpha} z^{(n)} - \left(a(x,t)\, z_x^{(n)}\right)_x + b(x,t) z^{(n)} = p^{(n-1)}(x,t), \\ z^{(n)}(x.0) = 0, \\ \int_\Omega x^k z^{(n)}(x,t)dx = 0, \qquad \forall t \in (0,T),\ k = \{0,1\}, \end{cases} \tag{P_4}$$

where

$$p^{(n-1)}(x,t) = f\left(x,t,u^{(n)},\frac{\partial u^{(n)}}{\partial x}\right) - f\left(x,t,u^{(n-1)},\frac{\partial u^{(n-1)}}{\partial x}\right)$$

with the condition

$$z^{(n)}(x.0) = 0 \text{ and } \int_\Omega x^k z^{(n)}(x,t)dx = 0 \qquad \forall t \in (0,T),\ k = \{0,1\}. \tag{34}$$

Lemma 5.1 *Assume that condition (34) holds, then for the linearized problem* (P_4)*, we have the following a priori estimate:*

$$\left\| Z^{(n)} \right\|_{L^2(0,1,H^1(0,1))} \le \lambda \left\| Z^{(n-1)} \right\|_{L^2(0,1,H^1(0,1))},$$

where λ *is a positive constant given by*

$$\lambda = \sqrt{\frac{5\phi^2}{2\varepsilon}},$$

for which $\varepsilon \prec\prec 1$.

Proof. Multiplying equation (P_4) by $\int_x^1 \left(\int_0^\varsigma z^n(\eta,t)\, d\eta\right) d\zeta$ and integrating the result over Q yield

$$\begin{aligned} &\int_Q \left(({}^{R}D_t^{\alpha} z^{(n)}(x,t) \cdot \int_x^1 \left(\int_0^\varsigma z^n(\eta,t)\, d\eta\right) d\zeta \right) dxdt \\ - &\int_Q (\frac{\partial}{\partial x}\left(a(x,t)\frac{\partial z^{(n)}(x,t)}{\partial x}\right) \cdot \int_x^1 \left(\int_0^\varsigma z^n(\eta,t)\, d\eta\right) d\zeta) dxdt \\ + &\int_Q b z^{(n)}(x,t) \cdot \int_x^1 \left(\int_0^\varsigma z^n(\eta,t)\, d\eta\right) d\zeta dxdt \\ = &\int_Q \left(p^{(n-1)}(x,t) \cdot \int_x^1 \left(\int_0^\varsigma z^n(\eta,t)\, d\eta\right) d\zeta \right) dxdt. \end{aligned} \tag{35}$$

By using the standard integration by parts for each term in (P_4) coupled with condition (34), we obtain

$$\begin{aligned} &\int_{Q^\tau}^{c} \left({}^{c}D_t^{\alpha} \int_0^x z^n(\zeta,t)\, d\zeta \right)^2 dxdt + a_1 \int_{Q^\tau} \left(z^n(x,t)\right)^2 dxdt \\ \le &\int_{Q^\tau} \left(\frac{a_3}{2} + \frac{\varepsilon}{2} + \frac{b}{2}\right) \left(z^n(x,t)\right)^2 dxdt + \frac{1}{2\varepsilon} \int_{Q^\tau} \left(p^{(n-1)}\right)^2 dxdt. \end{aligned} \tag{36}$$

On the other hand, by applying the operator $\Im_x^*$ to equation (36), we get

$$ {}^cD_t^{\alpha}\left(\Im_x^* Z^n\right) + a(x,t)\frac{\partial Z^{(n)}(x,t)}{\partial x} = \Im_x^*\left(p^{(n-1)}\right). \tag{37}$$

Consequently, by taking into account condition (37), multiplying the obtained equality by $\frac{\partial Z^n}{\partial x}$, and then integrating the result over $\Omega^\tau = (0,1)\times(0,\tau)$, where $0 \le \tau \le T$, we obtain

$$\begin{aligned} \int_{Q^\tau} \left({}^cD_t^{\alpha} Z^n\right)\cdot Z^n dxdt + \int_{Q^\tau} a(x,t)\left(\frac{\partial Z^{(n)}(x,t)}{\partial x}\right)^2 dxdt \\ = \int_{Q^\tau} \Im_x^*\left(p^{(n-1)}(x,t)\right)\frac{\partial Z^n}{\partial x} dxdt. \end{aligned} \tag{38}$$

Now, by using Lemmas 2.2, 2.3, 2.4, and 5.1 coupled with the Cauchy inequality with ε, we obtain

$$\begin{aligned} & \int_{Q^\tau}\left({}^cD_t^{\frac{\alpha}{2}} Z^n\right)^2 dxdt + \int_{Q^\tau} a(x,t)\left(\frac{\partial Z^{(n)}(x,t)}{\partial x}\right)^2 dxdt \\ \le\ & \frac{1}{2\varepsilon}\int_{Q^\tau} \Im_x^*\left(p^{(n-1)}(x,t)\right)^2 dxdt + \frac{\varepsilon}{2}\int_{Q^\tau}\left(\frac{\partial Z^n}{\partial x}\right)^2 dxdt. \end{aligned} \tag{39}$$

Combining the last two inequalities (38) and (39) gives

$$\begin{aligned} & \int_{Q^\tau}^{c}\left({}^cD_t^{\alpha}\int_0^x z^n(\zeta,t)\,d\zeta\right)^2 dxdt + \int_{Q^\tau}\left({}^cD_t^{\frac{\alpha}{2}} Z^n\right)^2 dxdt \\ + \ & a_1\int_{Q^\tau}\left(z^n(x,t)\right)^2 dxdt + \int_{Q^\tau} a(x,t)\left(\frac{\partial Z^{(n)}(x,t)}{\partial x}\right)^2 dxdt \\ \le\ & \frac{1}{2\varepsilon}\int_{Q^\tau}\left(p^{(n-1)}\right)^2 dxdt + \frac{1}{2\varepsilon}\int_{Q^\tau}\Im_x^*\left(p^{(n-1)}(x,t)\right)^2 dxdt \\ + \ & \int_{Q^\tau}\left(\frac{a_3}{2}+\frac{\varepsilon}{2}+\frac{b}{2}\right)\left(z^n(x,t)\right)^2 dxdt + \frac{\varepsilon}{2}\int_{Q^\tau}\left(\frac{\partial Z^n}{\partial x}\right)^2 dxdt. \end{aligned} \tag{40}$$

By eliminating two first integrals on the left-hand-side of inequality (40) and using the Cauchy inequality with ε, we get

$$\begin{aligned} & \left(a(x,t) - \frac{a_3}{2} - \frac{\varepsilon}{2} - \frac{b}{2}\right)\left\{\int_{Q^\tau}\left(\left(z^n(x,t)\right)^2 + \left(\frac{\partial Z^{(n)}(x,t)}{\partial x}\right)^2\right) dxdt\right\} \\ \le\ & \frac{1}{2\varepsilon}\left(\int_{Q^\tau}\left(p^{(n-1)}\right)^2 dxdt + \int_{Q^\tau}\Im_x^*\left(p^{(n-1)}(x,t)\right)^2 dxdt\right). \end{aligned} \tag{41}$$

Therefore, we have the estimate

$$\begin{aligned} \left\|\Im_x^* p^{(n-1)}\right\|^2_{L^2(0,1)} &\le \frac{1}{4}(1-0)^2\left\|p^{(n-1)}\right\|^2_{L^2(0,1)} \\ &\le \frac{1}{4}\left\|p^{(n-1)}\right\|^2_{L^2(0,1)} \end{aligned} \tag{42}$$

and

$$\begin{aligned}
&\int\limits_{Q^\tau} \left(p^{(n-1)}\right)^2 dxdt \\
\leq\ & \phi^2 \int\limits_{Q^\tau} \left(\left|z^{(n-1)}(x,t)\right| + \left|\frac{\partial Z^{(n-1)}(x,t)}{\partial x}\right|\right)^2 dxdt \\
\leq\ & \phi^2 \int_0^\tau \left\|z^{(n-1)}(\cdot,t)\right\|^2_{L^2(0,1)} + \left\|\frac{\partial Z^{(n-1)}(\cdot,t)}{\partial x}\right\|^2_{L^2(0,1)} dtdx. \qquad (43)
\end{aligned}$$

Now, substituting (41) and (42) into (44) yields

$$\begin{aligned}
&\left(a(x,t) - \frac{a_3}{2} - \frac{\varepsilon}{2} - \frac{b}{2}\right) \int_0^\tau \left\{\left\|z^{(n)}(\cdot,t)\right\|^2_{L^2(0,1)} + \left\|\frac{\partial Z^{(n)}(\cdot,t)}{\partial x}\right\|^2_{L^2(0,1)}\right\} dtdx \\
\leq\ & \frac{5\phi^2}{2\varepsilon} \int_0^\tau \left\{\left\|z^{(n-1)}(\cdot,t)\right\|^2_{L^2(0,1)} + \left\|\frac{\partial Z^{(n-1)}(\cdot,t)}{\partial x}\right\|^2_{L^2(0,1)}\right\} dtdx. \qquad (44)
\end{aligned}$$

Since $a(x,t) - \frac{a_3}{2} - \frac{\varepsilon}{2} - \frac{b}{2} \geq 0$, we find

$$\begin{aligned}
&\int_0^\tau \left\{\left\|z^{(n)}(\cdot,t)\right\|^2_{L^2(0,1)} + \left\|\frac{\partial Z^{(n)}(\cdot,t)}{\partial x}\right\|^2_{L^2(0,1)}\right\} dt \\
&\frac{5\phi^2}{2\varepsilon} \int_0^\tau \left\{\left\|z^{(n-1)}(\cdot,t)\right\|^2_{L^2(0,1)} + \left\|\frac{\partial Z^{(n-1)}(\cdot,t)}{\partial x}\right\|^2_{L^2(0,1)}\right\} dtdx.
\end{aligned}$$

The right-hand side here is independent of τ, and hence we shall replace the left-hand side by the upper bound with respect to τ to obtain the desired inequality, i.e.,

$$\left\|Z^{(n)}\right\|^2_{L^2(0,1,H^1(0,1))} \leq \frac{5\phi^2}{2\varepsilon} \left\|Z^{(n-1)}\right\|^2_{L^2(0,1,H^1(0,1))}.$$

This forms the criteria of convergence of the series $\sum\limits_{n=1}^{\infty} z^{(n)}$, which converges if $\frac{5\phi^2}{2\varepsilon} < 1$, that is, if $\phi = \sqrt{\frac{2\varepsilon}{5}}$. Since $Z^{(n)}(x,t) = u^{(n+1)}(x,t) - u^n(x,t)$, it follows that the sequence $(u^n)_{n\in N}$ will be defined by

$$u^{(n)}(x,t) = \sum_{n=1}^{n-1} z^{(i)} + u^{(0)}(x,t),$$

which converges to an element $u \in L^2\left(0,1,H^1(0,1)\right)$.

Therefore, we have established the following result.

Theorem 5.2 *Under the condition (34), the solution for problem (P_2) is unique.*

Proof. Suppose that u_1 and u_2 in $L^2\left(0,1,H^1(0,1)\right)$ are two solutions of (P_4), then $Z = u_1 - u_2$ satisfies $Z \in L^2\left(0,1,H^1(0,1)\right)$. As a result, we have

$${}^cD_t^\alpha u(x,t) - \frac{\partial}{\partial x}(a(x,t)\frac{\partial u(x,t)}{\partial x}) + bu(x,t) = \vartheta(x,t)$$

and

$$\int_{\Omega} x^k z^{(n)}(x,t)dx = 0 \qquad \forall t \in (0,T)\,,\ k = \{0,1\},$$

where

$$\vartheta(x,t) = f\left(x,t,u_1,\frac{\partial u_1}{\partial x}\right) - f\left(x,t,u_2,\frac{\partial u_1}{\partial x}\right).$$

Following the same procedure as in establishing the proof of Lemma 5.1, we get

$$\left\|Z^{(n)}\right\|_{L^2(0,1,H^1(0,1))} \leq \lambda \left\|Z^{(n-1)}\right\|_{L^2(0,1,H^1(0,1))},$$

where λ is the same constant as in Lemma 5.1. Since $\lambda < 1$, then we obtain

$$(1-\lambda)\left\|Z^{(n-1)}\right\|_{L^2(0,1,H^1(0,1))},$$

form which we conclude that $u_1 = u_2$ in $L^2\left(0,1,H^1\left(0,1\right)\right)$.

6 Conclusion

This paper has explained the important factors needed to ensure a solution stands out and fits well within a certain type of mathematical space. This is particularly relevant to a set of problems involving equations with fractions and reactions that change over time. We have figured out these factors by using certain mathematical estimates and techniques. By building upon previous work and using a step-by-step method, we have confirmed that there is indeed a unique solution to these tricky equations. There will be future research and applications on fractional partial differential equations.

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