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Numerical Solution of Fractional Hopfield Neural Networks Using Reproducing Kernel Hilbert Space Method

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Abstract: Artificial neural networks (ANN) consist of a group of the virtual neurons that are designed by computerized programs which use a variety of mathematical fractional equations. In this paper, we introduce the Reproducing Kernel Hilbert Space (RKHS) method for solving some certain fractional differential systems in the artificial neural networks field, which is the Hopfield network, using the conformable derivative.

Keywords: *Reproducing Kernel Hilbert Space Method (RKHSM), fractional derivative, artificial neural networks, differential systems, chaotic attractors.*

Mathematics Subject Classification (2010): 46E22, 26A33, 92B20, 34A30, 70K55.

1 Introduction

Artificial Neural Networks (ANN) is a recently emerging powerful computer-aided design (CAD) technology for modeling devices and circuits. These networks consist of a set of virtual neurons that are generated by computer programs that use a number of fractional mathematical equations to process the data that come from the neurons. The Hopfield network is a variety of recurrent artificial neural networks. Its idea arose from the behavior of particles in a magnetic field such that each particle is communicated (completely linked) with another particle by magnetic forces. This is referred to as activation in the

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case of neurons. As a result, both particles or neurons spin, encouraging one another to continue this rotation. The Hopfield neural networks come in two versions: binary and continuous. In the binary form, all neurons are linked to each other, there is no connection from a neuron to itself, in the continuous version, all connections, including self-connections, are allowed. The Hopfield network neurons will allow each other to rotate while they are in a spinning state. The movement of the particles is to process information, so they will be in the activation situation. For example, if two particles are in a rotating state to process information, this is known as binary activation in the Hopfield neural networks [1].

For obtaining the solution of the Hopfield neural network equation systems, we propose the reproducing kernel Hilbert space method which was used for the first time at the beginning of the 20th century by S. Zaremba for the harmonic and biharmonic functions to find solutions for boundary value problems (BVPs). The RKHS approach is a valuable framework for creating numerical solutions in applied sciences. This theory has been successfully extended to a variety of important applications in numerical analysis, computational mathematics, image processing, machine learning, probability and statistics, and finance [2–5], and it has been shown to be very effective in different fields of integrative equations [6,7], integrative differential equations [8–12] and partial differential equations [13, 14], and others [15–19], especially when the derivative order is fractional.

Recently, a new definition in the fractional calculus has been introduced concerning the conformable fractional derivative. This concept is a natural extension of the first-order derivative, and it satisfies some of the properties which are lost in the other fractional definitions such as those of the derivative of product and quotient of two functions formulas, and the chain rule.

This paper is organized as follows. In Section 2, some basic definitions and concepts are presented. We construct the reproducing kernel Hilbert spaces and present the structure of the analytical and approximate solutions in Section 3. In Sections 4 and 5, the convergence and error estimator are discussed to provide a number of numerical results to demonstrate the efficiency and accuracy of the reproducing kernel Hilbert space method. Finally, in Section 6, a short conclusion is provided.

2 Preliminaries and Backgrounds

In this section, we present some concepts and meanings of the conformable fractional derivative and the critical RKHS materials that will be used in this study.

Definition 2.1 Let the function $f : [0, \infty) \longrightarrow \mathbb{R}$, then the conformable fractional derivative of f of order $0 < \alpha \leq 1$ is given by

$$(\mathcal{D}^{\alpha} f)(x) = \lim_{\epsilon \to 0} \frac{f(x + \epsilon x^{1-\alpha}) - f(x)}{\epsilon}.$$

Moreover, if f is α-differentiable in some $(0, \alpha)$ and $\lim_{x \to 0} f^{(\alpha)}(x)$ exists, then $f^{(\alpha)}(0) = \lim_{x \to 0^+} f^{(\alpha)}(x)$.

Definition 2.2 Let $a \in (0, 1)$ and $f : [0, \infty) \longrightarrow \mathbb{R}$ be an $\alpha-$fractional integral function, then the "conformable fractional integral" of f is given by

$$\mathcal{I}_a^{\alpha}(f)(x) = \mathcal{I}_a^1(x^{\alpha-1} f) = \int_a^x \frac{f(t)}{t^{1-\alpha}} dt,$$

where the integral is the usual Riemann integral.

Theorem 2.1 *For $x \geq a$ and f being any continuous function in the domain of $\mathcal{I}^\alpha$, we have*

$$\mathcal{D}^\alpha(\mathcal{I}_a^\alpha f)(x) = f(x).$$

Proof. See [20].

Theorem 2.2 *If f is differentiable and $\alpha \in (0,1]$, then for $\forall x > a$, we have*

$$\mathcal{I}_a^\alpha \mathcal{D}_a^\alpha f(x) = f(x) - f(a).$$

Definition 2.3 Let $\mathcal{X}$ be a nonempty set, then the function $\mathcal{K} : \mathcal{X} \times \mathcal{X} \longrightarrow \mathbb{C}$ is a reproducing kernel of the Hilbert space $\mathcal{H}$ if and only if

1. $\mathcal{K}(.,x) \in \mathcal{X}, \forall x \in \mathcal{X}$,
2. $\forall x \in \mathcal{X}, \forall u \in \mathcal{H} : \langle u(.), \mathcal{K}(.,x)\rangle = u(x)$.

Here, the second condition is "the reproducing property"; the value of the function u at the point x is reproduced by the inner product of u with $\mathcal{K}(.,x)$. For instance, the reproducing kernel is unique, symmetric and positive definite.

Definition 2.4 The space $\Pi_2^m[a,b]$ is defined by

$$\Pi_2^m[a,b] = \left\{u \mid u^{(i)} \text{ are absolutely continuous, } i = 1,2,...,m-1 \text{ and } u^{(m)} \in L^2[a,b]\right\}.$$

The inner product and the norm of $\Pi_2^m[a,b]$ are given by

$$\langle u,v\rangle_{\Pi_2^m} = \textstyle\sum_{i=0}^{m-1} u^{(i)}(a)v^{(i)}(a) + \int_a^b u^{(m)}(t)v^{(m)}(t)dt$$

with

$$\|u\|_{\Pi_2^m} = \sqrt{\langle u,u\rangle_{\Pi_2^m}}.$$

3 Statement and Solution of the Problem

Consider the general HNN problem as follows:

$$\begin{cases} \mathcal{D}^\alpha X(\tau) &= -X(\tau) + WF(\tau,X), \\ X(0) &= A_0, \end{cases} \tag{1}$$

where $X = (x_1(\tau), x_2(\tau), \cdots, x_n(\tau))^T \in \mathbb{R}^n$, $W = (w_{ij}) \in M_{n\times n}$, $F = (f_1(X), f_2(X), \cdots, f_n(X))$, and $A_0 = (a_1, a_2, \cdots, a_n)$ such that a^η, $\eta = 1,...,n$, are constants.

$\mathcal{D}^\alpha$ represents the conformable fractional derivative, and $\tau \in T = [a,b]$, f_η are non-linear (generally non-linear) continuous functions. To find the approximate solutions of the problem (1), we utilize the RKHS algorithm over the long interval $T = [a,b]$.
Consider the spaces $\Pi_2^1[a,b]$, $\Pi_2^2[a,b]$ which are defined, respectively, by

$$\Pi_2^1[a,b] = \left\{u \mid u \text{ is absolutely continuous, and } u' \in L^2[a,b]\right\},$$

$$\Pi_2^2[a,b] = \left\{u \mid u,\ u' \text{ are absolutely continuous, and } u,\ u', u'' \in L^2[a,b],\ u(a) = 0\right\}.$$

We give the inner product and the norm in the above spaces, respectively, as follows:

$$\begin{cases} \langle u, v\rangle_{\Pi_2^1} = u(a)v(a) + \int_a^b u'(s)v'(s)ds, \\ \|u\|_{\Pi_2^1} = \sqrt{\langle u, u\rangle_{\Pi_2^1}}, \\ \langle u, v\rangle_{\Pi_2^2} = u(a)v(a) + u'(a)v'(a) + \int_a^b u''(s)v''(s)ds, \\ \|u\|_{\Pi_2^2} = \sqrt{\langle u, u\rangle_{\Pi_2^2}}. \end{cases}$$

Suppose that the interval $T = [a, b]$ is divided into M subintervals $[\gamma^{m-1}, \gamma^m],\ m = 1, ..., M$, of equal step size $z = \frac{b-a}{M}$, by using the nodes $\gamma^m = mz$. Firstly, we apply the RKHS approach over the interval $[a, \gamma^1]$ to find the numerical solution. For the subintervals $[\gamma^{m-1}, \gamma^m],\ m \geq 2$, we apply the RKHS method directly after using the initial conditions obtained over $[a, \gamma^1]$. Repeate the process and then generate a sequence of approximate solutions over $\{[a, \gamma^1], [\gamma^1, \gamma^2], ...[\gamma^{M-1}, \gamma^M]\}$.

3.1 Implementation of the process

The method steps can be summarized in the following points:

- Homogenize the initial conditions, and construct the space Π_2^2 in which each function satisfies the homogenous initial conditions of (1), then use the space Π_2^1.
- Take $\mathcal{K}$ and $\mathcal{R}$ as the reproducing kernel functions of the spaces Π_2^2 and Π_2^1, respectively, which are defined by

$$\mathcal{R}_\tau(t) = \frac{1}{\sinh(b-a)}[\cosh(t + \tau - b - a) + \cosh|t - \tau| - b - a],$$

  and

$$\mathcal{K}_\tau(t) = \frac{1}{6}\begin{cases} (\tau - a)(2a^2 - \tau^2 + 3t(2+\tau) - a(6 + 3t + \tau))\ \tau \leq t, \\ (t - a)(2a^2 - t^2 + 3\tau(2+t) - a(6 + 3\tau + t))\ \tau > t, \end{cases}$$

  and define the linear bounded operator $L : \Pi_2^2 \to \Pi_2^1$ such that $LX(\tau) = \mathcal{D}^\alpha X(\tau)$.
- Apply the conformable fractional integral to both sides and using $X(a) = 0$, we get

$$\begin{cases} X(\tau) = \mathcal{F}(\tau, X), \\ X(a) = 0. \end{cases} \tag{2}$$

- Choose a countable dense set $\{t_i\}_{i=1}^{\infty}$ from $[a, b]$ for the space $\Pi_2^2[a, b]$, and so define the complete system $\psi_i(\tau) = L^*\phi_i(\tau)$, where $\phi_i(\tau) = \mathcal{R}_{t_i}(\tau)$, and L^* is the adjoint operator of L.
- Derive an orthonormal functions system $\{\overline{\psi}_i(\tau)\}_{i=1}^{\infty}$ of the space $\Pi_2^2[a, b]$ from the Gram-Schmidt orthogonalization process of $\{\psi_i(\tau)\}_{i=1}^{\infty}$ as follows:

$$\overline{\psi}_i(\tau) = \sum_{k=1}^{i} B_{ik}\psi_k(\tau),\ i = 1, 2, \cdots,$$

such that B_{ij} are positive orthogonalization coefficents which are given by

$$B_{11} = \frac{1}{\|\Psi_1\|}, \ B_{ii} = \frac{1}{\sqrt{\|\Psi_i\|^2 - \sum_{k=1}^{i-1}(C_{ik})^2}}, \ B_{ij} = \frac{-\sum_{k=1}^{i-1} C_{ik}B_{kj}}{\sqrt{\|\Psi_i\|^2 - \sum_{k=1}^{i-1}(C_{ik})^2}}, \ j < i,$$

where $C_{ik} = \langle \Psi_i, \Psi_k \rangle_{\mathcal{W}_2^2}$.

Lemma 3.1 *$\psi_i(\tau)$ can be written as follows:*

$$\psi_i(\tau) = L_t \mathcal{K}_t(\tau) \mid_{\tau=t_i}.$$

Proof. See [19].

Theorem 3.1 *If L is an invertible operator, and if $\{t_i\}_{i=1}^{\infty}$ is dense on $[a, b]$, then $\{\psi_i(\tau)\}_{i=1}^{\infty}$ is the complete function system of the space $\Pi_2^2[a, b]$.*

Proof. See [19].

Theorem 3.2 *For every $X(\tau) \in \Pi_2^2[a, b]$, the series $\sum_{i=0}^{\infty} \left\langle X(\tau), \overline{\psi}_i(\tau) \right\rangle_{\pi_2^2} \overline{\psi}_i(\tau)$ are convergent in the sense of $\|.\|_{\Pi_2^2[a,b]}$. And if $\{t_i\}_{i=1}^{\infty}$ is a dense subset on $[a, b]$, then the solutions of (1) are given by*

$$X(\tau) = \sum_{i=1}^{\infty} \sum_{k=1}^{i} B_{ik} \mathcal{F}(\tau_k, X(\tau_k)) \overline{\psi}_i(\tau). \tag{3}$$

Proof. The steps of the proof are detailed in [19]. The next algorithm explains the implementation of the procedure for solving a system of differential equations in numerical form in terms of their network nodes based on the RKHS method.

3.2 Existence and uniqueness

Let the continuous function $F = (f_1, f_2, ..., f_n)$ and the Banach space $E = \{(\tau, X) \in \mathbb{R} \times \mathbb{R}^n | \tau \in J, \ X \in B\}$, where

$$\begin{array}{rcl} J & = & [0, T], \\ B & = & \{X \in \mathbb{R}^n | \ \|X - X_0\| \leq b\}. \end{array}$$

Suppose that F satisfies the Lipchitz condition, i.e., $\forall \ (\tau, X), \ (\tau, Y) \in E, \ \exists K \geq 0$ such that

$$\|F(\tau, X) - F(\tau, Y)\|_2 \leq K \|(\tau, X) - (\tau, Y)\|_2.$$

Theorem 3.3 *If F satisfies the following conditions:*

1. *F satisfies the Lipchitz condition.*
2. *$\|F(\tau, X(\tau))\|_2 \leq C$.*
3. *$T^\alpha \leq \min\left\{a^\alpha + \frac{\alpha b}{b + C\|W\|_2}, a^\alpha + \frac{\alpha}{2b(1 + K\|W\|_2)}\right\}$,*

then the system (1) has a unique solution on $[0, T]$.

Proof. For all $(\tau, X) \in E$, and by applying the conformable fractional integral to both sides of system (1), we get

$$X(\tau) = \int_a^\tau \frac{-X(s) + WF(s, X(s))}{s^{1-\alpha}} ds = \varphi(\tau, X(\tau)).$$

We need to show that $\varphi(\tau, X(\tau))$ is a map from E to E. Let $(\tau, X(\tau)) \in E$, for all $\tau \leq T$, we have

$$\begin{array}{rcl} \|\varphi(\tau, X(\tau)) - \varphi(\tau_0, X_0(\tau))\|_2 & = & \|\varphi(\tau, X(\tau))\|_2 = \left\| \int_a^\tau \frac{-X(s)+WF(s,X(s))}{s^{1-\alpha}} ds \right\|_2 \\ & \leq & [\|X\|_2 + \|W\|_2 \cdot \|F\|_2] \left(\frac{T^\alpha - a^\alpha}{\alpha}\right) \\ & \leq & [\|X\|_2 + C\|W\|_2] \left(\frac{T^\alpha - a^\alpha}{\alpha}\right) \end{array}$$

since $T^\alpha \leq a^\alpha + \frac{\alpha b}{b + C\|W\|_2}$ and $\|F\|_2 \leq C$, so $\|\varphi(\tau, X(\tau))\|_2 \leq b$. Then $\varphi(\tau, X(\tau))$ is a map from E to E.

Now, we prove that $\varphi(\tau, X(\tau))$ is a contraction. Let $(\tau, X(\tau)),\ (\tau, Y(\tau)) \in E$, we have

$$\begin{array}{rcl} \|\varphi(\tau, X(\tau)) - \varphi(\tau, Y(\tau))\|_2 & = & \left(\frac{T^\alpha - a^\alpha}{\alpha}\right) \|X - Y\|_2 + \left(\frac{T^\alpha - a^\alpha}{\alpha}\right) \|W\|_2 \|F(\tau, X(\tau)) - F(\tau, Y(\tau))\|_2 \\ & \leq & \left(\frac{T^\alpha - a^\alpha}{\alpha}\right) \|X - Y\|_2 + K\left(\frac{T^\alpha - a^\alpha}{\alpha}\right) \|W\|_2 \|X - Y\|_2 \\ & = & \left(\frac{T^\alpha - a^\alpha}{\alpha}\right) \|X - Y\|_2 \left[1 + K\|W\|_2\right] \end{array}$$

since $T^\alpha \leq \frac{\alpha}{2b(1 + K\|W\|_2)} + a^\alpha$, then $\left(\frac{T^\alpha - a^\alpha}{\alpha}\right)[1 + K\|W\|_2] \leq \frac{1}{2}$, so $\varphi(\tau, X(\tau))$ is a contraction. Thus, by the Banach fixed point theorem, there is a unique fixed point, then the solution is unique on $[0, T]$.

Algorithm 3.1 *Use the following stages to approximate the solutions of the problem (1) based on the RKHS method.*
Input: *The endpoints of* $[a, b]$, *the unit truth interval* $[a, b]$, *the integers* n *and* m, *the kernel function* $\mathcal{K}_t(\tau)$, *the differential operator* L , *the initial condition* A_0 , *and the function* F.
Output: *Approximate solution* $X_n(\tau)$.

- ***Stage A:*** *Fixed* $\tau \in [a, b]$ *and set* $t \in [a, b]$
  for $i = 1, ..., n$ *do*
  - ***stage 1:*** *set* $t_i = a + \frac{i-1}{n-1}$;
  - ***stage 1:*** *if* $t \leq \tau$ *let*
    $$\mathcal{K}_\tau(t) = \frac{1}{6}(t - a)(2a^2 - t^2 + 6\tau + 3t\tau - a(6 + t + 3\tau));$$
    else let
    $$\mathcal{K}_\tau(t) = \frac{1}{6}(\tau - a)(2a^2 - \tau^2 + 6t + 3\tau t - a(6 + \tau + 3t)).$$
  - ***stage 2:*** *For* $i = 1, ..., n$ *do*
    set

$$\psi_i(\tau) = L_t\mathcal{K}_\tau(t)|_{\tau=t_i}.$$

Output the orthogonal functions system $\psi_i(\tau)$.

- ***Stage B:*** *Obtain the orthogonalization coefficients B_{ij} as follows:*

  For $i = 1, ..., n$;
  For $j = 1, ..., i$ *set* $C_{ik} = \langle\psi_i, \psi_j\rangle_{\Pi_2^2}$ *and* $B_{11} = \frac{1}{Sqrt(C_{11})}$.
  Output C_{ij} and B_{11}.

- ***Stage C:*** *For* $i = 1, ..., n$, *set* $B_{ii} = (\|\psi_i\|^2_{\Pi_2^2} - \sum_{k=1}^{i-1}(C_{ik})^2)^{\frac{-1}{2}}$;
  else if $j \neq i$ *set* $B_{ij} = -(\sum_{k=1}^{i-1} C_{ik}B_{kj}).(\|\psi_i\|^2_{\Pi_2^2} - \sum_{k=1}^{i-1}(C_{ik})^2)^{\frac{-1}{2}}$.
  Output the orthogonalization coefficients B_{ij}.

- ***Stage D:*** *For* $i = 1, ..., n$ *set* $\overline{\psi}_i(t) = \sum_{k=1}^{i} B_{ik}\psi_i(t)$.
  Output the orthonormal functions system $\overline{\psi}_i(t)$.

- ***Stage E:*** *Set* $\tau_1 = 0$ *and choose* $X_0(\tau_1) = 0$;
  For $i = 1, ..., n$; *set*

$$\lambda_i = \sum_{k=1}^{i} B_{nk}\mathcal{F}(\tau_k, X_{k-1}(\tau_k))$$

set

$$X_n(\tau) = \sum_{i=1}^{n} \lambda_i\overline{\psi}_i(\tau).$$

Outcome the numerical solutions $X_n(\tau)$.

4 Numerical Experiments

Example 4.1 Consider the following HNN system:

$$\left\{\begin{array}{rcl} \mathcal{D}^\alpha X & = & -X + WF(X), \\ X(0) & = & [-0.109, -0.832, 1.721]^T, \end{array}\right. \tag{4}$$

where

$$W = \begin{pmatrix} 2 & -1.2 & 0 \\ 2 & 1.7 & 1.15 \\ -4.75 & 0 & 1.1 \end{pmatrix}, \quad \text{and } F = \tanh(X).$$

Using the RKHS method for solving the system (4) and taking $n = 500$, we get the following results.

Example 4.2 Consider the following system:

$$\left\{\begin{array}{rcl} \mathcal{D}^\alpha X & = & -X + WF(X), \\ X(0) & = & [1.048, -0.2233, -0.3150]^T, \end{array}\right. \tag{5}$$

where

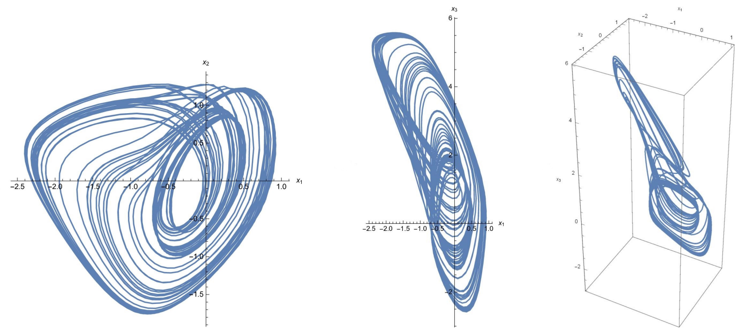


Figure 1: Chaotic attractors for $\alpha = 1$.

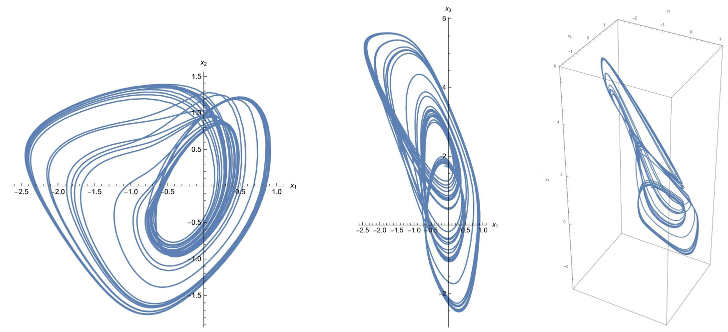


Figure 2: Chaotic attractors for $\alpha = 0.9$.

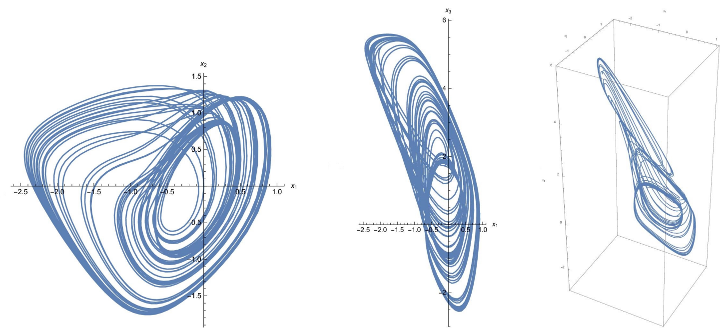
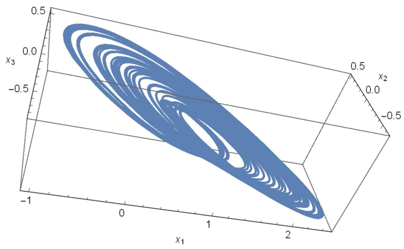


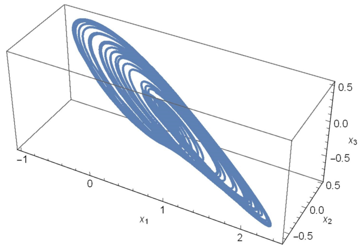
Figure 3: Chaotic attractors for $\alpha = 0.95$.

$$W = \begin{pmatrix} -1.4 & 1.2 & -7 \\ 1.1 & 0 & 2.8 \\ -k & -2 & 4 \end{pmatrix}, \quad \text{and } F = \tanh(X).$$

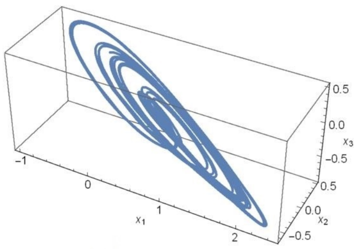
Using the RKHS method for solving the system (5) and taking $n = 250$, we get the following results.



(a)



(b)



(c)

Figure 4: Chaotic attractors of HNN for Example 4.2 such that: (a): $\alpha = 0.95,\ k = 0.95/$ (b): $\alpha = 0.85,\ k = 0.95/$ and (c): $\alpha = 0.75,\ k = 0.95$.

Example 4.3 Consider the following HNN system:

$$\begin{cases} \mathcal{D}^\alpha X &= -X + WF(X), \\ X(0) &= [0.0225, 0.1788, -4.831]^T, \end{cases} \tag{6}$$

where

$$W = \begin{pmatrix} 3.4 & -1.6 & 0.7 \\ 2.5 & 0 & 0.95 \\ k & 0.5 & 0 \end{pmatrix}, \quad \text{and } F = \tanh(X).$$

Using the RKHS method for solving the system (6) and taking $n = 300$, we get the following results.

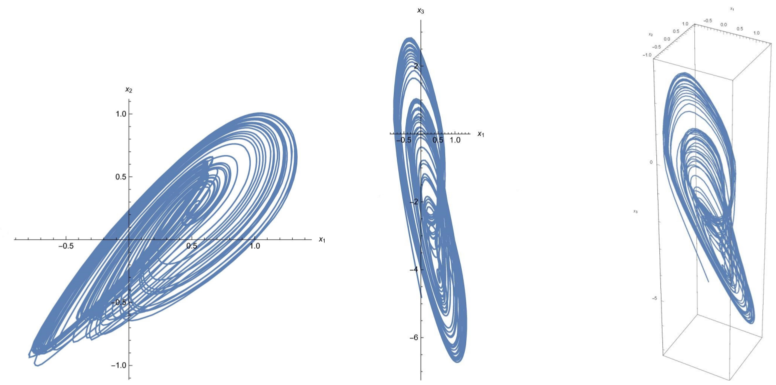


Figure 5: Chaotic attractors for $\alpha = 1$ and $k = -9$.

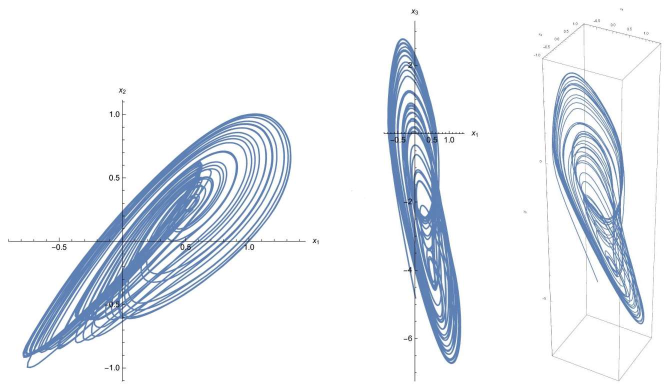


Figure 6: Chaotic attractors for $\alpha = 0.95$ and $k = -9$.

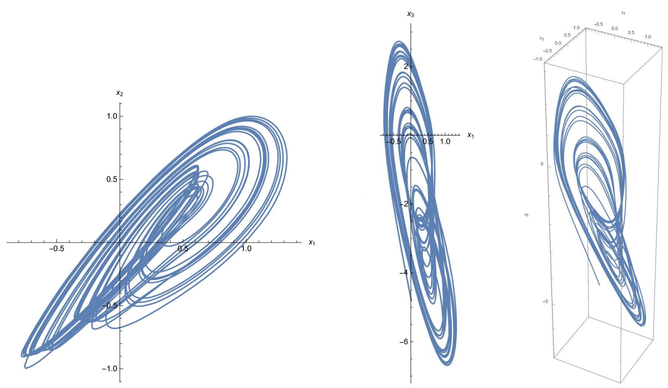


Figure 7: Chaotic attractors for $\alpha = 0.9$ and $k = -9$.

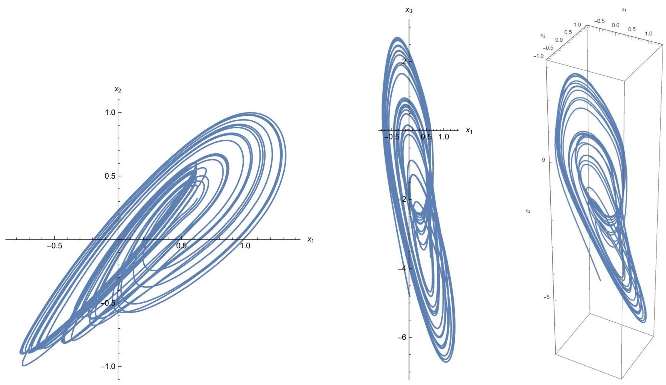


Figure 8: Chaotic attractors for $\alpha = 0.85$ and $k = -9$.

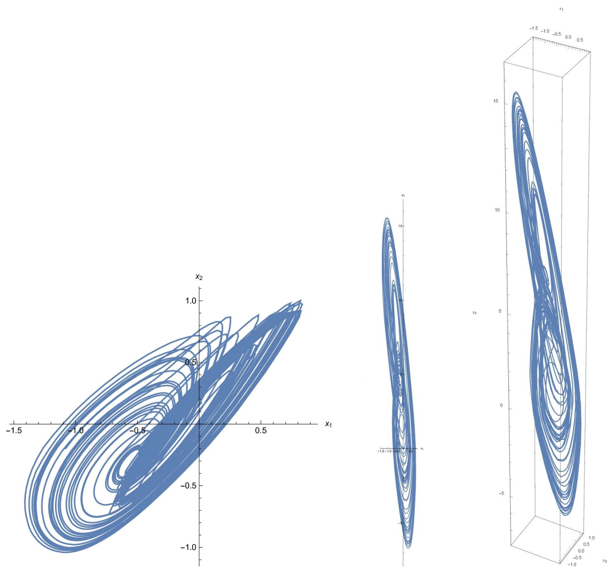


Figure 9: Chaotic attractors for $\alpha = 1$ and $k = -20$.

5 Conclusion

The Fractional Hopfield Neural Networks enhance complex nonlinear dynamics modeling by incorporating fractional calculus concepts, capturing long-term dependencies and memory effects more accurately, which is crucial for the investigation of real-world phenomena.

In this paper, we effectively used the RKHS method to solve numerical equation systems for the Hopfield Neural Network Equation Systems. We have demonstrated the accuracy and efficiency of the method in solving the systems of Hopfield neural network equations through the results we got in the numerical examples. In the future, we recommend that more research be done on the RKHS method for neural network systems. We also expect good results by solving the systems of Hopfield neural network equations with the Caputo and Atangana-Baleanu fractional derivatives.

References

[1] D. Kriesel. *A Brief Introduction to Neural Networks, chapter eight: Hopfield Neural Networks*. Published online, 2005. [Germany]

[2] M. Cui and Y. Lin. *Nonlinear Numerical Analysis in the Reproducing Kernel Space*. Nova Science, New York, NY, USA, 2009.

[3] A. Berlinet and C. Thomas-Agnan. *Reproducing Kernel Hilbert Spaces in Probability and Statistics*. Kluwer Academic Publishers, New York, 2001.

[4] A. Daniel. *Reproducing Kernel Spaces and Applications*. Springer, Basel, Switzerland, 2003.

[5] H.L. Weinert. *Reproducing Kernel Hilbert Spaces: Applications in Statistical Signal Processing*. Hutchinson Ross, 1982.

[6] H. Beyrami, T. Lotfi and K. Mahdiani. Stability and error analysis of the reproducing kernel Hilbert space method for the solution of weakly singular Volterra integral equation on graded mesh. *Applied Numerical Mathematics.* **120** (2017) 197–214.

[7] S. F. Javan, S. Abbasbandy and M. A. F. Araghi. *Application of Reproducing Kernel Hilbert Space Method for Solving a Class of Nonlinear Integral Equations*. Mathematical Problems in Engineering, 2017.

[8] O. A. Arqub, M. Al-Smadi and N. Shawagfeh. Solving Fredholm integro-differential equations using reproducing kernel Hilbert space method. *Applied Mathematics and Computation* **219** (17) (2013) 8938–8948.

[9] M. Al-Smadi, O. Abu Arqub and S. Momani. *A computational method for two-point boundary value problems of fourth-order mixed integrodifferential equations*. Mathematical Problems in Engineering, 2013.

[10] S. Bushnaq, B. Maayah and M. Ahmad. Reproducing kernel hilbert space method for solving fredholm integro-differential equations of fractional order. *Italian Journal of Pure and Applied Mathematics* **36** (2016) 307–318.

[11] A. AlHabees, B. Maayah and S. Bushnaq. Solving fractional proportional delay integro-dierential equations of first order by reproducing kernel hilbert space method. *Global Journal of Pure and Applied Mathematics* **12** (4) (2016) 3499–3516.

[12] S. Bushnaq, B. Maayah, S. Momani and A. Alsaedi. *A reproducing kernel Hilbert space method for solving systems of fractional integrodifferential equations*. Abstract and Applied Analysis, 2014.

[13] O. Abu Arqub. Fitted reproducing kernel Hilbert space method for the solutions of some certain classes of time-fractional partial differential equations subject to initial and Neumann boundary conditions. *Computers and Mathematics with Applications* **73** (6)(2017) 1243–1261.

[14] M. Klimek. Stationarity-conservation laws for fractional differential equations with variable coefficients. *Journal of Physics A: Mathematical and General* **35** (31)(2002) 6675–6693.

[15] Z. Altawallbeh, M. H. AL-Smadi and R. Abu-Gdairi. Approximate solution of second-order integro-differential equation of Volterra type in RKHS method. *International Journal of Mathematical Analysis* **7** (2013) 2145–2160.

[16] B. Maayah, S. Bushnaq, M. Ahmad and S. Momani. Computational method for solving nonlinear voltera integro-differential equations. *Journal of Computational and Theoretical Nanoscience* **13** (11) (2016) 7802–7806.

[17] O. Abu Arqub. *Reproducing kernel algorithm for the analytical-numerical solutions of nonlinear systems of singular periodic boundary value problems.* Mathematical Problems in Engineering, 2015.

[18] Z. Altawallbeh, M. Al-Smadi, I. Komashynska and A. Ateiwi. Numerical Solutions of Fractional Systems of Two-Point BVPs by Using the Iterative Reproducing Kernel Algorithm. *Ukrainian Mathematical Journal* **70** (5)(2018) 687–701.

[19] Y. Chellouf, B. Maayah, S. Momani, A. Alawneh and S. Alnabulsi. Numerical solution of fractional differential equations with temporal two-point bvps using reproducing kernal Hilbert space method. *AIMS Mathematics* **6** (4)(2021) 3465–3485.

[20] R. Khalil, M. Al Horani, A. Yousef and M. Sababheh. A new definition of fractional derivative. *Journal of Computational and Applied Mathematics* **264** (2014) 65–70.

[21] T. Abdeljawad. On conformable fractional calculus. *Journal of Computational and Applied Mathematics* **279** (2015) 57–66.

[22] N. Allouch, S. Hamani and J. Henderson. Boundary Value Problem for Fractional q-Difference Equations. *Nonlinear Dynamics and Systems Theory* **24** (2) (2024) 111–122.

[23] M. Abu Hammad, S. Alshorm, S. Rasem and L. Abed. Conformable Fractional Inverse Gamma Distribution. *Nonlinear Dynamics and Systems Theory* **24** (2) (2024) 159–167.

[24] M. Moumni and M. Tilioua. A Neural Network Approximation for a Model of Micromagnetism. *Nonlinear Dynamics and Systems Theory* **22** (4) (2022) 432–446.