Nonlinear Dynamics and Systems Theory, **24** (5) (2024) 473–484

A Dynamic Problem with Wear Involving Thermoviscoelastic Materials with a Long Memory

Chahira Guenoune, Aziza Bachmar * and Souraya Boutechebak

Department of Mathematics, Setif 1 University, 19000 Setif, Algeria.

Received: February 2, 2024; Revised: October 9, 2024

Abstract: We consider a dynamic contact problem with friction in thermo-viscoelasticity with long memory body. The body is in contact with an obstacle. The contact is frictional and bilateral with a moving rigid foundation which results in the wear of the contacting surface. The problem is formulated as a coupled system of an elliptic variational inequality for the displacement and the heat equation for the temperature. We establish a variational formulation for the model and we prove the existence of a unique weak solution to the problem. The proof is based on a classical existence and uniqueness result for parabolic inequalities, differential equations and fixed point arguments.

Keywords: *frictional contact; thermo-visco-elastic; fixed point; dynamic process; variational inequality; wear.*

Mathematics Subject Classification (2010): 74M10, 74M15, 74F15, 49J40, 70k75, 93-10.

1 Introduction

Scientific research and recent papers in mechanics are articulated around two main components, one devoted to the laws of behavior and the other devoted to boundary conditions imposed on the body. The boundary conditions reflect the binding of the body with the outside world. Recent researches use coupled laws of behavior between mechanical and electric effects or between mechanical and thermal effects. For the case of coupled laws of behavior between mechanical and electric effects, general models can be found in [5,6]. For the case of coupled laws of behavior between mechanical and thermal effects, the transmission problem in thermo-viscoplasticity is studied in [3], the contact problem

* Corresponding author: mailto:aziza_bechmar@yahoo.fr

with adhesion for thermo-viscoplasticity is considered in [1]. Situations of contact between deformable bodies are very common in the industry and everyday life. Contact of braking pads with wheels, tires with roads, pistons with skirts or the complex metal forming processes are just a few examples. The constitutive laws with internal variables have been used in various publications in order to model the effect of internal variables in the behavior of real bodies like metals, rocks, polymers and so on, for which the rate of deformation depends on the internal variables. Some of the internal state variables considered by many authors are the spatial display of dislocation, the work-hardening of materials. In this paper, we consider a general model for the dynamic process of bilateral frictional contact between a deformable body and an obstacle, which results in the wear of the contacting surface. The material obeys a thermo-viscoelasticity constitutive law with long memory body. We derive a variational formulation of the problem which includes a variational second order evolution inequality. We establish the existence of a unique weak solution of the problem. The idea is to reduce the second order nonlinear evolution inequality of the system to the first order evolution inequality. After this, we use classical results for first order nonlinear evolution inequalities and equation, a parabolic variational inequality and the fixed point arguments.

The paper is structured as follows. In Section 2, we present the thermo-visco-elastic contact model with friction and provide comments on the contact boundary conditions. In Section 3, we list the assumptions on the data and derive the variational formulation. In Section 4, we present our main existence and uniqueness results.

2 Problem Statement

Problem P: Find a displacement field $u : \Omega\times[0.T] \to \mathbb{R}^d$, a stress field $\sigma : \Omega\times[0.T] \to \mathbb{S}^d$, the temperature $\theta : \Omega \times [0.T] \to \mathbb{R}_+$ and the wear $\omega : \Gamma_3 \times [0.T] \to \mathbb{R}_+$ such that

$$\begin{aligned} &\sigma = \mathcal{A}(\varepsilon(u(t))) + \mathcal{G}(\varepsilon(\dot{u}(t))) + \int_0^t \mathcal{B}(t-s)\varepsilon(u(s))ds - \theta(t)\mathcal{M}, \\ &\text{ in } \Omega \times [0.T], \end{aligned} \tag{1}$$

$$\rho\ddot{u} = Div\ \sigma + f_0, \quad \text{in } \Omega \times [0.T], \tag{2}$$

$$\dot{\theta} - div(K\nabla\theta) = -\mathcal{M}.\nabla\dot{u} + q, \ \text{in } \Omega \times [0.T], \tag{3}$$

$$u = 0, \text{on } \Gamma_1 \times [0.T], \tag{4}$$

$$\sigma\nu = h, \text{on } \Gamma_2 \ \times [0.T], \tag{5}$$

$$\begin{cases} \sigma_\nu = -\alpha\left|\dot{u}_\nu\right|, \ \ |\sigma_\tau| = -\mu\sigma_\nu, \\ \sigma_\tau = -\lambda\left(\dot{u}_\tau - v^*\right), \ \lambda \geq 0, \ \dot{\omega} = -kv^*\sigma_\nu, \ k > 0. \text{ on } \Gamma_3 \ \times [0.T], \end{cases} \tag{6}$$

$$-k_{ij}\frac{\partial\theta}{\partial\nu}\nu_j = k_e(\theta-\theta_R) - h_\tau(|\dot{u}_\tau|), \ \text{on}\Gamma_3 \times [0.T], \tag{7}$$

$$\theta = 0, \ \text{in } \Gamma_1 \cup \ \Gamma_2 \times [0.T], \tag{8}$$

$$u(0) = u_0, \dot{u}(0) = u_1, \theta(0) = \theta_0, \omega(0) = \omega_0, \ \text{in } \Omega, \tag{9}$$

where (1) is the thermo-visco-elastic constitutive law with long memory, we denote $\varepsilon(u)$ (respectively, $\mathcal{A}, \mathcal{G}, \xi, \xi^*$) the linearized strain tensor (respectively, the elasticity tensor, the viscosity nonlinear tensor, the third order piezoelectric tensor and its transpose), (2) represents the equation of motion, where ρ represents the mass density, we mention that $Div\sigma$ is the divergence operator, (3) represents the evolution equation of the heat field, (4) and (5) are the displacement and traction boundary conditions, (6) describes the bilateral frictional contact with wear described above on the potential contact surface Γ_3, (7) is the pointwise heat exchange condition on the contact surface, where k_{ij} are the components of the thermal conductivity tensor, v_j are the normal components of the outward unit normal v, k_e is the heat exchange coefficient, θ_R is the known temperature of the foundation. (8) represents the temperature boundary conditions. Finally, (9) represents the initial conditions.

3 Variational Formulation and Preliminaries

For a weak formulation of the problem, first, we introduce some notation. The indices i, j, k, l range from 1 to d and summation over repeated indices is implied. An index that follows the comma represents the partial derivative with respect to the corresponding component of the spatial variable, e.g., $u_{i.j} = \frac{\partial u_i}{\partial x_j}$. We also use the following notations:

$$\begin{gathered} H = \mathbb{L}^2(\Omega)^d = \{u = (u_i)/u_i \in \mathbb{L}^2(\Omega)\}, \\ \mathcal{H} = \sigma = (\sigma_{ij})/\sigma_{ij} = \sigma_{ji} \in \mathbb{L}^2(\Omega), \\ H_1 = u = (u_i)/\varepsilon(u) \in \mathcal{H} = H^1(\Omega)^d \\ \mathcal{H}_1 = \sigma \in \mathcal{H}/Div\sigma \in H. \end{gathered}$$

The operators of deformation ε and divergence Div are defined by

$$\varepsilon(u) = (\varepsilon_{ij}(u)), \varepsilon_{ij}(u) = \tfrac{1}{2}(u_{i,j} + u_{j,i}), Div\sigma = (\sigma_{ij,j}).$$

The spaces $H, \mathcal{H}, H_1$ and $\mathcal{H}_1$ are real Hilbert spaces endowed with the canonical inner products given by

$$\begin{gathered} (u,v)_H = \textstyle\int_\Omega u_i v_i dx, \forall u, v \in H, \\ (\sigma,\tau)_{\mathcal{H}} = \textstyle\int_\Omega \sigma_{ij}\tau_{ij} dx, \forall \sigma, \tau \in \mathcal{H}, \\ (u,v)_{H_1} = (u,v)_H + (\varepsilon(u), \varepsilon(v))_{\mathcal{H}}, \forall u, v \in H_1, \\ (\sigma,\tau)_{\mathcal{H}_1} = (\sigma,\tau)_{\mathcal{H}} + (Div\sigma, Div\tau)_H, \sigma, \tau \in \mathcal{H}_1. \end{gathered}$$

We denote by $|\cdot|_H$ (respectively, $|\cdot|_{\mathcal{H}}$, $|\cdot|_{H_1}$and $|\cdot|_{\mathcal{H}1}$) the associated norm on the space H (respectively, $\mathcal{H}$, H_1 and $\mathcal{H}_1$).

Let $H_\Gamma = (H^{1/2}(\Gamma))^d$ and $\gamma : H^1(\Gamma)^d \to H_\Gamma$ be the trace map. For every element $v \in (H^1(\Gamma))^d$, we also use the notation v to denote the trace map γv of v on Γ, and we denote by v_ν and v_τ the normal and tangential components of v on Γ given by

$$v_\nu = v.\nu, v_\tau = v - v_\nu.$$

Similarly, for a regular (say $\mathcal{C}^1$) tensor field $\sigma : \Omega \to \mathbb{S}^d$, we define its normal and tangential components by

$$\sigma_\nu = (\sigma\nu).\nu, \sigma_\tau = \sigma\nu - \sigma_\nu.$$

We use standard notation for the $\mathbb{L}^p$ and the Sobolev spaces associated with Ω and Γ and, for a function $\psi \in H^1(\Omega)$, we still write ψ to denote its trace on Γ. We recall that the summation convention applies to a repeated index.

When σ is a regular function, the following Green's type formula holds:

$$(\sigma, \varepsilon(v))_{\mathcal{H}} + (Div\sigma, v)_H = \int_\Gamma \sigma\nu.vda \quad \forall v \in H_1. \tag{10}$$

Next, we define the space

$$V = \{u \in H_1 / \ u = 0 \ \text{ on } \Gamma_1\}.$$

Since $meas \ (\Gamma_1) > 0$, the following Korn's inequality holds:

$$|\varepsilon(u)|_{\mathcal{H}} \geq c_K \, |v|_{H_1} \, \forall v \in V, \tag{11}$$

where $c_K > 0$ is a constant which depends only on Ω and Γ_1. On the space V, we use the inner product

$$(u, v)_V = (\varepsilon(u), \varepsilon(v))_{\mathcal{H}}, \tag{12}$$

let $|.|_V$ be the associated norm. It follows by (12) that the norms $|.|_{H_1}$ and $|.|_V$ are equivalent norms on V and therefore, $(V, \ |.|_V)$ is a real Hilbert space. Moreover, by the Sobolev trace theorem, there exists a constant c_0 depending only on the domain Ω, Γ_1 and Γ_3 such that

$$|v|_{\mathbb{L}^2(\Gamma_3)^d} \leq c_0 \, |v|_V \, \forall v \in V. \tag{13}$$

Finally, for a real Banach space $(X, \ |.|_X)$, we use the usual notation for the space $\mathbb{L}^p \, (0.T, X)$ and $W^{k.p} \, (0.T, X)$, where $1 \leq p \leq \infty, k = 1, 2......$; we also denote by $C \, (0.T, X)$ and $C^1 \, (0.T, X)$ the spaces of continuous and continuously differentiable function on $[0.T]$ with values in X, with the respective norms:

$$|x|_{C(0.T,X)} = \max_{t\in[0.T]} |x \, (t)|_X \ ,$$

$$|x|_{C^1(0.T,X)} = \max_{t\in[0.T]} |x \, (t)|_X + \max_{t\in[0.T]} \left|\dot{x} \, (t)\right|_X .$$

In what follows, we assume the following assumptions on the problem P. The elasticity operator $\mathcal{A} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d$ satisfies

$$\begin{cases} (a) \, \exists \ L_{\mathcal{A}} > 0 \text{ such that } : |\mathcal{A} \, (x, \varepsilon_1) - \mathcal{A} \, (x, \varepsilon_2)| \leq L_{\mathcal{A}} \, |\varepsilon_1 - \varepsilon_2| \\ \forall \ \varepsilon_1 \ , \varepsilon_2 \in \mathbb{S}^d, \ a. \ e. \ x \in \Omega, \\ (c) \ \text{ The mapping } x \to \mathcal{A} \, (x, \varepsilon) \ \text{ is Lebesgue measurable in } \Omega \text{ for all } \varepsilon \in \mathbb{S}^d, \\ (d) \ \text{ The mapping } x \to \mathcal{A} \, (x, 0) \in \mathcal{H}. \end{cases} \tag{14}$$

The viscosity operator $\mathcal{G} : \Omega \times \mathbb{S}^d \ \times \mathbb{S}^d \to \mathbb{S}^d$ satisfies

$$\begin{cases} (a) \ \exists \ L_{\mathcal{G}} > 0 : |\mathcal{G}(x, \varepsilon_1) - \mathcal{G}(x, \varepsilon_2)| \leq L_{\mathcal{G}} |\varepsilon_1 - \varepsilon_1|, \forall \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d, \ a.e. \ x \in \Omega, \\ (b) \ \exists m_{\mathcal{G}} > 0 : (\mathcal{G} \, (x, \varepsilon_1) - \mathcal{G} \, (x, \varepsilon_2) \, , \varepsilon_1 - \varepsilon_2) \geq m_{\mathcal{G}} \, |\varepsilon_1 - \varepsilon_2|^2 \, , \forall \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d, \\ (c) \text{ the mapping } x \to \mathcal{G} \, (x, \varepsilon) \ \text{ is Lebesgue measurable in } \Omega \text{ fo rall } \varepsilon \in \mathbb{S}^d, \\ (d) \text{ the mapping } x \mapsto \mathcal{G}(x, 0) \in \ \mathcal{H}. \end{cases} \tag{15}$$

The relaxation tensor $\mathcal{B} : [0.T] \times \Omega \times \mathbb{S}^d \to \mathbb{S}^d$ such that $(t, x, \tau) \mapsto (\mathcal{B}_{ijkh} \, (t, x) \, \tau_{kh})$ satisfies

$$\begin{cases} (a) \mathcal{B}_{ijkh} \in W^{1.\infty}(0.T, \mathbb{L}^\infty(\Omega)), \\ (b) \mathcal{B} \, (t) \, \sigma \cdot \tau = \sigma \cdot \mathcal{B} \, (t) \, \tau, \forall \sigma, \tau \in \mathbb{S}^d, \ p.p.t \in [0.T] \, , \ a.e. \text{in } \Omega. \end{cases} \tag{16}$$

The function $h_\tau : \Gamma_3 \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ satisfies

$$\begin{cases} (a) \ \exists \ L_\tau > 0 \ : \ |h_\tau(x, r_1) - \ h_\tau(x, r_2)| \leq L_h |r_1 - r_2| \quad \forall r_1, r_2 \in \mathbb{R}_+, \ \ a.e. x \in \Gamma_3, \\ (b) \ x \longrightarrow \ h_\tau \, (x, r) \in \mathbb{L}^2(\Gamma_3) \ \text{ is Lebesgue measurable in} \Gamma_3, \forall r \in \mathbb{R}_+. \end{cases} \tag{17}$$

The mass density ρ satisfies

$$\rho \in \mathbb{L}^\infty(\Omega) \text{ there exists } \rho^* > 0 \text{ such that } \rho(x) \geq \rho^*, a.e.x \in \Omega. \tag{18}$$

The body forces, surface tractions, the densities of electric charges, and the functions α and μ satisfy

$$\begin{cases} f_0 \in \mathbb{L}^2(0.T, H), h \in \mathbb{L}^2(0.T, \mathbb{L}^2(\Gamma_2)^d), \\ q \in W^{1.\infty}(0.T, \mathbb{L}^2(\Omega)), \theta_R \in W^{1.\infty}(0.T, \mathbb{L}^2(\Gamma_3)), k_e \in \mathbb{L}^\infty(\Omega, \mathbb{R}_+), \\ \mathcal{M} = (m_{ij}), m_{ij} = m_{ji} \in \mathbb{L}^\infty(\Omega), \\ \begin{cases} K = (k_{i,j});\ k_{ij} = k_{ji} \in \mathbb{L}^\infty(\Omega), \\ \forall c_k > 0, \forall (\xi_i) \in \mathbb{R}^d, k_{ij}\xi_i\xi_j \geq c_k\xi_i\xi_i. \end{cases} \\ \alpha \in \mathbb{L}^\infty(\Gamma_3), \alpha(x) \geq \alpha^* > 0, a.e. \text{on } \Gamma_3, \\ \mu \in \mathbb{L}^\infty(\Gamma_3), \mu(x) > 0, a.e. \text{on } \Gamma_3. \end{cases} \tag{19}$$

The initial data satisfies

$$u_0 \in V, u_1 \in \mathbb{L}^2(\Omega), \theta_0 \in \mathbb{L}^2(\Omega), \omega_0 \in \mathbb{L}^\infty(\Gamma_3). \tag{20}$$

We use a modified inner product on $H = \mathbb{L}^2(\Omega)^d$ given by

$$((u, v)) = (\rho u, v)_{\mathbb{L}^2(\Omega)^d}, \forall u.v \in H.$$

That is, it is weighted with ρ. We let H be with the associated norm

$$\|v\|_H = (\rho v, v)^{\frac{1}{2}}_{\mathbb{L}^2(\Omega)^d},\ \forall v \in H.$$

We use the notation $(.,.)_{V' \times V}$ to represent the duality pairing between V' and V . Then we have

$$(u, v)_{V' \times V} = ((u, v)), \forall u \in H, \forall v \in V.$$

It follows from assumption (18) that $\|.\|_H$ and $|.|_H$ are equivalent norms on H, and also, the inclusion mapping of $(V, |.|_V)$ into $(H, \|.\|_H)$ is continuous and dense. We denote by V' the dual space of V. Identifying H with its own dual, we can write the Gelfand triple $V \subset H = H' \subset V'$.

We define the space

$$E = \{\gamma \in H^1(\Omega)/\gamma = 0 \text{ on } \Gamma_1 \cup \Gamma_2\}. \tag{21}$$

We define the function $f(t) \in V$ by

$$(f(t), v)_V = \int_\Omega f_0(t) v dx + \int_{\Gamma_2} h(t) v da, \forall v \in V, t \in [0.T],$$

for all u, $v \in V$, $\psi \in W$ and $t \in [0.T]$, and note that condition (19) implies that

$$f \in \mathbb{L}^2(0.T, V'). \tag{22}$$

We consider the wear functional $j : V \times V \to \mathbb{R}$,

$$j(u, v) = \int_{\Gamma_3} \alpha\, |u_\nu|\, (\mu\, |v_\tau - v^*|) da. \tag{23}$$

Finally, we consider $\phi : V \times V \to \mathbb{R}$,

$$\phi(u,v) = \int_{\Gamma_3} \alpha \, |u_\nu| \, v_\nu da, \forall v \in V. \tag{24}$$

We define for all $\varepsilon > 0$,

$$j_\varepsilon(g,v) = \int_{\Gamma_3} \alpha \, |g_\nu| \, (\mu\sqrt{|v_\tau - v^*|^2 + \varepsilon^2}) da, \forall v \in V.$$

We define $Q : [0,T] \to E^{'}$; $K : E \to E^{'}$ and $R : V \to E^{'}$ by

$$(Q(t),\mu)_{E' \times E} = \int_{\Gamma_3} k_e \theta_R(t) \mu ds + \int_\Omega q\mu dx, \forall \mu \in E, \tag{25}$$

$$(K\tau,\mu)_{E' \times E} = \sum_{i,j=1}^{d} \int_\Omega k_{ij} \frac{\partial \tau}{\partial x_j} \frac{\partial \mu}{\partial x_i} dx + \int_{\Gamma_3} k_e \tau \mu ds, \forall \mu \in E, \tag{26}$$

$$(Rv,\mu)_{E' \times E} = \int_{\Gamma_3} h_\tau(|v_\tau|) \mu dx - \int_\Omega (\mathcal{M}.\nabla v) \mu dx, \forall v \in V, \tau, \mu \in E. \tag{27}$$

Using the above notation and Green's formula, we derive the following variational formulation of mechanical problem P.

Problem PV : Find a displacement field $u : \Omega \times [0.T] \to V$, a stress field $\sigma : \Omega \times [0.T] \to \mathbb{S}^d$, the temperature $\theta : \Omega \times [0.T] \to \mathbb{R}_+$ and the wear $\omega : \Gamma_3 \times [0.T] \to \mathbb{R}_+$ such that

$$\begin{aligned}&(\ddot{u}(t), w - \dot{u}(t))_{V' \times V} + (\sigma(t), \varepsilon(w - \dot{u}(t)))_{\mathcal{H}} + j(\dot{u}, w) - j(\dot{u}, \dot{u}(t)) + \\ &\phi(\dot{u}, w) - \phi(\dot{u}, \dot{u}(t)) \geq \big(f(t), w - \dot{u}(t)\big), \ \forall u, w \in V,\end{aligned} \tag{28}$$

$$\dot{\theta}(t) + K\theta(t) = R\dot{u}(t) + Q(t), \text{ on } E' \tag{29}$$

$$\dot{\omega} = -kv^* \sigma_\nu. \tag{30}$$

4 Existence and Uniqueness Result

Our main result which states the unique solvability of Problem PV is the following.

Theorem 4.1 *Let the assumptions* (14) − (20) *hold. Then Problem PV has a unique solution* $(u, \sigma, \theta, \omega)$ *which satisfies*

$$u \in C^1(0.T, H) \cap W^{1.2}(0.T, V) \cap W^{2.2}(0.T, V'), \tag{31}$$

$$\sigma \in \mathbb{L}^2(0.T, \mathcal{H}_1), Div\sigma \in \mathbb{L}^2(0.T, V'), \tag{32}$$

$$\theta \in W^{1.2}(0.T, E') \cap \mathbb{L}^2(0.T, E) \cap C(0.T, \mathbb{L}^2(\Omega)), \tag{33}$$

$$\omega \in C^1(0.T, \mathbb{L}^2(\Gamma_3)). \tag{34}$$

We conclude that under the assumptions (14) − (20), the mechanical problem (1) − (9) has a unique weak solution with the regularity (31) − (34). The proof of this theorem will be carried out in several steps. It is based on the arguments of first order nonlinear evolution inequalities, evolution equations, a parabolic variational inequality, and fixed point arguments.

First step: Let $g \in \mathbb{L}^2(0.T, V)$ and $\eta \in \mathbb{L}^2(0.T, V')$ be given, we deduce a variational formulation of Problem PV.

Problem $PV_{g\eta}$: Find a displacement field $u_{g\eta} : [0.T] \to V$ such that

$$\begin{cases}(\ddot{u}_{g\eta}(t), w - \dot{u}_{g\eta}(t))_{V' \times V} + (\mathcal{G}\varepsilon(\dot{u}_{g\eta}(t)), w - \dot{u}_{g\eta}(t))_{V' \times V} + (\eta(t), w - \dot{u}_{g\eta})_{V' \times V} \\ j(g, w) - j(g, \dot{u}_{g\eta}(t)) \geq (f(t), w - \dot{u}_{g\eta}(t))_{V' \times V}, \ \forall \, w \in V, t \in [0.T], \\ u_{g\eta}(0) = u_0, \dot{u}_{g\eta}(0) = u_1.\end{cases} \tag{35}$$

We define $f_\eta(t) \in V$ for $a.e.t \in [0.T]$ by

$$(f_\eta(t), w)_{V' \times V} = (f(t) - \eta(t), w)_{V' \times V}, \forall w \in V. \tag{36}$$

From (22), we deduce that

$$f_\eta \in \mathbb{L}^2(0.T, V'). \tag{37}$$

Let now $u_{g\eta} : [0.T] \to V$ be the function defined by

$$u_{g\eta}(t) = \int_0^t v_{g\eta}(s)ds + u_0, \forall t \in [0.T]. \tag{38}$$

We define the operator $G : V' \to V$ by

$$(Gv, w)_{V' \times V} = (\mathcal{G}\varepsilon(v(t)), \varepsilon(w))_{\mathcal{H}}, \forall v, w \in V. \tag{39}$$

Lemma 4.1 *For all $g \in \mathbb{L}^2(0.T, V)$ and $\eta \in \mathbb{L}^2(0.T, V')$, $PV_{g\eta}$ has a unique solution with the regularity*

$$v_{g\eta} \in C(0.T, H) \cap \mathbb{L}^2(0.T, V) \textit{ and } \quad \dot{v}_{g\eta} \in \mathbb{L}^2(0.T, V'). \tag{40}$$

Proof. The proof from (35) nonlinear second order evolution inequalities is given in [2, 4, 7, 8].

In the second step, we use the displacement field $u_{g\eta}$ to consider the following variational problem.

Second step: We use the displacement field $u_{g\eta}$ to consider the following variational problem.

Problem $P_{\theta g\eta}$: Find $\theta_{g\eta} \in E$ such that

$$\dot{\theta}_{g\eta}(t) + K\theta_{g\eta}(t) = R\dot{u}_{g\eta}(t) + Q(t), \text{ on } E'. \tag{41}$$

Lemma 4.2 *Under the assumptions* $(14) - (20)$, *the problem $P_{\theta g\eta}$ has a unique solution*

$$\theta_{g\eta} \in W^{1.2}(0.T, E') \cap \mathbb{L}^2(0.T, E) \cap C(0.T, \mathbb{L}^2(\Omega)).$$

Proof. Since we have the Gelfand triple $E \subset \mathbb{L}^2(\Omega) \subset E'$, we use a classical result for first order evolution equations given in [9] to prove the unique solvability of (41). Now, we have $\theta_0 \in \mathbb{L}^2(\Omega)$. The operator K is linear and continuous, so $a(\tau, \mu) = (K\tau, \mu)_{E' \times E}$ is bilinear, continuous and coercive, we use the continuity of $a(.,.)$ and from (19), we deduce that

$$a(\tau, \mu) = (K\tau, \mu)_{E' \times E} \leq |k|_{\mathbb{L}^\infty(\Omega)^{d \times d}} |\nabla\tau|_E |\nabla\mu|_E + |k_e|_{\mathbb{L}^\infty(\Gamma_3)} |\tau|_{\mathbb{L}^2(\Gamma_3)} |\mu|_{\mathbb{L}^2(\Gamma_3)}$$
$$\leq C |\tau|_E |\mu|_E .$$

We have

$$a(\tau, \tau) = (K\tau, \tau)_{E' \times E} = \sum_{i,j=1}^{d} \int_\Omega k_{ij} \frac{\partial\tau}{\partial x_j} \frac{\partial\tau}{\partial x_i} dx + \int_{\Gamma_3} k_e \tau^2 ds.$$

By(19), there exists a constant $C > 0$ such that

$$(K\tau, \tau)_{E' \times E} \geq C\, |\tau|_E^2 .$$

We have $\theta_0 \in \mathbb{L}^2(\Omega)$. Let

$$F(t) \in E' : \ (F(t), \tau)_{E' \times E} = (R\dot{u}_{g\eta}(t) + Q(t), \tau) \quad \forall \tau \in E.$$

Under the assumptions (17), (19), we have

$$\int_0^T |R\dot{u}|_{E'}^2 \, dt < \infty, \quad \int_0^T |Q(t)|_{\mathbb{E}'}^2 \, dt < \infty, \quad \int_0^T |F|_{E'}^2 \, dt < \infty.$$

We find

$$F \in \mathbb{L}^2(0.T, E^{'}).$$

By a classical result for first order evolution equations,

$$\exists! \theta_{g\eta} \in W^{1.2}(0.T, E^{'}) \cap \mathbb{L}^2(0.T, E) \cap C(0.T, \mathbb{L}^2(\Omega)).$$

Consider the operator

$$\begin{aligned}
&\Lambda : \mathbb{L}^2(0.T, V \times V') \to \mathbb{L}^2(0.T, V \times V'),\\
&\Lambda(g, \eta) = (\Lambda_1(g), \Lambda_2(\eta)), \forall g \in \mathbb{L}^2(0.T, V), \forall \eta \in \mathbb{L}^2(0.T, V'),\\
&\Lambda_1(g) = v_{g\eta},\\
&(\Lambda_2(\eta), w)_{V' \times V} = (\mathcal{A}(\varepsilon(u(t))), \varepsilon(w))_{\mathcal{H}} + (\textstyle\int_0^t \mathcal{B}(t-s)\varepsilon(u(s))ds - \theta(t)\mathcal{M}, \varepsilon(w))_{\mathcal{H}}\\
&+\phi(g, w),\\
&|\Lambda(g_2, \eta_2) - \Lambda(g_1, \eta_2)|_{\mathbb{L}^2(0.T; V \times V')}^2 = |(\Lambda_1(g_2), \Lambda_2(\eta_2)) - (\Lambda_1(g_1), \Lambda_2(\eta_1))|_{\mathbb{L}^2(0.T; V \times V')}^2,\\
&= |\Lambda_1(g_2) - \Lambda_1(g_1)|_{\mathbb{L}^2(0.T; V \times V')}^2 + |\Lambda_2(\eta_2) - \Lambda_2(\eta_1)|_{\mathbb{L}^2(0.T; V \times V')}^2 .
\end{aligned} \tag{42}$$

We have the following result.

Lemma 4.3 *The mapping $\Lambda : \mathbb{L}^2(0.T, V \times V') \to \mathbb{L}^2(0.T, V \times V')$ has a unique element $(g^*, \eta^*) \in \mathbb{L}^2(0.T, V \times V')$ such that*

$$\Lambda(g^*, \eta^*) = (g^*, \eta^*). \tag{43}$$

Proof. Let $(g_i, \eta_i) \in \mathbb{L}^2(0.T, V \times V')$. We use the notation (u_i, φ_i). For $(g, \eta) = (g_i, \eta_i), i = 1.2$, let $t \in [0.T]$. We have

$$\Lambda_1(g) = v_{g\eta}. \tag{44}$$

So

$$|g_1(t) - g_2(t)|_V^2 \leq |v_1(t) - v_2(t)|_V^2 . \tag{45}$$

It follows that

$$\begin{aligned}
&(\dot{v}_1(t) - \dot{v}_2(t), v_1(t) - v_2(t)) + (\mathcal{G}\varepsilon(v_1(t)) - \mathcal{G}\varepsilon(v_2(t)), \varepsilon(v_1(t)) - \varepsilon(v_2(t))) +\\
&(\eta_1(t) - \eta_2(t), v_1(t) - v_2(t)) + j(g_1, v_1(t)) - j(g_1, v_2(t)) - j(g_2, v_1(t)) + j(g_2, v_2(t)) \leq 0.
\end{aligned} \tag{46}$$

From the definition of the functional j given by (23), and using (13), (19), we have

$$j(g_2, v_2(t)) - j(g_2, v_1(t)) - j(g_1, v_2(t)) + j(g_1, v_1(t)) \leq C\, |g_1 - g_2|_V \, |v_1 - v_2|_V . \tag{47}$$

Integrating inequality (46) with respect to time, using the initial conditions $v_2(0) = v_1(0) = v_0$, using (13), (15), using Cauchy-Schwartz's inequality and the inequalities $2ab \leq \frac{C}{m_{\mathcal{G}}}a^2 + \frac{m_{\mathcal{G}}}{C}b^2$ and $2ab \leq \frac{1}{m_{\mathcal{G}}}a^2 + m_{\mathcal{G}}b^2$, by Gronwall's inequality, we find

$$|v_1(t) - v_2(t)|_V^2 \leq C(\int_0^t |g_1(s) - g_2(s)|_V^2 \, ds + \int_0^t |\eta_1(s) - \eta_2(s)|_{V'}^2 \, ds). \tag{48}$$

So

$$|g_1 - g_2|_V^2 \leq C(\int_0^t |g_1(s) - g_2(s)|_V^2 \, ds + \int_0^t |\eta_1(s) - \eta_2(s)|_{V'}^2 \, ds). \tag{49}$$

And we have

$$(\Lambda_2(\eta), w)_{V' \times V} = (\mathcal{A}(\varepsilon(u(t))), \varepsilon(w))_{\mathcal{H}} + (\int_0^t \mathcal{B}(t-s)\varepsilon(u(s))ds - \theta(t)\mathcal{M}, \varepsilon(w))_{\mathcal{H}} + \phi(g, w). \tag{50}$$

From the definition of the functional ϕ given by (24), and using (13), (19), we have

$$\phi(g_1, v_2(t)) - \phi(g_1, v_1(t)) - \phi(g_2, v_2(t)) + \phi(g_2, v_1(t)) \leq C\,|g_1 - g_2|_V\,|v_1 - v_2|_V\,. \tag{51}$$

So

$$\begin{gathered} |\eta_1(t) - \eta_2(t)|_{V'}^2 \leq C(|u_1(t) - u_2(t)|_V^2 + \int_0^t |u_1(s) - u_2(s)|_V^2 \, ds + \\ |\theta_1(t) - \theta_2(t)|_{\mathbb{L}^2(\Omega)}^2 + |g_1(t) - g_2(t)|_V^2). \end{gathered} \tag{52}$$

By (46), using the inequalities $2ab \leq \frac{2C}{m_{\mathcal{G}}}a^2 + \frac{m_{\mathcal{G}}}{2C}b^2$ and $2ab \leq \frac{2}{m_{\mathcal{G}}}a^2 + \frac{m_{\mathcal{G}}}{2}b^2$, we find

$$\begin{aligned} &\tfrac{1}{2}\,|v_1(t) - v_2(t)|_V^2 + m_{\mathcal{G}} \int_0^t |v_1(s) - v_2(s)|_V^2 \, ds \leq \tfrac{1}{m_{\mathcal{G}}} \int_0^t |\eta_1(s) - \eta_2(s)|_{V'}^2 \, ds + \\ &+ \tfrac{m_{\mathcal{G}}}{4} \int_0^t |v_1(s) - v_2(s)|_V^2 \, ds + C \times \tfrac{C}{m_{\mathcal{G}}} \int_0^t |g_1(s) - g_2(s)|_V^2 \, ds + \\ &C \times \tfrac{m_{\mathcal{G}}}{4C} \int_0^t |v_1(s) - v_2(s)|_V^2 \, ds. \end{aligned} \tag{53}$$

So

$$\int_0^t |v_1(s) - v_2(s)|_V^2 \, ds \leq C(\int_0^t |\eta_1(s) - \eta_2(s)|_{V'}^2 \, ds + \int_0^t |g_1(s) - g_2(s)|_V^2 \, ds). \tag{54}$$

By (41), we find

$$\begin{aligned} &\left(\dot{\theta}_1(t) - \dot{\theta}_2(t), \theta_1(t) - \theta_2(t)\right)_{E' \times E} + (K(\theta_1) - K(\theta_2), \theta_1(t) - \theta_2(t))_{E' \times E} \\ &= (R(v_1) - R(v_2), \theta_1(t) - \theta_2(t))_{E' \times E}\,. \end{aligned} \tag{55}$$

We integrate (55) over [0.T], we use the initial conditions $\theta_1(0) = \theta_2(0) = \theta_0$, and we use the coercive of K and the Lipschitz continuity of R to deduce that

$$\begin{gathered} \tfrac{1}{2}\,|\theta_1(t) - \theta_2(t)|_{\mathbb{L}^2(\Omega)}^2 + C \int_0^t |\theta_1(s) - \theta_2(s)|_{\mathbb{L}^2(\Omega)}^2 \, ds \leq \\ C\left(\int_0^t \, |v_1(s) - v_2(s)|_V\,|\theta_1(s) - \theta_2(s)|_{\mathbb{L}^2(\Omega)} \, ds\right). \end{gathered}$$

Using the inequality $2ab \leq \frac{1}{2}a^2 + 2b^2$, we find

$$\begin{gathered} \tfrac{1}{2}\,|\theta_1(t) - \theta_2(t)|_{\mathbb{L}^2(\Omega)}^2 + C \int_0^t |\theta_1(s) - \theta_2(s)|_{\mathbb{L}^2(\Omega)}^2 \, ds \leq \\ \tfrac{C}{4} \int_0^t \, |v_1(s) - v_2(s)|_V \, ds + C\,|\theta_1(s) - \theta_2(s)|_{\mathbb{L}^2(\Omega)} \, ds. \end{gathered}$$

Also,

$$|\theta_1(t) - \theta_2(t)|^2_{\mathbb{L}^2(\Omega)} \leq C \int_0^t |v_1(s) - v_2(s)|^2_V \, ds. \tag{56}$$

By (54), we find

$$|\theta_1(t) - \theta_2(t)|^2_{\mathbb{L}^2(\Omega)} \leq C \left(\int_0^t |\eta_1(s) - \eta_2(s)|^2_{V'} \, ds + \int_0^t |g_1(s) - g_2(s)|^2_V \, ds \right). \tag{57}$$

So,

$$|\eta_1(t) - \eta_2(t)|^2_{V'} \leq C \left(\int_0^t |g_1(s) - g_2(s)|^2_V \, ds + \int_0^t |\eta_1(s) - \eta_2(s)|^2_{V'} \, ds \right). \tag{58}$$

Also,

$$|u_1(t) - u_2(t)|^2_V + \int_0^t |u_1(s) - u_2(s)|^2_V \, ds \leq C(\int_0^t |v_1(s) - v_2(s)|^2_V + \int_0^t |u_1(s) - u_2(s)|^2_V) ds. \tag{59}$$

And

$$|u_1(t) - u_2(t)|^2_V \geq 0.$$

$$\textstyle\int_0^t \int_0^s |u_1(r) - u_2(r)|^2_V \, dr ds \geq 0.$$

So,

$$|u_1 - u_2|^2_V + \int_0^t |u_1 - u_2|^2_V \, ds \leq C \int_0^t (|v_1(s) - v_2(s)|^2_V + |u_1(s) - u_2(s)|^2_V + \\ + \int_0^s |u_1(r) - u_2(r)|^2_V \, dr) ds.$$

By Gronwall's inequality, and using (54), we have

$$|u_1 - u_2|^2_V + \int_0^t |u_1 - u_2|^2_V \, ds \leq C(\int_0^t |\eta_1(s) - \eta_2(s)|^2_{V'} \, ds + \int_0^t |g_1(s) - g_2(s)|^2_V \, ds). \tag{60}$$

And using (49) and (58), we find

$$|\Lambda(g_1, \eta_1) - \Lambda(g_2, \eta_2)|^2_{\mathbb{L}^2(0.T; V \times V')} \leq C \int_0^t |(g_1, \eta_1) - (g_2, \eta_2)|^2_{V \times V'} \, ds. \tag{61}$$

Thus, for m sufficiently large, Λ^m is a contraction on $\mathbb{L}^2(0.T, V \times V')$ and so Λ has a unique fixed point in this Banach space. We consider the operator $\mathcal{L} : C(0.T, \mathbb{L}^2(\Gamma_3)) \to C(0.T, \mathbb{L}^2(\Gamma_3))$,

$$\mathcal{L}\omega(t) = -kv^* \int_0^t \sigma_\nu(s) ds, \forall t \in [0.T]. \tag{62}$$

Lemma 4.4 *The operator $\mathcal{L} : C(0.T, \mathbb{L}^2(\Gamma_3)) \to C(0.T, \mathbb{L}^2(\Gamma_3))$ has a unique element $\omega^* \in C(0.T, \mathbb{L}^2(\Gamma_3))$ such that*

$$\mathcal{L}\omega^* = \omega^*.$$

Proof. Using (62), we have

$$|\mathcal{L}\omega_1(t) - \mathcal{L}\omega_2(t)|^2_{\mathbb{L}^2(\Gamma_3)} \leq k\upsilon^* \int_0^t |\sigma_1(s) - \sigma_2(s)|^2_{\mathcal{H}}\, ds. \tag{63}$$

From (1), we have

$$\begin{aligned} |\mathcal{L}\omega_1(t) - \mathcal{L}\omega_2(t)|^2_{\mathbb{L}^2(\Gamma_3)} \leq C \textstyle\int_0^t (|u_1(t) - u_2(t)|^2_V + \int_0^t |u_1(s) - u_2(s)|^2_V\, ds + \\ + |\theta_1(t) - \theta_2(t)|^2_{\mathbb{L}^2(\Omega)}) ds. \end{aligned} \tag{64}$$

By (56) and (59), we find

$$|u_1 - u_2|^2_V + \int_0^t |u_1 - u_2|^2_V\, ds + |\theta_1(t) - \theta_2(t)|^2_{\mathbb{L}^2(\Omega)} \leq \int_0^t |v_1(s) - v_2(s)|^2_V\, ds. \tag{65}$$

So,

$$\begin{aligned} |u_1 - u_2|^2_V + \textstyle\int_0^t |u_1 - u_2|^2_V\, ds + |\theta_1(t) - \theta_2(t)|^2_{\mathbb{L}^2(\Omega)} \leq C(\int_0^t |v_1(s) - v_2(s)|^2_V\, ds + \\ + |\omega_1(t) - \omega_2(t)|^2_{\mathbb{L}^2(\Gamma_3)}). \end{aligned}$$

So, we have

$$|u_1 - u_2|^2_V + \int_0^t |u_1 - u_2|^2_V\, ds + |\theta_1(t) - \theta_2(t)|^2_{\mathbb{L}^2(\Omega)} \leq C\, |\omega_1(t) - \omega_2(t)|^2_{\mathbb{L}^2(\Gamma_3)}. \tag{66}$$

By (64), we find $|\mathcal{L}\omega_1(t) - \mathcal{L}\omega_2(t)|_{\mathbb{L}^2(\Gamma_3)} \leq C \int_0^t |\omega_1(s) - \omega_2(s)|_{\mathbb{L}^2(\Gamma_3)}\, ds$. Thus, for m sufficiently large, $\mathcal{L}^m$ is a contraction on $C(0.T, \mathbb{L}^2(\Gamma_3))$ and so $\mathcal{L}$ has a unique fixed point in this Banach space. Now, we have all the ingredients to prove Theorem 4.1.

Existence. Let $(g^*, \eta^*) \in \mathbb{L}^2(0.T, V \times V')$ be the fixed point of Λ defined by (42), let $\omega^* \in C(0.T, \mathbb{L}^2(\Gamma_3))$ be the fixed point of $\mathcal{L}\omega^*$ defined by (62), and let $(u, \theta) = (u_{g^*\eta^*}, \theta_{g^*\eta^*})$ be the solutions of Problems $PV_{g^*\eta^*}$ and $P_{\theta g\eta}$. It results from (35), (41) that $(u_{g^*\eta^*}, \theta_{g^*\eta^*})$ is the solution of Problem PV. Properties (31) – (34) follow from Lemmas 4.1 and 4.2.

Uniqueness. The uniqueness of the solution is a consequence of the uniqueness of the fixed point of the operators $\Lambda, \mathcal{L}$ defined by (42), (62), and the unique solvability of the Problems $PV_{g\eta}$ and $P_{\theta g\eta}$. This completes the proof.

5 Concluding Remarks and Perspectives

This paper studies electromechanical and thermomechanical contact problems with or without friction within the framework of the mechanics of continuous media. This involves extending the previous results on the existence and uniqueness of the solution by new techniques. We will also try to obtain the properties of the solution for different boundary conditions (traction displacement, contact with or without friction, contact with adhesion, etc.) In the study of this type of problem, there is also a question of developing numerical methods for the resolution of the nonlinear equations concerned.

References

[1] A. A. Abdelaziz and S. Boutechebak. Analysis of a dynamic thermo-elastic-viscoplastic contact problem. *Electron. J. Qual. Theory Differ. Equ.* **71** (2013) 1–17. MR3151718.

[2] A. Bachmar, S. Boutechebak and T. Serrar. Variational Analysis of a Dynamic Electroviscoelastic Problem with Friction. *J. Sib. Fed. Univ. Math & Phy.* **12** (1) (2019) 68–78.

[3] I. Boukaroura and S. Djabi. A Dynamic Tresca's Frictional contact problem with damage for thermo elastic-viscoplastic bodies. Stud. Univ. Babes-Bolyai Math. **64** (3) (2019) 433–449.

[4] G. Duvaut and J. L. Lions. *Inequalities in Mechanics and Physics.* Springer-Verlag, Berlin, 1988.

[5] R. D. Mindlin. Elasticity, piezoelectricity and crystal lattice dynamics. *J. Elasticity* **2** (1972) 217–280.

[6] R. D. Mindlin. Polarization gradient in elastic dielectrics. *Int. J. Solids Structures* **4** (1968) 637–663.

[7] K. I. Saffidine and S.Mesbahi. Existence Result for Positive Solution of a Degenerate Reaction-Diffusion System via a Method of Upper and Lower Solutions. *Nonlinear Dyn. Syst. Theory* **21** (4) (2021) 434–445.

[8] M. Selmani and L. Selmani. *Frictional Contact Problem for Elastic-Viscoplastic Materials with Thermal Effect.* Berlin, Helberg, 2013.

[9] M. Sofonea, W. Han and M. Shillor. *Analysis and Approximation of Contact Problems with Adhesion or Damage.* Chapman Hall/ CRC, New York, 2006.