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Relationship between Persymmetric Solutions and Minimal Persymmetric Solutions of $AXA^{(*)} = B$

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Abstract: The minimal rank persymmetric solution of the quaternion matrix equation $AXA^{(*)} = B$ is defined as the matrix X that minimizes the rank of the difference $AXA^{(*)} - B$ or, equivalently, $r\big(AXA^{(*)} - B\big) = \min$, where B is persymmetric. In this paper, we focus on the quaternion matrix equation $AXA^{(*)} = B$. Our aim is to investigate the inclusion relationships between two sets, Ω_1 and Ω_2, where Ω_1, Ω_2 are, respectively, the set of persymmetric solutions and the set of minimal rank persymmetric solutions of the quaternion matrix equation $AXA^{(*)} = B$. Then, we deduce the necessary and sufficient conditions for the following relations to hold: $\Omega_1 \cap \Omega_2 \neq \varnothing$, $\Omega_1 \subseteq \Omega_2$ and $\Omega_1 \supseteq \Omega_2$.

Keywords: *linear system; persymmetric solution; Moore-Penrose inverse; rank.*

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1 Introduction

Throughout this paper, $\mathbb{R}$ and $\mathbb{C}$ stand for the real number field and the complex number field, respectively. Let $\mathbb{H}^{m\times n}$ be the set of $m\times n$ matrices over the real quaternion algebra:

$$\mathbb{H} = \left\{a_0 + a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k} \mid \boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{ijk} = -1,\ a_0, a_1, a_2, a_3 \in \mathbb{R}\right\}.$$

The symbols A^* and $r(A)$ stand for the conjugate transpose and the rank of A, respectively. I_m denotes the identity matrix of order m. The Moore-Penrose generalized

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inverse of a given matrix $A \in \mathbb{H}^{m\times n}$ is defined to be the unique matrix $A^{+} \in \mathbb{H}^{n\times m}$ which satisfies the following four matrix equations:

$$\text{(a) } AXA = A, \text{ (b) } XAX = X, \text{ (c) } (AX)^{*} = AX, \text{ (d) } (XA)^{*} = XA.$$

The Moore-Penrose inverse has been the subject of many studies (see [1], [8]).

Furthermore, F_A and E_A stand for the two projectors $F_A = I_n - A^{+}A$ and $E_A = I_m - AA^{+}$ induced by $A \in \mathbb{H}^{m\times n}$. We denote the $n \times n$ permutation matrix whose elements along the southwest–northeast diagonal are ones and whose remaining elements are zeros by V_n.

In 1843, Irish mathematician Sir William Rowan Hamilton made a significant contribution when he introduced quaternions. However, it is important to note that the quaternion algebra H is an associative noncommutative division algebra over the real number field $\mathbb{R}$. It has applications in diverse fields such as computer science, orbital mechanics, signal and color image processing, and control theory, see e.g. [7], [10], [16], [19].

Consider the quaternion matrix equation

$$AXA^{(*)} = B, \tag{1}$$

where $A \in \mathbb{H}^{m\times n}$, $B = B^{(*)} \in \mathbb{H}^{m\times m}$ are given matrices, and $X \in \mathbb{H}^{n\times n}$ is an unknown matrix.

The two main objectives of matrix theory, which has focused on matrix equations, are first to establish the conditions in which a solution exists and then to provide a general solution for the problems. The efforts are focused on defining the behaviors of the solutions based on the identification of the general solution. Numerous criteria are included in this, namely, ranks, ranges, norms, definiteness, etc.

In information theory, engineering systems, linear estimation theory, linear system theory, and other fields, persymmetric and perskew-symmetric matrices are just as helpful as symmetric and skew-symmetric matrices. For example, in [14], Wang et al. provided symmetric, persymmetric, and centrosymmetric solutions for certain systems of quaternion matrix equations, in [17], Xie and Sheng presented the problem of generating a matrix A with specified eigenpair, where A is an anti-symmetric and persymmetric matrix, in [15], Wang et al. derived the expressions of the least squares Hermitian persymmetric and bisymmetric solution for the quaternion matrix equation $AXB = C$ by using the semi-tensor product of matrices and the real vector representation of the quaternion matrix. For further related works, one may refer to [9], [4], [10].

Wang et al. in [13] gave the following definition.

Definition 1.1 Let $A = (a_{ij}) \in \mathbb{H}^{m\times n}$, $A^{*} = (\overline{a}_{ji}) \in \mathbb{H}^{n\times m}$and $A^{(*)} = (\overline{a}_{m-j+1,n-i+1}) \in \mathbb{H}^{n\times m}$, where $\overline{a}_{ji}$ is the conjugate of the quaternion a_{ji}. Then $A^{(*)} = V_n A^{*} V_m$. We say that the matrix $A = (a_{ij}) \in \mathbb{H}^{n\times n}$ is symmetric if $A = A^{*}$, the matrix $A = (a_{ij}) \in \mathbb{H}^{n\times n}$ is persymmetric if $A = A^{(*)}$ and the matrix $A = (a_{ij}) \in \mathbb{H}^{n\times n}$ is perskew-symmetric if $A = -A^{(*)}$.

The rank of a matrix is a quantity that plays an important role in characterizing the algebraic properties of matrices, this concept has been the subject of research by many authors, Tian proposed the notion of least-rank solutions to matrix equations in [11, 12] based on the minimal rank of the linear matrix function $A - BXC$ over the field $\mathbb{C}$. In [2], Guerarra studied positive and negative definite submatrices in an Hermitian least-rank solution of the matrix equation $AXA^* = B$. Further, in [3], she investigated necessary and sufficient conditions for the matrix equation $AXB = C$ to have a Hermitian Re-positive or Re-negative definite solution. In [18], Xu et al. used the Moore-Penrose inverse to deduce the necessary and sufficient conditions for the existence of Hermitian (skew-Hermitian), Re-nonnegative (Re-positive) definite, and Re-nonnegative (Re-positive) definite least-rank solutions to $AXB = C$ and presented explicit representations of the general solutions in cases for which the solvability conditions were satisfied.

Motivated by the works mentioned above, and in view of the applications of and interest in quaternion matrices, in this study, we investigate the inclusion relationships between two sets, as well as the set of persymmetric solutions and the set of minimal rank persymmetric solutions of the quaternion matrix equation (1).

The following lemma is due to Matsagalia and Styan [8], and can be easily generalized to $\mathbb{H}$.

Lemma 1.1 *Let $A \in \mathbb{H}^{s\times r}$, $B \in \mathbb{H}^{s\times k}$, $C \in \mathbb{H}^{l\times r}$, $D \in \mathbb{H}^{l\times k}$. Then*

$$r\begin{bmatrix} A, & B \end{bmatrix} = r(A) + r(E_A B) = r(B) + r(E_B A), \tag{2}$$

$$r\begin{bmatrix} A \\ C \end{bmatrix} = r(A) + r(CF_A) = r(C) + r(AF_C), \tag{3}$$

$$r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = r(B) + r(C) + r(E_B A F_C). \tag{4}$$

The following formulas are derived from (2), (3), and (4):

$$r\begin{bmatrix} A & BF_N \\ E_R C & 0 \end{bmatrix} = r\begin{bmatrix} A & B & 0 \\ C & 0 & R \\ 0 & N & 0 \end{bmatrix} - r(N) - r(R).$$

$$r\begin{bmatrix} M & L \\ E_R A & E_R B \end{bmatrix} = r\begin{bmatrix} M & L & 0 \\ A & B & R \end{bmatrix} - r(R),$$

$$r\begin{bmatrix} M & AF_N \\ L & BF_N \end{bmatrix} = r\begin{bmatrix} M & A \\ L & B \\ O & N \end{bmatrix} - r(N).$$

2 Relationship between Persymmetric Solutions and Minimal Persymmetric Solutions of $AXA^{(*)} = B$

It is well known that the persymmetric solution and the minimal persymmetric solution of the quaternion equation (1) are not necessarely unique, throughout this section, we adopt the following notations:

$$\Omega_1 = \left\{ X \in \mathbb{H}^{n\times n} \;/\; AXA^{(*)} = B \right\}, \tag{5}$$

$$\Omega_2 = \left\{ X_m \in \mathbb{H}^{n\times n} \;/\; r\left(B - AXA^{(*)}\right) = \min \right\}. \tag{6}$$

Notably, the two sets equations in (5) and (6) may not necessarily be equal. In this section, we focus on the necessary and sufficient conditions for the following relations to hold:

$$\Omega_1 \cap \Omega_2 \neq \emptyset, \;\; \Omega_1 \subseteq \Omega_2 \text{ and } \Omega_1 \supseteq \Omega_2.$$

We need the following lemmas.

Lemma 2.1 *[5] Let $A \in \mathbb{H}^{m\times n}$, $B = B^{(*)} \in \mathbb{H}^{m\times m}$. Then the matrix equation (1) has a persymmetric solution if and only if $\mathcal{R}(B) \subseteq \mathcal{R}(A)$. In this case, the general persymmetric solution can be expressed as*

$$X = A^{+}BA^{+^{(*)}} + F_A V + V^{(*)} F_A^{(*)},$$

where $V \in \mathbb{H}^{n\times n}$ is arbitrary.

Lemma 2.2 *[5] Let $A \in \mathbb{H}^{m\times n}$, $B = B^{(*)} \in \mathbb{H}^{m\times m}$ be given. Then the expression of general minimal rank persymmetric solution of (1) is given by*

$$X = -S^{(*)}M^{+}S + S_1^{(*)}U^{(*)} + US_1, \tag{7}$$

where

$$M = \begin{bmatrix} A & B \\ 0 & A^{(*)} \end{bmatrix}, \; S = \begin{bmatrix} 0 \\ I_n \end{bmatrix}, \; S_1 = E_M S,$$

and $U \in \mathbb{H}^{n\times n}$ is arbitrary.

Khan et al. in [5] [Theorem 2.4], derived the minimal rank formula of the quaternion matrix expression

$$p(X,Y) = G - LX - (LX)^{(*)} - CYC^{(*)}$$

with respect to the pair of matrices X and $Y = Y^{(*)}$, where $G \in \mathbb{H}^{m\times m}$, $L \in \mathbb{H}^{m\times n}$ and $C \in \mathbb{H}^{m\times m}$ are given, $X \in \mathbb{H}^{n\times m}$ and $Y = Y^{(*)} \in \mathbb{H}^{m\times m}$ are variable matrices, and then derived the following result.

Lemma 2.3 *[5] Let $G = G^{(*)} \in \mathbb{H}^{m\times m}$, $L \in \mathbb{H}^{m\times n}$ be given, and $X \in \mathbb{H}^{n\times m}$ be a variable matrix. Then*

$$\min_{X\in\mathbb{H}^{n\times m}} r\left[G - LX - (LX)^{(*)}\right] = r\begin{bmatrix} L & G \\ 0 & L^{(*)} \end{bmatrix} - 2r(L). \tag{8}$$

The following lemma is due to Liu and Tian in [6], and can be easily generalized to $\mathbb{H}$.

Lemma 2.4 *Let $D \in \mathbb{H}^{s\times r}$, $N \in \mathbb{H}^{s\times l}$ and $K \in \mathbb{H}^{k\times r}$ be matrices such that $\mathcal{R}\left(K^{(*)}\right) \subset \mathcal{R}(N)$. Then*

$$\max_{X\in\mathbb{C}^{l\times k}} r\left(D - NXK - (NXK)^{(*)}\right) = \min\left\{ r\begin{bmatrix} D & N \end{bmatrix}, r\begin{bmatrix} K & D \\ 0 & K^{(*)} \end{bmatrix} \right\}, \tag{9}$$

$$\min_{X\in\mathbb{C}^{l\times k}} r\left(D - NXK - (NXK)^{(*)}\right) = 2r\begin{bmatrix} D & N \end{bmatrix} + r\begin{bmatrix} K & D \\ 0 & K^{(*)} \end{bmatrix} - 2r\begin{bmatrix} D & N \\ K & 0 \end{bmatrix}. \tag{10}$$

The main result of this work is the following theorem.

Theorem 2.1 *Let $A \in \mathbb{H}^{m\times n}$ and $B = B^{(*)} \in \mathbb{H}^{m\times m}$ be given, and suppose that X, X_m are persymmetric solutions and minimal persymmetric solution of Eq. (1), respectively, and let Ω_1 and Ω_2 be as given in (5) and (6), respectively. Then the following hold:*
(a) Eq. (1) has persymmetric solutions and minimal rank solutions, that is, $\Omega_1 \cap \Omega_2 \neq \emptyset$ if and only if

$$r\begin{bmatrix} A & B & 0 & 0 & 0 \\ 0 & A^{(*)} & 0 & 0 & 0 \\ A & 0 & 0 & 0 & B \\ 0 & 0 & B & A & 0 \\ 0 & 0 & A^{(*)} & 0 & A^{(*)} \end{bmatrix} = 2r\left(A\right) + 2r\begin{bmatrix} B \\ A^{(*)} \end{bmatrix}.$$

(b) All the persymmetric solutions of Eq. (1) are minimal rank persymmetric solutions of Eq. (1), that is, $\Omega_1 \subseteq \Omega_2$ if and only if

$$r\begin{bmatrix} A & B & 0 & 0 \\ 0 & A^{(*)} & 0 & A^{(*)} \\ A & 0 & 0 & -B \\ 0 & 0 & A & B \end{bmatrix} = r\left(A\right) + 2r\begin{bmatrix} A & B \end{bmatrix} \quad or \quad r\begin{bmatrix} A & B \end{bmatrix} = r\begin{bmatrix} A & B \\ 0 & A^{(*)} \end{bmatrix}.$$

(c) All the minimal rank persymmetric solutions of Eq. (1) are persymmetric solutions of Eq. (1), that is, $\Omega_1 \supseteq \Omega_2$ if and only if

$$r\begin{bmatrix} A & B & 0 & 0 & 0 \\ 0 & A^{(*)} & 0 & 0 & 0 \\ A & 0 & 0 & 0 & B \\ 0 & 0 & A & B & 0 \\ 0 & 0 & 0 & A^{(*)} & 0 \end{bmatrix} = 2r\begin{bmatrix} A & B \\ 0 & A^{(*)} \end{bmatrix} \quad or\ A = 0.$$

Proof. (a) The intersection $\Omega_1 \cap \Omega_2 \neq \emptyset$ means that Eq. (1) has minimal rank persymmetric solutions and persymmetric solutions, which implies the minimum rank of the matrix expression $X - X_m$ is zero, that is,

$$\min_{X\in\Omega_1,\ X_m\in\Omega_2} r\left(X - X_m\right) = 0. \tag{11}$$

According to Lemmas 2.1 and 2.2, the expressions for the general persymmetric solution and the minimal-rank solution of the matrix equation (1) can be written as follows:

$$X = A^{+}BA^{+^{(*)}} + F_AV + V^{(*)}F_A^{(*)},$$
$$X_m = -S^{(*)}M^{+}S + S_1^{(*)}U^{(*)} + US_1.$$

We can write the expression $X - X_m$ as follows:

$$\begin{aligned} X - X_m &= A^{+}BA^{+^{(*)}} + F_AV + V^{(*)}F_A^{(*)} + S^{(*)}M^{+}S - S_1^{(*)}U^{(*)} - US_1 \\ &= A^{+}BA^{+^{(*)}} + S^{(*)}M^{+}S + \begin{bmatrix} F_A & -S_1^{(*)} \end{bmatrix}\begin{bmatrix} V \\ U^{(*)} \end{bmatrix} \\ &+ \left(\begin{bmatrix} F_A & -S_1^{(*)} \end{bmatrix}\begin{bmatrix} V \\ U^{(*)} \end{bmatrix}\right)^{(*)} \\ &= G + HW + \left(HW\right)^{(*)}, \end{aligned} \tag{12}$$

where $G=A^{+}BA^{+^{(*)}}+S^{(*)}M^{+}S$, $H=\begin{bmatrix} F_A & -S_1^{(*)} \end{bmatrix}$ and $W=\begin{bmatrix} V \\ U^{(*)} \end{bmatrix}$ is arbitrary with appropriate size.

Applying (8) in Lemma 2.3 to (12) yields

$$\min_{X\in\Omega_1,\ X_m\in\Omega_2} r\left(X-X_m\right)=\min_{W} r\left(G+HW+(HW)^{(*)}\right)=r\begin{bmatrix} H & G \\ 0 & H^{(*)} \end{bmatrix}-2r\left(H\right).$$

By applying Lemma 1.1 and three elementary block matrix operations and simplifying by $AA^{+}B=B$, we obtain

$$r\begin{bmatrix} H & G \\ 0 & H^{(*)} \end{bmatrix}=r\begin{bmatrix} F_A & -S_1^{(*)} & A^{+}BA^{+^{(*)}}+S^{(*)}M^{+}S \\ 0 & 0 & F_A^{(*)} \\ 0 & 0 & -S_1 \end{bmatrix}$$

$$=r\begin{bmatrix} A^{+}BA^{+^{(*)}}+S^{(*)}M^{+}S & F_A & S^{(*)}F_{M^{(*)}} \\ E_{A^{(*)}} & 0 & 0 \\ E_M S & 0 & 0 \end{bmatrix}$$

$$=\begin{bmatrix} 0 & I_n & S^{(*)} & 0 & 0 \\ I_n & 0 & 0 & A^{(*)} & 0 \\ S & 0 & 0 & 0 & M \\ AA^{+}BA^{+^{(*)}} & A & 0 & 0 & 0 \\ MM^{+}S & 0 & M & 0 & 0 \end{bmatrix}-2r\left(A\right)-2r\left(M\right)$$

$$=\begin{bmatrix} 0 & I_n & S^{(*)} & 0 & 0 \\ I_n & 0 & 0 & A^{(*)} & 0 \\ S & 0 & 0 & 0 & M \\ 0 & A & 0 & B & 0 \\ 0 & 0 & M & 0 & M \end{bmatrix}-2r\left(A\right)-2r\left(M\right)$$

$$=r\begin{bmatrix} 0 & I_n & I_n & 0 & 0 & 0 & 0 \\ I_n & 0 & 0 & 0 & A^{(*)} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & A & B \\ I_n & 0 & 0 & 0 & 0 & 0 & A^{(*)} \\ 0 & A & 0 & 0 & B & 0 & 0 \\ 0 & 0 & A & B & 0 & A & B \\ 0 & 0 & 0 & A^{(*)} & 0 & 0 & A^{(*)} \end{bmatrix}-2r\left(A\right)-2r\left(M\right)$$

$$=r\begin{bmatrix} 0 & I_n & 0 & 0 & 0 & 0 & 0 \\ I_n & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & A & B \\ 0 & 0 & 0 & 0 & -A^{(*)} & 0 & A^{(*)} \\ 0 & 0 & A & 0 & B & 0 & 0 \\ 0 & 0 & -A & B & 0 & A & B \\ 0 & 0 & 0 & A^{(*)} & 0 & 0 & A^{(*)} \end{bmatrix}-2r\left(A\right)-2r\left(M\right)$$

$$=2n+\begin{bmatrix} A & B & 0 & 0 & 0 \\ 0 & A^{(*)} & 0 & 0 & 0 \\ A & 0 & 0 & 0 & B \\ 0 & 0 & B & A & 0 \\ 0 & 0 & A^{(*)} & 0 & A^{(*)} \end{bmatrix}-2r\left(A\right)-2r\left(M\right), \tag{13}$$

and

$$\begin{aligned} r(H) &= r\begin{bmatrix} F_A & S_1^{(*)} \end{bmatrix} \\ &= r\begin{bmatrix} I_n & S^{(*)} \\ A & 0 \\ 0 & M \end{bmatrix} - r(A) - r(M) \\ &= n + r(A) + r\begin{bmatrix} B \\ A^{(*)} \end{bmatrix} - r(A) - r(M) \\ &= n + r\begin{bmatrix} B \\ A^{(*)} \end{bmatrix} - r(M). \end{aligned} \tag{14}$$

By substituting (13) and (14) into (11), we obtain (a).

b) Note that $\Omega_1 \subseteq \Omega_2$ means that all the persymmetric solutions of Eq. (1) are minimal rank persymmetric solutions, hence the inclusion $\Omega_1 \subseteq \Omega_2$ is equivalent to

$$\max_{X\in\Omega_1}\min_{X_m\in\Omega_2} r(X - X_m) = 0. \tag{15}$$

Applying (8) in Lemma 2.3 to the matrix expression $X - X_m$ yields

$$\begin{aligned} \min_{X_m\in\Omega_2} r(X - X_m) &= \min_{X_m\in\Omega_2} r\left(X + S^{(*)}M^+S - S_1^{(*)}U^{(*)} - US_1\right) \\ &= r\begin{bmatrix} S_1^{(*)} & X + S^{(*)}M^+S \\ 0 & S_1 \end{bmatrix} - 2r(S_1). \end{aligned} \tag{16}$$

The 2×2 block matrix in (16) can be written as

$$\begin{bmatrix} S_1^{(*)} & X + S^{(*)}M^+S \\ 0 & S_1 \end{bmatrix} = \begin{bmatrix} S_1^{(*)} & A^+BA^{+^{(*)}} + F_AV + V^{(*)}F_A^{(*)} + S^{(*)}M^+S \\ 0 & S_1 \end{bmatrix}$$

$$= \left(\begin{array}{c} \begin{bmatrix} S_1^{(*)} & A^+BA^{+^{(*)}} + S^{(*)}M^+S \\ 0 & S_1 \end{bmatrix} + \begin{bmatrix} F_A \\ 0 \end{bmatrix} V \begin{bmatrix} 0 & I_n \end{bmatrix} \\ + \left(\begin{bmatrix} F_A \\ 0 \end{bmatrix} V \begin{bmatrix} 0 & I_n \end{bmatrix} \right)^{(*)} \end{array} \right). \tag{17}$$

Applying (9) to (17) yields

$$\begin{aligned} &\max_{X\in\Omega_1} \begin{bmatrix} S_1^{(*)} & X + S^{(*)}M^+S \\ 0 & S_1 \end{bmatrix} \\ &= \min \left\{ \begin{array}{c} r\begin{bmatrix} S_1^{(*)} & A^+BA^{+^{(*)}} + S^{(*)}M^+S & F_A \\ 0 & S_1 & 0 \end{bmatrix}, \\ r\begin{bmatrix} I_n & S_1^{(*)} & A^+BA^{+^{(*)}} + S^{(*)}M^+S \\ 0 & 0 & S_1 \\ 0 & 0 & I_n \end{bmatrix} \end{array} \right\} \\ &= \min \left\{ 2n + r\begin{bmatrix} A & B & 0 & 0 \\ 0 & A^{(*)} & 0 & A^{(*)} \\ A & 0 & 0 & -B \\ 0 & 0 & A & B \end{bmatrix} - r(A) - 2r(M),\, 2n \right\}. \end{aligned} \tag{18}$$

Furthermore we have

$$\begin{aligned} r(S_1) &= r(E_M S) \\ &= \begin{bmatrix} S & M \end{bmatrix} - r(M) \\ &= \begin{bmatrix} 0 & A & B \\ I_n & 0 & A^{(*)} \end{bmatrix} - r(M) \\ &= n + r\begin{bmatrix} A & B \end{bmatrix} - r(M). \end{aligned} \tag{19}$$

Substituting (19) into (16) and combining (18) and (16) yield

$$\begin{aligned} &\max_{X\in\Omega_1}\min_{X_m\in\Omega_2} r(X - X_m) \\ &= \min\left\{ \begin{array}{c} r\begin{bmatrix} A & B & 0 & 0 \\ 0 & A^{(*)} & 0 & A^{(*)} \\ A & 0 & 0 & -B \\ 0 & 0 & A & B \end{bmatrix} - r(A) - 2r\begin{bmatrix} A & B \end{bmatrix}, \\ -2r\begin{bmatrix} A & B \end{bmatrix} + 2r(M) \end{array} \right\}. \end{aligned} \tag{20}$$

Finally, by substituting (20) into (15), we obtain the desired results in (b).

c) The inclusion $\Omega_1 \supseteq \Omega_2$ means that all the minimal rank persymmetric solutions of Eq. (1) are persymmetric solutions, then the inclusion $\Omega_1 \supseteq \Omega_2$ is equivalent to

$$\max_{X_m\in\Omega_2}\min_{X\in\Omega_1} r(X - X_m) = 0. \tag{21}$$

Then we have

$$\min_{X\in\Omega_1} r(X - X_m) = \min_{V}\left(A^+BA^{+^{(*)}} - X_m + F_A V + V^{(*)}F_A^{(*)}\right). \tag{22}$$

Applying (8) to (22) yields

$$\min_{X\in\Omega_1} r(X - X_m) = r\begin{bmatrix} F_A & A^+BA^{+^{(*)}} - X_m \\ 0 & F_A^{(*)} \end{bmatrix} - 2r(F_A). \tag{23}$$

From Lemma 1.1, we have

$$r(F_A) = r\begin{bmatrix} I_n \\ A \end{bmatrix} - r(A) = n - r(A). \tag{24}$$

The 2×2 block matrix on the right-hand side of (23) can be rewritten as

$$\begin{aligned} & r\begin{bmatrix} F_A & A^+BA^{+^{(*)}} - X_m \\ 0 & F_A^{(*)} \end{bmatrix} = r\begin{bmatrix} F_A & A^+BA^{+^{(*)}} + S^{(*)}M^+S - S_1^{(*)}U^{(*)} - US_1 \\ 0 & F_A^{(*)} \end{bmatrix} \\ &= r\begin{bmatrix} F_A & A^+BA^{+^{(*)}} + S^{(*)}M^+S \\ 0 & F_A^{(*)} \end{bmatrix} + \begin{bmatrix} -S_1^{(*)} \\ 0 \end{bmatrix} U^{(*)}\begin{bmatrix} 0 & I_n \end{bmatrix} \\ &+ \left(\begin{bmatrix} -S_1^{(*)} \\ 0 \end{bmatrix} U^{(*)}\begin{bmatrix} 0 & I_n \end{bmatrix}\right)^{(*)}. \end{aligned} \tag{25}$$

Hence, by applying (9) to (25), we obtain

$$\max_{X_m \in \Omega_2} r \begin{bmatrix} F_A & A^+BA^{+^{(*)}} - X_m \\ 0 & F_A^{(*)} \end{bmatrix}$$
$$= \min \left\{ \begin{array}{l} r \begin{bmatrix} F_A & A^+BA^{+^{(*)}} + S^{(*)}M^+S & -S_1^{(*)} \\ 0 & F_A^{(*)} & 0 \end{bmatrix}, \\ \begin{bmatrix} I_n & F_A & A^+BA^{+^{(*)}} + S^{(*)}M^+S \\ 0 & 0 & F_A^{(*)} \\ 0 & 0 & I_n \end{bmatrix} \end{array} \right\}$$
$$= \min \left\{ 2n + r \begin{bmatrix} A & B & 0 & 0 & 0 \\ 0 & A^{(*)} & 0 & 0 & 0 \\ A & 0 & 0 & 0 & B \\ 0 & 0 & A & B & 0 \\ 0 & 0 & 0 & A^{(*)} & 0 \end{bmatrix} - 2r(A) - 2r(M), \, 2n \right\}. \tag{26}$$

Substituting (24) into (23) and combining (26) and (23) yield

$$\max_{X_m \in \Omega_2} \min_{X \in \Omega_1} r(X - X_m)$$
$$= \min \left\{ r \begin{bmatrix} A & B & 0 & 0 & 0 \\ 0 & A^{(*)} & 0 & 0 & 0 \\ A & 0 & 0 & 0 & B \\ 0 & 0 & A & B & 0 \\ 0 & 0 & 0 & A^{(*)} & 0 \end{bmatrix} - 2r(M), \, r(A) \right\}. \tag{27}$$

By substituting (27) into (21), we obtain the desired results in (c).

3 Conclusion

In the previous sections, we have studied some algebraic characterizations of relationships between two sets consisting of two types of solutions for the same quaternion matrix equation $AXA^{(*)} = B$. These two types are, respectively, the persymmetric solutions and the minimal rank persymmetric solutions. Further, the persymmetric solutions exist under the solvability conditions where the matrix equations will be consistent, otherwise, the minimal rank persymmetric solutions always exist. Also, the results obtained clearly illustrate the fundamental characteristics and attributes of some prominent linear matrix equations and their relationships when we have used the matrix rank method, which is considered a useful and effective approach for solving various matrix equality and matrix set inclusion problems.

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