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Conformable Fractional Khalouta Transform and Its Applications to Fractional Differential Equations

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Abstract: In 2023, the author [6] introduced a new integral transform called the Khalouta transform which is a generalization of many well-known integral transforms. In this paper, our aim is to generalize the formula of the Khalouta transform to the conformable fractional order. Moreover, we present and prove some main properties and theorems related to the conformable fractional Khalouta transform. In order to illustrate the validity, efficiency, and applicability of the proposed technique, we apply the conformable fractional Khalouta transform to solve some fractional differential equations. Finally, the results show that our new technique is powerful, effective, and applicable for the both conformable fractional problems.

Keywords: *fractional differential equations; Khalouta transform; conformable fractional derivative; exact solutions.*

Mathematics Subject Classification (2010): 34A08, 26A33, 70K75, 93-06.

1 Introduction

Fractional differential equations are a very important mathematial tool for modeling many applications in real life sciences and engineering such as fluid dynamics, mathematical biology, electrical circuits, optics, quantum mechanics, biophysics, wave theory, polymers, continuum mechanics, etc. [1, 4, 5, 7, 11–13]. There are many definitions of fractional derivatives and integrals used in many applications and natural phenomena such as Riemann–Liouville [10], Liouville–Caputo [9], Caputo–Fabrizio [3], Atangana–Baleanu [2] derivatives and so on. In 2014, Khalil et al. [8] introduced a new definition of the fractional derivative which is called the conformable fractional derivative, and it

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is very easily computed compared with other previous definitions. In recent years, many mathematics researchers have been interested in solving fractional differential equations using different fractional integral transform. The main objective of this paper is to extend the definition of the Khalouta transform [6] to a fractional order in the sense of the conformalble derivative and to give new interesting results for solving various types of conformalble fractional differential equations.

2 Basic Notions and Preliminaries

Definition 2.1 [8] The conformable fractional derivative of order α is defined for a function $u : [0, +\infty) \longrightarrow \mathbb{R}$ by

$$\mathcal{C}^{(n\alpha)}u(t) = \frac{d^{n\alpha}}{dt^{n\alpha}}u(t) = \lim_{\varepsilon \to 0} \frac{u^{[\alpha]-1}(t + \varepsilon t^{[\alpha]-\alpha}) - u^{[\alpha]-1}(t)}{\varepsilon}, t > 0,$$

where $n - 1 < n\alpha \leq n, n \in \mathbb{N}$ and $[\alpha]$ is the smallest integer greater than or equal to α, provided $\mathcal{C}^{(n\alpha)}u(0) = \lim_{t\to 0^+} \mathcal{C}^{(n\alpha)}u(t)$ is $n-$differentiable and $\lim_{t\to 0^+} \mathcal{C}^{(n\alpha)}u(t)$ exists.

As a special case, if $0 < \alpha \leq 1$, then we have

$$\mathcal{C}^{(\alpha)}u(t) = \frac{d^{\alpha}}{dt^{\alpha}}u(t) = \lim_{\varepsilon \to 0} \frac{u(t + \varepsilon t^{1-\alpha}) - u(t)}{\varepsilon}, t > 0,$$

provided $\mathcal{C}^{(\alpha)}u(0) = \lim_{t\to 0^+} \mathcal{C}^{(\alpha)}u(t)$ is $\alpha-$differentiable and $\lim_{t\to 0^+} \mathcal{C}^{(\alpha)}u(t)$ exists.

The most important and useful rule is that: If $u(t)$ is an $n-$differentiable function at $t > 0$ and $n - 1 < n\alpha \leq n$, $n \in \mathbb{N}$, then

$$\mathcal{C}^{(n\alpha)}u(t) = t^{[\alpha]-\alpha}u^{[\alpha]}(t).$$

Definition 2.2 [8] The conformable fractional integral of order α is defined for a function $u : [0, +\infty) \longrightarrow \mathbb{R}$ by

$$\mathcal{I}^{(\alpha)}u(t) = \int_0^t u(t)d_{\alpha}t = \int_0^t u(t)t^{\alpha-1}dt, 0 < \alpha \leq 1, t > 0.$$

If $u : [0, +\infty) \longrightarrow$ is an $\alpha-$differentiable function and $0 < \alpha \leq 1$, then

$$\mathcal{C}^{(\alpha)}u(t) = t^{1-\alpha}u'(t), \tag{1}$$

and

$$\mathcal{C}^{(\alpha)}\mathcal{I}^{(\alpha)}u(t) = u(t).$$

3 Conformable Fractional Khalouta Transform

Definition 3.1 The conformable fractional Khalouta transform of a piecewise continuous function $u : [0, +\infty) \longrightarrow \mathbb{R}$ of exponential order is defined on the set

$$\mathcal{S}_{\alpha} = \left\{u(t) : \exists K, \vartheta_1, \vartheta_2 > 0, |u(t)| < K \exp\left(\alpha\vartheta_j \left|t^{\alpha}\right|\right), \text{ if } t^{\alpha} \in (-1)^j \times [0, \infty)\right\}$$

by the following integral:

$$\begin{aligned} \mathbb{KH}_\alpha\left[u(t)\right] &= \mathcal{K}_\alpha(s,\gamma,\eta) = \frac{s}{\gamma\eta}\int_0^\infty \exp\left(-\frac{st^\alpha}{\gamma\eta\alpha}\right)u(t)t^{\alpha-1}dt \\ &= \lim_{\sigma\to\infty}\frac{s}{\gamma\eta}\int_0^\sigma \exp\left(-\frac{st^\alpha}{\gamma\eta\alpha}\right)u(t)t^{\alpha-1}dt, \end{aligned}$$

where $s>0, \gamma>0$ and $\eta>0$ are the Khalouta transform variables, σ is a real number and the integral is taken along the line $t=\sigma$.

Theorem 3.1 *Let $u:[0,+\infty)\longrightarrow\mathbb{R}$ be a real value function such that*

$$\mathbb{KH}_\alpha\left[u(t)\right] = \mathcal{K}_\alpha(s,\gamma,\eta),$$

and $0<\alpha\leq 1$, then

$$\mathcal{K}_\alpha(s,\gamma,\eta) = \mathbb{KH}\left[u\left((\alpha t)^{\frac{1}{\alpha}}\right)\right].$$

Proof. Using Definition 3.1, we get

$$\begin{aligned} \mathcal{K}_\alpha(s,\gamma,\eta) &= \frac{s}{\gamma\eta}\int_0^\infty \exp\left(-\frac{st^\alpha}{\gamma\eta\alpha}\right)u(t)t^{\alpha-1}dt \\ x &= \frac{t^\alpha}{\alpha} \Longrightarrow dx = t^{\alpha-1}dt \text{ and } t=(\alpha x)^{\frac{1}{\alpha}} \\ &= \frac{s}{\gamma\eta}\int_0^\infty \exp\left(-\frac{sx}{\gamma\eta}\right)u((\alpha x)^{\frac{1}{\alpha}})dx \\ &= \mathbb{KH}\left[u\left((\alpha t)^{\frac{1}{\alpha}}\right)\right]. \end{aligned}$$

Theorem 3.2 *Let $u:[0,+\infty)\longrightarrow\mathbb{R}$ be an $\alpha-$differentiable function and $0<\alpha\leq 1$, then*

$$\mathbb{KH}_\alpha\left[\mathcal{C}^{(\alpha)}u(t)\right] = \frac{s}{\gamma\eta}\mathbb{KH}_\alpha\left[u(t)\right] - \frac{s}{\gamma\eta}u(0).$$

Proof. Using Definition 3.1 and equation (1), we have

$$\begin{aligned} \mathbb{KH}_\alpha\left[\mathcal{C}^{(\alpha)}u(t)\right] &= \frac{s}{\gamma\eta}\int_0^\infty \exp\left(-\frac{st^\alpha}{\gamma\eta\alpha}\right)\mathcal{C}^{(\alpha)}u(t)t^{\alpha-1}dt \\ &= \frac{s}{\gamma\eta}\int_0^\infty \exp\left(-\frac{st^\alpha}{\gamma\eta\alpha}\right)t^{1-\alpha}u'(t)t^{\alpha-1}dt \\ &= \frac{s}{\gamma\eta}\int_0^\infty \exp\left(-\frac{st^\alpha}{\gamma\eta\alpha}\right)u'(t)dt. \end{aligned}$$

With integration by parts, we get

$$\begin{aligned} \mathbb{KH}_\alpha\left[\mathcal{C}^{(\alpha)}u(t)\right] &= \frac{s}{\gamma\eta}\left(\lim_{\sigma\to\infty}\left[\exp\left(-\frac{st^\alpha}{\gamma\eta\alpha}\right)u(t)\right]_0^\sigma \right. \\ &\quad \left. +\frac{s}{\gamma\eta}\int_0^\infty \exp\left(-\frac{st^\alpha}{\gamma\eta\alpha}\right)u(t)t^{\alpha-1}dt\right) \\ &= \frac{s}{\gamma\eta}\left(-u(0)+\mathbb{KH}_\alpha\left[u(t)\right]\right) \\ &= \frac{s}{\gamma\eta}\mathbb{KH}_\alpha\left[u(t)\right] - \frac{s}{\gamma\eta}u(0). \end{aligned}$$

Theorem 3.3 *Let $u : [0, +\infty) \longrightarrow \mathbb{R}$ be an $n-$differentiable function and $0 < \alpha \leq 1$, then*

$$\mathbb{KH}_\alpha \left[\mathcal{C}^{(n\alpha)} u(t) \right] = \frac{s^n}{\gamma^n \eta^n} \mathbb{KH}_\alpha \left[u(t) \right] - \frac{s^n}{\gamma^n \eta^n} u(0).$$

Proof. The proof follows using the induction process on n and Theorem 3.2.

4 Conformable Fractional Khalouta Transform for Some Functions

Theorem 4.1 *Let $a, b, c \in \mathbb{R}$ and $0 < \alpha \leq 1$, then*

$$1)\ \mathbb{KH}_\alpha \left[b \right] = b.$$

$$2)\ \mathbb{KH}_\alpha \left[t^c \right] = \left(\frac{\alpha\gamma\eta}{s} \right)^{\frac{c}{\alpha}} \Gamma\left(\frac{c}{\alpha} + 1 \right).$$

$$3)\ \mathbb{KH}_\alpha \left[\frac{t^{n\alpha}}{\alpha^n} \right] = \left(\frac{\gamma\eta}{s} \right)^n \Gamma\left(n + 1 \right).$$

$$4)\ \mathbb{KH}_\alpha \left[\exp\left(a\frac{t^\alpha}{\alpha} \right) \right] = \frac{s}{s - a\gamma\eta}.$$

$$5)\ \mathbb{KH}_\alpha \left[\sin\left(a\frac{t^\alpha}{\alpha} \right) \right] = \frac{as\gamma\eta}{s^2 + a^2\gamma^2\eta^2}.$$

$$6)\ \mathbb{KH}_\alpha \left[\sinh\left(a\frac{t^\alpha}{\alpha} \right) \right] = \frac{as\gamma\eta}{s^2 - a^2\gamma^2\eta^2}.$$

$$7)\ \mathbb{KH}_\alpha \left[\cos\left(a\frac{t^\alpha}{\alpha} \right) \right] = \frac{s^2}{s^2 + a^2\gamma^2\eta^2}.$$

$$8)\ \mathbb{KH}_\alpha \left[\cosh\left(a\frac{t^\alpha}{\alpha} \right) \right] = \frac{s^2}{s^2 - a^2\gamma^2\eta^2}.$$

Proof. Using Theorem 3.1, we get

1)

$$\mathbb{KH}_\alpha \left[b \right] = \mathbb{KH} \left[b \right] = b.$$

2)

$$\begin{aligned} \mathbb{KH}_\alpha \left[t^c \right] &= \mathbb{KH} \left[(\alpha t)^{\frac{c}{\alpha}} \right] = \alpha^{\frac{c}{\alpha}} \mathbb{KH} \left[t^{\frac{c}{\alpha}} \right] \\ &= \left(\frac{\alpha\gamma\eta}{s} \right)^{\frac{c}{\alpha}} \Gamma\left(\frac{c}{\alpha} + 1 \right). \end{aligned}$$

3) If we put $c = n\alpha$, then

$$\mathbb{KH}_\alpha \left[t^{n\alpha} \right] = \mathbb{KH} \left[(\alpha t)^{\frac{n\alpha}{\alpha}} \right] = \alpha^n \mathbb{KH} \left[t^n \right].$$

So

$$\mathbb{KH}_\alpha \left[\frac{t^{n\alpha}}{\alpha^n} \right] = \mathbb{KH} \left[t^n \right] = \left(\frac{\gamma\eta}{s} \right)^n \Gamma\left(n + 1 \right).$$

4)

$$\begin{aligned}\mathbb{KH}_{\alpha}\left[\exp\left(a\frac{t^{\alpha}}{\alpha}\right)\right] &= \mathbb{KH}\left[\exp\left(a\frac{\left((\alpha t)^{\frac{1}{\alpha}}\right)^{\alpha}}{\alpha}\right)\right]\\ &= \mathbb{KH}\left[\exp\left(at\right)\right]=\frac{s}{s-a\gamma\eta}.\end{aligned}$$

5)

$$\begin{aligned}\mathbb{KH}_{\alpha}\left[\sin\left(a\frac{t^{\alpha}}{\alpha}\right)\right] &= \mathbb{KH}\left[\sin\left(a\frac{\left((\alpha t)^{\frac{1}{\alpha}}\right)^{\alpha}}{\alpha}\right)\right]\\ &= \mathbb{KH}\left[\sin\left(at\right)\right]=\frac{as\gamma\eta}{s^{2}+a^{2}\gamma^{2}\eta^{2}}.\end{aligned}$$

6)

$$\begin{aligned}\mathbb{KH}_{\alpha}\left[\sinh\left(a\frac{t^{\alpha}}{\alpha}\right)\right] &= \mathbb{KH}\left[\sinh\left(a\frac{\left((\alpha t)^{\frac{1}{\alpha}}\right)^{\alpha}}{\alpha}\right)\right]\\ &= \mathbb{KH}\left[\sinh\left(at\right)\right]=\frac{as\gamma\eta}{s^{2}-a^{2}\gamma^{2}\eta^{2}}.\end{aligned}$$

7)

$$\begin{aligned}\mathbb{KH}_{\alpha}\left[\cos\left(a\frac{t^{\alpha}}{\alpha}\right)\right] &= \mathbb{KH}\left[\cos\left(a\frac{\left((\alpha t)^{\frac{1}{\alpha}}\right)^{\alpha}}{\alpha}\right)\right]\\ &= \mathbb{KH}\left[\cos\left(at\right)\right]=\frac{s^{2}}{s^{2}+a^{2}\gamma^{2}\eta^{2}}.\end{aligned}$$

8)

$$\begin{aligned}\mathbb{KH}_{\alpha}\left[\cosh\left(a\frac{t^{\alpha}}{\alpha}\right)\right] &= \mathbb{KH}\left[\cosh\left(a\frac{\left((\alpha t)^{\frac{1}{\alpha}}\right)^{\alpha}}{\alpha}\right)\right]\\ &= \mathbb{KH}\left[\cosh\left(at\right)\right]=\frac{s^{2}}{s^{2}-a^{2}\gamma^{2}\eta^{2}}.\end{aligned}$$

5 Properties of Conformable Fractional Khalouta Transform

Theorem 5.1 *Let $u, v : [0, +\infty) \longrightarrow \mathbb{R}$ be given functions such that*

$$\mathbb{KH}_{\alpha}\left[u(t)\right] = \mathcal{K}_{\alpha}(s,\gamma,\eta),$$

and

$$\mathbb{KH}_{\alpha}\left[v(t)\right] = \mathcal{H}_{\alpha}(s,\gamma,\eta),$$

and let $\lambda, \mu \in \mathbb{R}$ and $0 < \alpha \leq 1$, then we have

1) *Linear property*

$$\mathbb{KH}_\alpha\left[\lambda u(t)\pm\mu v(t)\right]=\lambda\mathbb{KH}_\alpha\left[u(t)\right]\pm\mu\mathbb{KH}_\alpha\left[v(t)\right].$$

2) *Shifting property*

$$\mathbb{KH}_\alpha\left[\exp\left(-a\frac{t^\alpha}{\alpha}\right)u(t)\right]=\frac{s}{s+a\gamma\eta}\mathcal{K}_\alpha(s,\frac{s}{s+a\gamma\eta},\eta).$$

3) *Integral property*

$$\mathbb{KH}_\alpha\left[\mathcal{I}^{(\alpha)}u(t)\right]=\frac{\gamma\eta}{s}\mathbb{KH}_\alpha\left[u(t)\right].$$

4) *Convolution property*

$$\mathbb{KH}_\alpha\left[\left(u*v\right)(t)\right]=\frac{\gamma\eta}{s}\mathcal{K}_\alpha(s,\gamma,\eta)\mathcal{H}_\alpha(s,\gamma,\eta),$$

where $u*v$ is a convolution of two functions defined by

$$\left(u*v\right)(t)=\int_0^t u(\tau)v(t-\tau)d\tau=\int_0^t u(t-\tau)v(\tau)d\tau.$$

Proof. 1) Using Definition 3.1, we get

$$\begin{aligned}\mathbb{KH}_\alpha\left[\lambda u(t)\pm\mu v(t)\right] &= \frac{s}{\gamma\eta}\int_0^\infty \exp\left(-\frac{st}{\gamma\eta}\right)(\lambda u(t)\pm\mu v(t))t^{\alpha-1}dt\\ &= \frac{s}{\gamma\eta}\int_0^\infty \exp\left(-\frac{st}{\gamma\eta}\right)\lambda u(t)t^{\alpha-1}dt\\ &\quad \pm\frac{s}{\gamma\eta}\int_0^\infty \exp\left(-\frac{st}{\gamma\eta}\right)\mu v(t))t^{\alpha-1}dt\\ &= \lambda\left(\frac{s}{\gamma\eta}\int_0^\infty \exp\left(-\frac{st}{\gamma\eta}\right)u(t)t^{\alpha-1}dt\right)\\ &\quad \pm\mu\left(\frac{s}{\gamma\eta}\int_0^\infty \exp\left(-\frac{st}{\gamma\eta}\right)v(t))t^{\alpha-1}dt\right)\\ &= \lambda\mathbb{KH}_\alpha\left[u(t)\right]\pm\mu\mathbb{KH}_\alpha\left[v(t)\right].\end{aligned}$$

2) Using Theorem 3.1, we get

$$\begin{aligned}\mathbb{KH}_\alpha\left[\exp\left(-a\frac{t^\alpha}{\alpha}\right)u(t)\right] &= \mathbb{KH}\left[\exp\left(-a\frac{\left((\alpha t)^{\frac{1}{\alpha}}\right)^\alpha}{\alpha}\right)u((\alpha t)^{\frac{1}{\alpha}})\right]\\ &= \mathbb{KH}\left[\exp\left(-at\right)u((\alpha t)^{\frac{1}{\alpha}})\right]\\ &= \frac{s}{\gamma\eta}\int_0^\infty \exp\left(-\frac{st}{\gamma\eta}\right)\exp\left(-at\right)u((\alpha t)^{\frac{1}{\alpha}})dt\\ &= \frac{s}{\gamma\eta}\int_0^\infty \exp\left(-\left(\frac{s}{\gamma\eta}+a\right)t\right)u((\alpha t)^{\frac{1}{\alpha}})dt\\ &= \frac{s}{\gamma\eta}\int_0^\infty \exp\left(-\left(\frac{s+a\gamma\eta}{\gamma\eta}\right)t\right)u((\alpha t)^{\frac{1}{\alpha}})dt. \qquad (2)\end{aligned}$$

If $(s+a\gamma\eta)\,t = sx$ and $t = \frac{sx}{s+a\gamma\eta}$, then we have $dt = \frac{s}{s+a\gamma\eta}dx$, so equation (2) becomes

$$\begin{aligned}\mathbb{KH}_\alpha\left[\exp\left(-a\frac{t^\alpha}{\alpha}\right)u(t)\right] &= \frac{s}{\gamma\eta}\int_0^\infty \exp\left(-\frac{sx}{\gamma\eta}\right)u\left(\left(\frac{\alpha sx}{s+a\gamma\eta}\right)^{\frac{1}{\alpha}}\right)\frac{s}{s+a\gamma\eta}dx\\ &= \frac{s}{s+a\gamma\eta}\left(\frac{s}{\gamma\eta}\int_0^\infty \exp\left(-\frac{sx}{\gamma\eta}\right)u\left(\left(\frac{\alpha sx}{s+a\gamma\eta}\right)^{\frac{1}{\alpha}}\right)dx\right)\\ &= \frac{s}{s+a\gamma\eta}\mathcal{K}_\alpha(s,\frac{s}{s+a\gamma\eta},\eta).\end{aligned}$$

3) Using Theorem 3.2, we get

$$\mathbb{KH}_\alpha\left[\mathcal{C}^{(\alpha)}\mathcal{I}^{(\alpha)}u(t)\right] = \frac{s}{\gamma\eta}\mathbb{KH}_\alpha\left[\mathcal{I}^{(\alpha)}u(t)\right] - \frac{s}{\gamma\eta}\mathcal{I}^{(\alpha)}u(0).$$

But $\mathcal{C}^{(\alpha)}\mathcal{I}^{(\alpha)}u(t) = u(t)$ and $\mathcal{I}^{(\alpha)}u(0) = 0$, so we have

$$\mathbb{KH}_\alpha\left[\mathcal{I}^{(\alpha)}u(t)\right] = \frac{\gamma\eta}{s}\mathbb{KH}_\alpha\left[u(t)\right].$$

4) Using Theorem 3.1 and the convolution definition, we get

$$\begin{aligned}\mathbb{KH}_\alpha\left[(u*v)(t)\right] &= \mathbb{KH}\left[(u*v)\left((\alpha t)^{\frac{1}{\alpha}}\right)\right]\\ &= \frac{s}{\gamma\eta}\int_0^\infty \exp\left(-\frac{st}{\gamma\eta}\right)(u*v)\left((\alpha t)^{\frac{1}{\alpha}}\right)dt \qquad (3)\\ &= \frac{s}{\gamma\eta}\int_0^\infty \exp\left(-\frac{st}{\gamma\eta}\right)\left(\int_0^t u\left(\alpha(t-\tau)^{\frac{1}{\alpha}}\right)v\left((\alpha\tau)^{\frac{1}{\alpha}}\right)d\tau\right)dt.\end{aligned}$$

From the region $R = \{(\tau,t)\in\mathbb{R}^2 : 0\le\tau\le t \text{ and } 0\le t\le+\infty\}$, we can change the order of integration, i.e., equation (3) becomes

$$\mathbb{KH}_\alpha\left[(u*v)(t)\right] = \frac{s}{\gamma\eta}\int_0^\infty\int_\tau^\infty \exp\left(-\frac{st}{\gamma\eta}\right)u\left(\alpha(t-\tau)^{\frac{1}{\alpha}}\right)v\left((\alpha\tau)^{\frac{1}{\alpha}}\right)d\tau dt. \qquad (4)$$

Substituting $x = t-\tau$ and $dx = dt$ in equation (4), we get

$$\begin{aligned}\mathbb{KH}_\alpha\left[(u_1*u_2)(t)\right] &= \frac{s}{\gamma\eta}\int_0^\infty\int_0^\infty \exp\left(-\frac{s(x+\tau)}{\gamma\eta}\right)u\left((\alpha x)^{\frac{1}{\alpha}}\right)v\left((\alpha\tau)^{\frac{1}{\alpha}}\right)d\tau dx\\ &= \frac{s}{\gamma\eta}\left(\int_0^\infty \exp\left(-\frac{sx}{\gamma\eta}\right)u\left((\alpha x)^{\frac{1}{\alpha}}\right)dx\right)\\ &\quad\times\left(\int_0^\infty \exp\left(-\frac{s\tau}{\gamma\eta}\right)v\left((\alpha\tau)^{\frac{1}{\alpha}}\right)d\tau\right)\\ &= \frac{\gamma\eta}{s}\left(\frac{s}{\gamma\eta}\int_0^\infty \exp\left(-\frac{sx}{\gamma\eta}\right)u\left((\alpha x)^{\frac{1}{\alpha}}\right)dx\right)\\ &\quad\times\left(\frac{s}{\gamma\eta}\int_0^\infty \exp\left(-\frac{s\tau}{\gamma\eta}\right)v\left((\alpha\tau)^{\frac{1}{\alpha}}\right)d\tau\right)\\ &= \frac{\gamma\eta}{s}\mathcal{K}_\alpha(s,\gamma,\eta)\mathcal{H}_\alpha(s,\gamma,\eta).\end{aligned}$$

6 Applications

Application 1 Consider the linear conformable fractional differential equation

$$\mathcal{C}^{(\alpha)}u(t) - u(t) = 1, 0 < \alpha \leq 1, \tag{5}$$

subject to the initial condition

$$u(0) = 0. \tag{6}$$

Applying the conformable fractional Khalouta transform on both sides of equation (5) and using Theorem 5.1, we get

$$\mathbb{KH}_\alpha \left[\mathcal{C}^{(\alpha)}u(t)\right] - \mathbb{KH}_\alpha \left[u(t)\right] = \mathbb{KH}_\alpha \left[1\right]. \tag{7}$$

When using Theorems 3.2 and 4.1, equation (7) becomes

$$\frac{s}{\gamma\eta}\mathbb{KH}_\alpha \left[u(t)\right] - \frac{s}{\gamma\eta}u(0) - \mathbb{KH}_\alpha \left[u(t)\right] = 1. \tag{8}$$

Substituting the initial condition (6) and simplifying equation (8), we have

$$\begin{aligned} \mathbb{KH}_\alpha \left[u(t)\right] &= \frac{\gamma\eta}{s-\gamma\eta} \\ &= \frac{s}{s-\gamma\eta} - 1. \end{aligned} \tag{9}$$

Taking the inverse conformable fractional Khalouta transform of both sides of equation (9), we get

$$u(t) = \exp\left(\frac{t^\alpha}{\alpha}\right) - 1. \tag{10}$$

For $\alpha = 1$, the result in equation (10) reduces to the exact solution for the standard form of equations (5) and (6) as follows:

$$u(t) = \exp(t) - 1.$$

Application 2 Consider the linear conformable fractional differential equation

$$\mathcal{C}^{(2\alpha)}u(t) - u(t) = \sin\left(\frac{2t^\alpha}{\alpha}\right), 0 < \alpha \leq 1, \tag{11}$$

subject to the initial conditions

$$u(0) = 2, \mathcal{C}^{(\alpha)}u(0) = 0. \tag{12}$$

Applying the conformable fractional Khalouta transform on both sides of equation (11) and using Theorem 5.1, we get

$$\mathbb{KH}_\alpha \left[\mathcal{C}^{(2\alpha)}u(t)\right] - \mathbb{KH}_\alpha \left[u(t)\right] = \mathbb{KH}_\alpha \left[\sin\left(\frac{2t^\alpha}{\alpha}\right)\right]. \tag{13}$$

When using Theorems 3.3 and 4.1, equation (13) becomes

$$\frac{s^2}{\gamma^2\eta^2}\mathbb{KH}_\alpha \left[u(t)\right] - \frac{s^2}{\gamma^2\eta^2}u(0) - \mathbb{KH}_\alpha \left[u(t)\right] = \frac{2s\gamma\eta}{s^2+4\gamma^2\eta^2}. \tag{14}$$

Substituting the initial conditions (12) and simplifying equation (14), we have

$$\begin{aligned}\mathbb{KH}_{\alpha}\left[u(t)\right] &= \frac{2s\gamma^{3}\eta^{3}}{\left(s^{2}+4\gamma^{2}\eta^{2}\right)\left(s^{2}-\gamma^{2}\eta^{2}\right)}+\frac{2s^{2}}{s^{2}-\gamma^{2}\eta^{2}} \\ &= -\frac{1}{5}\frac{2s\gamma\eta}{s^{2}+4\gamma^{2}\eta^{2}}+\frac{2}{5}\frac{s\gamma\eta}{s^{2}-\gamma^{2}\eta^{2}}+\frac{2s^{2}}{s^{2}-\gamma^{2}\eta^{2}}. \qquad (15)\end{aligned}$$

Taking the inverse conformable fractional Khalouta transform of both sides of equation (15), we get

$$u(t)=-\frac{1}{5}\sin\left(\frac{2t^{\alpha}}{\alpha}\right)+\frac{2}{5}\sinh\left(\frac{t^{\alpha}}{\alpha}\right)+2\cosh\left(\frac{t^{\alpha}}{\alpha}\right). \qquad (16)$$

For $\alpha=1$, the result in equation (16) reduces to the exact solution for the standard form of equations (11) and (12) as follows:

$$u(t)=-\frac{1}{5}\sin\left(2t\right)+\frac{2}{5}\sinh\left(t\right)+2\cosh\left(t\right).$$

Application 3 Consider the linear conformable fractional differential equation

$$\mathcal{C}^{(3\alpha)}u(t)+\mathcal{C}^{(\alpha)}u(t)=\frac{t^{\alpha}}{\alpha}, 0<\alpha\leq 1, \qquad (17)$$

subject to the initial conditions

$$u(0)=0, \mathcal{C}^{(\alpha)}u(0)=0, \mathcal{C}^{(2\alpha)}u(0)=0. \qquad (18)$$

Applying the conformable fractional Khalouta transform on both sides of equation (17) and using Theorem 5.1, we get

$$\mathbb{KH}_{\alpha}\left[\mathcal{C}^{(3\alpha)}u(t)\right]+\mathbb{KH}_{\alpha}\left[\mathcal{C}^{(\alpha)}u(t)\right]=\mathbb{KH}_{\alpha}\left[\frac{t^{\alpha}}{\alpha}\right]. \qquad (19)$$

When using Theorems 3.3 and 4.1, equation (19) becomes

$$\frac{s^{3}}{\gamma^{3}\eta^{3}}\mathbb{KH}_{\alpha}\left[u(t)\right]-\frac{s^{3}}{\gamma^{3}\eta^{3}}u(0)+\frac{s}{\gamma\eta}\mathbb{KH}_{\alpha}\left[u(t)\right]-\frac{s}{\gamma\eta}u(0)=\frac{\gamma\eta}{s}\Gamma\left(2\right). \qquad (20)$$

Substituting the initial conditions (18) and simplifying the equation (20), we have

$$\begin{aligned}\mathbb{KH}_{\alpha}\left[u(t)\right] &= \frac{\gamma^{4}\eta^{4}}{s^{2}\left(s^{2}+\gamma^{2}\eta^{2}\right)} \\ &= \frac{\gamma^{2}\eta^{2}}{s^{2}}+\frac{s^{2}}{s^{2}+\gamma^{2}\eta^{2}}-1 \\ &= \frac{1}{2}\frac{\gamma^{2}\eta^{2}}{s^{2}}\Gamma\left(3\right)+\frac{s^{2}}{s^{2}+\gamma^{2}\eta^{2}}-1. \qquad (21)\end{aligned}$$

Taking the inverse conformable fractional Khalouta transform of both sides of equation (21), we get

$$u(t)=\frac{t^{2\alpha}}{2\alpha^{2}}+\cos\left(\frac{t^{\alpha}}{\alpha}\right)-1. \qquad (22)$$

For $\alpha=1$, the result in equation (22) reduces to the exact solution for the standard form of equations (17) and (18) as follows:

$$u(t)=\frac{t^{2}}{2}+\cos\left(t\right)-1.$$

7 Conclusion

In this paper, we carefully proposed a new conformable fractional integral transform known as the conformable fractional Khalouta transform which presents a promising tool for solving fractional differential equations. The conformable fractional Khalouta transform was successfully applied to find solutions of conformable fractional differential equations. It can be concluded that the proposed methodology is very powerful and effective in finding analytical solutions for wide categories of fractional differential equations.

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