NONLINEAR DYNAMICS AND SYSTEMS THEORY
An International Journal of Research and Surveys
Published by InforMath Publishing Group since 2001

Volume 24 Number 5 2024

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Founded by A.A. Martynyuk in 2001.
Registered in Ukraine Number: KB No 5267 / 04.07.2001.

NONLINEAR DYNAMICS AND SYSTEMS THEORY

An International Journal of Research and Surveys

Impact Factor from SCOPUS for 2023: SJR – 0.298, SNIP – 1.032 and CiteScore – 1.6

Nonlinear Dynamics and Systems Theory (ISSN 1562–8353 (Print), ISSN 1813–7385 (Online)) is an international journal published under the auspices of the S.P. Timoshenko Institute of Mechanics of National Academy of Sciences of Ukraine and Curtin University of Technology (Perth, Australia). It aims to publish high quality original scientific papers and surveys in areas of nonlinear dynamics and systems theory and their real world applications.

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PUBLICATION AND SUBSCRIPTION INFORMATION

Nonlinear Dynamics and Systems Theory will have 6 issues in 2024, printed in hard copy (ISSN 1562–8353) and available online (ISSN 1813–7385), by InforMath Publishing Group, Nesterov str., 3, Institute of Mechanics, Kiev, MSP 680, Ukraine, 03057. Subscription prices are available upon request from the Publisher, EBSCO Information Services (`mailto:journals@ebsco.com`), or website of the Journal: `http://e-ndst.kiev.ua`. Subscriptions are accepted on a calendar year basis. Issues are sent by airmail to all countries of the world. Claims for missing issues should be made within six months of the date of dispatch.

ABSTRACTING AND INDEXING SERVICES

Papers published in this journal are indexed or abstracted in: Mathematical Reviews / MathSciNet, Zentralblatt MATH / Mathematics Abstracts, PASCAL database (INIST–CNRS) and SCOPUS.

Nonlinear Dynamics and Systems Theory, **24** (5) (2024) 431–441

On Localization of Spectrum of an Integro-Differential Convection-Diffusion-Reaction Operator

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Received: April 10, 2024; Revised: October 1, 2024

Abstract: This paper explores the spectral properties of a non-self-adjoint integral-differential operator defined on an unbounded domain. The operator is governed by the Dirichlet-type conditions. We utilize the pseudo-spectral theory to demonstrate that the operator's spectrum is localized in the real numbers.

Keywords: *non-self-adjoint operators; unbounded operators; spectral analysis; integral-differential operators.*

Mathematics Subject Classification (2010): 47Axx, 58Jxx, 35Gxx.

1 Introduction

Non-self-adjoint and unbounded operators are fundamental in numerous branches of physics and chemistry, where phenomena like convection, diffusion, and reactions are widespread, see [1–3] and references therein. In this study, we focus on the spectral analysis of a non-self-adjoint integral-differential operator of convection-diffusion-reaction type, defined on an unbounded domain and subject to the Dirichlet-type conditions. The operator under consideration, denoted as L, is defined by the expression

$$L\xi = -\Delta\xi + \begin{pmatrix} -y \\ -x \end{pmatrix} \cdot \nabla\xi + (x^2 + y^2)\xi + \int_\Gamma k(x, y, z, t)\xi(z, t)dzdt.$$

Convection equations can be considered as dynamic systems [4–6], where the state of the system evolves over time. They describe the transport of a quantity under the effect of a velocity field and can be analysed using the theory of dynamical systems and

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semi-groups of operators. This approach makes it possible to study the stability and asymptotic behaviour of the solutions. The spectral analysis of the associated operator provides information about the propagation of the initial perturbations.

This study is distinguished by the unbounded and non-self-adjoint characteristics of the operator, which render it a subject of great interest within this field of research [7, 8]. The primary contributions of this study lie in utilizing the pseudo-spectral theory, see [9, 10], to demonstrate that the spectrum of the operator L is localized in $\mathbb{R}$. This innovative approach provides a promising alternative to the traditional spectral theory, with potential implications across various application domains. Our methodology is based mainly on the pseudo-spectral theory, splitting the spatial domain into finite-dimensional domains, then returning to the limit and recovering all the spectral properties. This technique was used in [11, 12].

Nevertheless, despite the notable advancements made, this study is subject to certain limitations, particularly with regard to the assumptions made about the integral operator within the integral-differential operator. These assumptions may prove challenging to verify in practice, although their relevance remains compelling.

The structure of this paper is designed to provide a comprehensive understanding of the problem under study. We begin by defining the theoretical framework in Section 2, then proceed to examine the restriction of the operator L to a bounded domain to localize its spectrum in Section 3. Next, in Section 4, we explore the relationship between the operator L and its restriction using the pseudo-spectral theory.

2 General Framework

Let Γ be an open unbounded set in $\mathbb{R}^2$ defined as follows:

$$\Gamma = \left\{(x, y) \in \mathbb{R}^2 : x > 0 \text{ and } -x < y < x\right\},$$

with its boundary denoted by $\partial\Gamma$. We define the space $L^2(\Gamma)$, the Hilbert space of complex-valued (classes of) functions defined almost everywhere on Γ, provided with their usual inner product $\langle \cdot, \cdot \rangle$. Let L be the integro-differential operator defined on $L^2(\Gamma)$ by

$$L\xi = -\Delta\xi + \begin{pmatrix} -y \\ -x \end{pmatrix} .\nabla\xi + (x^2 + y^2)\xi + \int_\Gamma k(x, y, z, t)\xi(z, t)dzdt,$$

where k is a real-valued function defined on $\Gamma \times \Gamma$, satisfying

$$(H) \left\| \begin{array}{l} i)\ \forall(x, y), (z, t) \in \Gamma : |k(x, y, z, t)| \leq k_1(x, y)k_2(z, t), \\ \\ ii)\ k_1 \in L_\infty(\Gamma) \text{ and } k_2 \in L^2(\Gamma), \\ \\ iii)\ \forall(x, y), (z, t) \in \Gamma, \quad e^{xy}k(x, y, z, t) = e^{zt}k(z, t, x, y). \end{array} \right.$$

The operator T is given as follows:

$$T\xi = -\Delta\xi + \begin{pmatrix} -y \\ -x \end{pmatrix} \cdot \nabla\xi + (x^2 + y^2),$$

where this operator falls into the category of convection-diffusion operators, see [13]. Additionally, the operator $\mathcal{K}$ is defined as follows:

$$\forall\xi \in L^2(\Gamma), \forall(x, y) \in \Gamma, \quad \mathcal{K}\xi(x, y) = \int_\Gamma k(x, y, z, t)\xi(z, t)\, dz\, dt$$

representing the integral operator associated with the reaction term [14]. The sesquilinear form q is defined as follows:

$$q(\xi_1,\xi_2)=\int_\Gamma \nabla\xi_1.\overline{\nabla\xi_2}dxdy+\int_\Gamma \begin{pmatrix} -y \\ -x \end{pmatrix}.\nabla\xi_1\bar{\xi_2}dxdy+\int_\Gamma (x^2+y^2)\xi_1\bar{\xi_2}dxdy+\langle\mathcal{K}\xi_1,\xi_2\rangle,$$

and $\mathcal{Q}$ the quadratic form associated with q, is

$$\mathcal{Q}(\xi)=\int_\Gamma |\nabla\xi|^2dxdy+\int_\Gamma \begin{pmatrix} -y \\ -x \end{pmatrix}.\nabla\xi\bar{\xi}dxdy+\int_\Gamma (x^2+y^2)|\xi|^2dxdy+\langle\mathcal{K}\xi,\xi\rangle.$$

Using the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}\left|\int_\Gamma \begin{pmatrix} -y\bar{\xi} \\ -x\bar{\xi} \end{pmatrix}.\nabla\xi dxdy\right| &\leq \int_\Gamma \left|\begin{pmatrix} -y\bar{\xi} \\ -x\bar{\xi} \end{pmatrix}\right|.|\nabla\xi|dxdy \\ &\leq \frac{1}{2}\left(\|\nabla\xi\|^2_{L^2(\Gamma)}+\int_\Gamma (x^2+y^2)|\xi|^2dxdy\right).\end{aligned}$$

Hence, q is a sectorial form defined on the linear space

$$V=H^1_0(\Gamma)\cap\Big\{\,\xi\in L^2(\Gamma):\int_\Gamma (x^2+y^2)|\xi|^2dxdy<+\infty\,\Big\}$$

and L is the operator associated with q and its domain is

$$D(L)=H^2(\Gamma)\cap H^1_0(\Gamma)\cap\Big\{\,\xi\in L^2(\Gamma):\int_\Gamma (x^2+y^2)|\xi|^2dxdy<+\infty\,\Big\}.$$

Consider the eigenvalue problem, which represents the main problem addressed in this paper:

$$(P)\begin{cases} \text{Find } (\lambda,\xi)\in(\mathbb{C},D(L)\backslash\{0\}): \\ L\xi=\lambda\xi \text{ on } \Gamma, \\ \xi=0 \text{ on } \partial\Gamma. \end{cases}$$

We define the decreasing family $\{\Gamma_\eta\}_{0<\eta<1}$ of open bounded sets of $\mathbb{R}^2$ as

$$\Gamma_\eta=\left\{(x,y)\in\mathbb{R}^2:\eta<x<\eta^{-1}\text{and}-(1-\eta)(x-\eta)<y<(1-\eta)(x-\eta)\right\},$$

this family converges to Γ when η tends to 0. For all $\eta\in]0,1[$, we define on $L^2(\Gamma_\eta)$ the sesquilinear form q_η by

$$\begin{aligned}q_\eta(\xi_1,\xi_2)=&\textstyle\int_{\Gamma_\eta}\nabla\xi_1.\overline{\nabla\xi_2}dxdy+\int_{\Gamma_\eta}\begin{pmatrix} -y \\ -x \end{pmatrix}.\nabla\xi_1\bar{\xi_2}dxdy \\ &\textstyle+\int_{\Gamma_\eta}(x^2+y^2)\xi_1\bar{\xi_2}dxdy+<\mathcal{K}_\eta\xi_1,\xi_2>,\end{aligned}$$

where

$$\mathcal{K}_\eta\xi=\int_{\Gamma_\eta}k_\eta(x,y,z,t)\xi(z,t)dzdt,$$

and k_η is the restriction of k in Γ_η. It is evident that $k_\eta\in L^2(\Gamma_\eta\times\Gamma_\eta)$. To avoid any confusion, $\langle\cdot,\cdot\rangle$ is the usual inner product defined on $L^2(\Gamma_\eta)$. We note also that q_η is a sectorial form defined on $H^1_0(\Gamma_\eta)$ and the operator associated with q_η is L_η , defined on

$$D(L_\eta)=H^2(\Gamma_\eta)\cap H^1_0(\Gamma_\eta).$$

3 Spectrum of L_η

This section will examine the spectrum of the operator L_η. The results are presented in Theorem 1. We begin by defining the inner product on $L^2(\Gamma_\eta)$ by

$$\langle \xi_1, \xi_2 \rangle_\eta = \int_{\Gamma_\eta} e^{xy} \xi_1(x,y) \overline{\xi_2(x,y)} dxdy,$$

where its associated norm is denoted by $\|.\|_\eta$, which is equivalent to the usual norm $\|.\|_{L^2(\Gamma_\eta)}$. Note that the spectrum $sp(L)$ is defined as

$$sp(L) = \left\{ z \in \mathbb{C} : (L - zI)^{-1} \text{ is not bounded operator } \right\},$$

and so, $sp_p(L)$ consists only of the eigenvalues of L. Finally, $sp_{ess}(L) = sp(L) \setminus sp_p(L)$.

Lemma 3.1 *For all $\eta \in]0,1[$, L_η is self-adjoint with respect to $\langle \cdot, \cdot \rangle_\eta$.*

Proof. Let $\xi \in D(L_\eta)$, for all $(x,y) \in \Gamma_\eta$, we define $\tilde{\xi}(x,y) = e^{\frac{xy}{2}} \xi(x,y)$. So, we obtain that

$$\Delta\tilde{\xi} = (\Delta\xi + y\partial_x\xi + x\partial_y\xi + \frac{1}{4}(x^2+y^2)\xi)e^{\frac{xy}{2}}.$$

Let $\xi_1, \xi_2 \in D(L_\eta)$. By the Green formula and using the above equation, we get

$$\langle T_\eta \xi_1, \xi_2 \rangle_\eta = \int_{\Gamma_\eta} \nabla\tilde{\xi_1}.\overline{\nabla\tilde{\xi_2}} dxdy + \int_{\Gamma_\eta} \frac{5}{4}(x^2+y^2)\tilde{\xi_1}\overline{\tilde{\xi_2}} dxdy. \tag{1}$$

On the other hand, under the assumption (H), we get

$$\langle \mathcal{K}_\eta \xi_1, \xi_2 \rangle_\eta = \langle \xi_1, \mathcal{K}_\eta \xi_2 \rangle_\eta. \tag{2}$$

Consequently, from (1) and (2), the operator L_η is self-adjoint with respect to $\langle \cdot, \cdot \rangle_\eta$.

□

As a result, $sp(L_\eta)$ is a real value. Because of the impossibility of extending the inner product $\langle \cdot, \cdot \rangle_\eta$ over $L^2(\Gamma)$, it is not possible to ensure that L is self-adjoint.

We define the coefficient $C_{PF} = \frac{d(\Gamma_\eta)}{\sqrt{2}}$, where d is a measure on $\mathbb{R}^2$. The coefficient C_{PF} is known as the Poincaré-Friedrich constant [15]. The following theorem localises the essential and point spectra in the real line for the operators L_η.

Theorem 3.1 *For all $\eta \in]0,1[$, the essential spectrum of L_η, $sp_{ess}(L_\eta)$ is included in $\left] \frac{\eta^2}{2} - \|k_\eta\|_{L^2(\Gamma_\eta \times \Gamma_\eta)}, +\infty \right[$, and the point spectrum of L_η, $sp_p(L_\eta)$ is included in $[C_{PF}^{-2} + \eta^2 - \|k_\eta\|_{L^2(\Gamma_\eta \times \Gamma_\eta)}, +\infty[$.*

Proof. For all $\eta \in]0,1[$ and $\xi \in D(L_\eta)$, we have

$$\begin{aligned} Re(\langle L_\eta \xi, \xi \rangle) &= \frac{1}{2}(\langle L_\eta \xi, \xi \rangle + \overline{\langle L_\eta \xi, \xi \rangle}) = \frac{1}{2}(\langle L_\eta \xi, \xi \rangle + \langle \xi, L_\eta \xi \rangle) \\ &= Re(\langle A_\eta \xi, \xi \rangle) + Re(\langle \mathcal{K}_\eta \xi, \xi \rangle). \end{aligned}$$

So, using the Green formula and integrating by parts, we get

$$\begin{aligned}\langle A_\eta \xi, \xi\rangle &= \int_{\Gamma_\eta} (-\Delta\xi - y\partial_x\xi - x\partial_y\xi + (x^2+y^2)\xi).\bar{\xi}dxdy \\ &= \int_{\Gamma_\eta} \nabla\xi.\nabla\bar{\xi}dxdy - \int_{\partial\Gamma_\eta} \bar{\xi}\frac{\partial\xi}{\partial n}ds + \int_{\Gamma_\eta} y\xi\partial_x\bar{\xi}dxdy \\ &\quad - \int_{\Gamma_\eta} y\partial_x(\xi\bar{\xi})dxdy + \int_{\Gamma_\eta} x\xi\partial_y\bar{\xi}dxdy - \int_{\Gamma_\eta} x\partial_y(\xi\bar{\xi})dxdy \\ &\quad + \int_{\Gamma_\eta} (x^2+y^2)|\xi|^2dxdy,\end{aligned}$$

since $\xi \in H_0^1(\Gamma_\eta)$, this implies $\xi = 0$ a.e on $\partial\Gamma_\eta$, we simplify certain terms as

$$\int_{\partial\Gamma_\eta} \bar{\xi}\frac{\partial\xi}{\partial n}ds = 0, \ \int_{\Gamma_\eta} x\partial_y(\xi\bar{\xi})dxdy = \int_\eta^{\frac{1}{\eta}} [x\xi\bar{\xi}]_{-(1-\eta)(x-\eta)}^{(1-\eta)(x-\eta)}dx = 0,$$

and

$$\int_{\Gamma_\eta} y\partial_x(\xi\bar{\xi})dxdy = \int_{\frac{1-\eta}{\eta}}^{-\frac{1-\eta}{\eta}} [y\xi\bar{\xi}]_{\frac{y}{1-\eta}}^{\frac{1}{\eta}}dy = 0.$$

So,

$$\langle A_\eta\xi, \xi\rangle = \int_{\Gamma_\eta} (|\nabla\xi|^2 d - y\xi\overline{\partial_x\xi} - x\xi\overline{\partial_y\xi} + (x^2+y^2)|\xi|^2)dxdy. \tag{3}$$

With the same argument, we find

$$\langle \xi, A_\eta\xi\rangle = \int_{\Gamma_\eta} (|\nabla\xi|^2 + y\xi\overline{\partial_x\xi} + x\xi\overline{\partial_y\xi} + (x^2+y^2)|\xi|^2)dxdy. \tag{4}$$

Thus, by adding (3) to (4) and using the Poincaré inequality, we get

$$Re(\langle A_\eta\xi, \xi\rangle) = \int_{\Gamma_\eta} (|\nabla\xi|^2 + (x^2+y^2)|\xi|^2)dxdy \geq (C_{PF}^{-2} + \eta^2)\,\|\xi\|^2_{L^2(\Gamma_\eta)}.$$

Now, let us estimate the term $Re(\langle \mathcal{K}_\eta\xi, \xi\rangle)$. By applying the Cauchy-Schwarz inequality twice, we obtain

$$\begin{aligned}|Re\,(\langle \mathcal{K}_\eta\xi, \xi\rangle)| &\leq |\langle \mathcal{K}_\eta\xi, \xi\rangle| \\ &\leq \left(\int_{\Gamma_\eta}\left(\int_{\Gamma_\eta} k_\eta(x,y,z,t)\xi(z,t)dzdt\right)^2 dxdy\right)^{\frac{1}{2}} \left(\int_{\Gamma_\eta} \xi^2(x,y)dxdy\right)^{\frac{1}{2}} \\ &\leq \left(\int_{\Gamma_\eta}\int_{\Gamma_\eta} k_\eta^2(x,y,z,t)dxdydzdt\right)^{\frac{1}{2}} \left(\int_{\Gamma_\eta} \xi^2(z,t)dzdt\right)^{\frac{1}{2}} \\ &\quad \times \left(\int_{\Gamma_\eta} \xi^2(x,y)dxdy\right)^{\frac{1}{2}} \\ &\leq \|k_\eta\|_{L^2(\Gamma_\eta\times\Gamma_\eta)}\|\xi\|^2_{L^2(\Gamma_\eta)}.\end{aligned}$$

Thus,

$$Re(\langle \mathcal{K}_\eta \xi, \xi \rangle) \geq -\|k_\eta\|_{L^2(\Gamma_\eta \times \Gamma_\eta)} \|\xi\|^2_{L^2(\Gamma_\eta)},$$

then

$$Re(\langle L_\eta \xi, \xi \rangle) \geq (C_{PF}^{-2} + \eta^2 - \|k_\eta\|_{L^2(\Gamma_\eta \times \Gamma_\eta)}) \|\xi\|^2_{L^2(\Gamma_\eta)}. \tag{5}$$

For all $\lambda \in \mathbb{R}$ such that $\lambda < C_{PF}^{-2} + \eta^2 - \|k_\eta\|_{L^2(\Gamma_\eta \times \Gamma_\eta)}$ and $\xi \in D(L_\eta)$, we have the result

$$\|(L_\eta - \lambda I)\xi\|_{L^2(\Gamma_\eta)} \geq (C_{PF}^{-2} + \eta^2 - \|k_\eta\|_{L^2(\Gamma_\eta \times \Gamma_\eta)} - \lambda) \|\xi\|_{L^2(\Gamma_\eta)}. \tag{6}$$

Indeed, for all $\lambda \in \mathbb{R}$,

$$\|(L_\eta - \lambda I)\xi\|^2_{L^2(\Gamma_\eta)} = \|L_\eta \xi\|^2_{L^2(\Gamma_\eta)} - 2\lambda Re(< L_\eta \xi, \xi >) + \lambda^2 \|\xi\|^2_{L^2(\Gamma_\eta)} \geq 0, \tag{7}$$

then $(Re(\langle L_\eta \xi, \xi \rangle))^2 \leq \|\xi\|^2_{L^2(\Gamma_\eta)} \|L_\eta \xi\|^2_{L^2(\Gamma_\eta)}$, which implies that

$$\|L_\eta \xi\|^2_{L^2(\Gamma_\eta)} \geq \frac{(Re(\langle L_\eta \xi, \xi \rangle))^2}{\|\xi\|^2_{L^2(\Gamma_\eta)}}, \ \xi \in D(L_\eta) \setminus \{0\}.$$

Injecting the last inequality in (7), we get

$$\|(L_\eta - \lambda I)\xi\|^2_{L^2(\Gamma_\eta)} \geq (\frac{(Re(\langle L_\eta \xi, \xi \rangle))}{\|\xi\|^2_{L^2(\Gamma_\eta)}} - \lambda)^2 \|\xi\|^2_{L^2(\Gamma_\eta)},$$

by (5), for all $\lambda \in \mathbb{R}$ such that $\lambda < C_{PF}^{-2} + \eta^2 - \|k_\eta\|_{L^2(\Gamma_\eta \times \Gamma_\eta)}$, we find

$$\|(L_\eta - \lambda I)\xi\|^2_{L^2(\Gamma_\eta)} \geq (C_{PF}^{-2} + \eta^2 - \|k_\eta\|_{L^2(\Gamma_\eta \times \Gamma_\eta)} - \lambda)^2 \|\xi\|_{L^2(\Gamma_\eta)},$$

we conclude (6).

The operator $L_\eta - \lambda I$ is injective for $\lambda < C_{PF}^{-2} + \eta^2 - \|k_\eta\|_{L^2(\Gamma_\eta \times \Gamma_\eta)}$, which means that the point spectrum of L_η is included in $[C_{PF}^{-2} + \eta^2 - \|k_\eta\|_{L^2(\Gamma_\eta \times \Gamma_\eta)}, +\infty[$.

Now, we define the problems (P_η) and $(\tilde{P}_\eta)$ as follows:

$$(P_\eta) \begin{cases} \text{for all } g \in L^2(\Gamma_\eta), \text{ find } \xi \in D(L_\eta) \backslash \{0\} : \\ L_\eta \xi - \lambda \xi = g \text{ on } \Gamma_\eta, \\ \xi = 0 \text{ on } \partial \Gamma_\eta \end{cases}$$

and

$$(\tilde{P}_\eta) \begin{cases} \text{find } \xi \in H_0^1(\Gamma_\eta) \backslash \{0\} : \\ a_{\lambda,\eta}(\xi, v) = l(v) \text{ for } v \in H_0^1(\Gamma_\eta), \end{cases}$$

where

$$a_{\lambda,\eta}(\xi, v) = \int_{\Gamma_\eta} \nabla \xi . \overline{\nabla v} dxdy + \int_{\Gamma_\eta} \begin{pmatrix} -y \\ -x \end{pmatrix} . \nabla \xi \bar{v} dxdy + \int_{\Gamma_\eta} (x^2 + y^2 - \lambda) \xi \bar{v} dxdy + \langle \mathcal{K}_\eta \xi, v \rangle,$$

and

$$l(v) = \int_{\Gamma_\eta} g \tilde{v} dxdy.$$

The sesquilinear form $a_{\lambda,\eta}(\cdot,\cdot)$ is continuous and coercive in $H_0^1(\Gamma_\eta)$ for $\lambda < \frac{\eta^2}{2} - \|k_\eta\|_{L^2(\Gamma_\eta\times\Gamma_\eta)}$. Indeed, using the Cauchy-Schawrz inequality

$$\begin{aligned}|a_{\lambda,\eta}(\xi,v)| &\leq \|\nabla\xi\|_{L^2(\Gamma_\eta)}\|\nabla v\|_{L^2(\Gamma_\eta)} + \frac{1}{2}\Big(\|\nabla\xi\|_{L^2(\Gamma_\eta)}\|\nabla v\|_{L^2(\Gamma_\eta)} + \\ &\int_{\Gamma_\eta}(x^2+y^2)\xi\bar{v}dxdy\Big) + \int_{\Gamma_\eta}(x^2+y^2-\lambda)\xi\bar{v}dxdy \\ &+\|k_\eta\|_{L^2(\Gamma_\eta\times\Gamma_\eta)}\|\xi\|_{L^2(\Gamma_\eta)}\|v\|_{L^2(\Gamma_\eta)} \\ &\leq \frac{3}{2}\|\nabla\xi\|_{L^2(\Gamma_\eta)}\|\nabla v\|_{L^2(\Gamma_\eta)} + C_1\|\xi\|_{L^2(\Gamma_\eta)}\|v\|_{L^2(\Gamma_\eta)} \\ &+\|k_\eta\|_{L^2(\Gamma_\eta\times\Gamma_\eta)}\|\xi\|_{L^2(\Gamma_\eta)}\|v\|_{L^2(\Gamma_\eta)},\end{aligned}$$

where $C_1 = \sup_{\Gamma_\eta}\left\{\frac{3}{2}(x^2+y^2)-\lambda\right\}$, we have

$$|a_{\lambda,\eta}(\xi,v)| \leq C(\eta,\lambda,\|k_\eta\|_{L^2(\Gamma_\eta\times\Gamma_\eta)})\|\xi\|_{H^1(\Gamma_\eta)}\|v\|_{H^1(\Gamma_\eta)}$$

with

$$C(\eta,\lambda,\|k_\eta\|_{L^2(\Gamma_\eta\times\Gamma_\eta)}) = \max\{\frac{3}{2}, C_1 + \|k_\eta\|_{L^2(\Gamma_\eta\times\Gamma_\eta)}\}.$$

For the coercivity of $a_{\lambda,\eta}(.,.)$, we have

$$\langle\mathcal{K}_\eta\xi,\xi\rangle \geq -\|k_\eta\|_{L^2(\Gamma_\eta\times\Gamma_\eta)}\|\xi\|^2_{L^2(\Gamma_\eta)}.$$

So,

$$\begin{aligned}a_{\lambda,\eta}(\xi,\xi) &= \int_{\Gamma_\eta}|\nabla\xi|^2dxdy + \int_{\Gamma_\eta}\begin{pmatrix}-y\\-x\end{pmatrix}.\nabla\xi\bar{\xi}dxdy \\ &+\int_{\Gamma_\eta}(x^2+y^2-\lambda)|\xi|^2dxdy + \langle\mathcal{K}_\eta\xi,\xi\rangle \\ &\geq \|\nabla\xi\|^2_{L^2(\Gamma_\eta)} - \frac{1}{2}\left(\|\nabla\xi\|^2_{L^2(\Gamma_\eta)} + \int_{\Gamma_\eta}(x^2+y^2)|\xi|^2dxdy\right) \\ &+\int_{\Gamma_\eta}(x^2+y^2-\lambda)|\xi|^2dxdy - \|k_\eta\|_{L^2(\Gamma_\eta\times\Gamma_\eta)}\|\xi\|^2_{L^2(\Gamma_\eta)} \\ &\geq \frac{1}{2}\|\nabla\xi\|^2_{L^2(\Gamma_\eta)} + \min_{\Gamma_\eta}\left\{\frac{1}{2}(x^2+y^2)-\lambda\right\}\|\xi\|^2_{L^2(\Gamma_\eta)} - \|k_\eta\|_{L^2(\Gamma_\eta\times\Gamma_\eta)}\|\xi\|^2_{L^2(\Gamma_\eta)} \\ &\geq \frac{1}{2}\|\nabla\xi\|^2_{L^2(\Gamma_\eta)} + \left(\frac{\eta^2}{2} - \lambda - \|k_\eta\|_{L^2(\Gamma_\eta\times\Gamma_\eta)}\right)\|\xi\|^2_{L^2(\Gamma_\eta)}.\end{aligned}$$

Then

$$a_{\lambda,\eta}(\xi,\xi) \geq \min\left\{\frac{1}{2}, \frac{\eta^2}{2} - \lambda - \|k_\eta\|_{L^2(\Gamma_\eta\times\Gamma_\eta)}\right\}\|\xi\|^2_{H_0^1(\Gamma_\eta)}.$$

Otherwise, the semilinear form l is continuous in $H_0^1(\Gamma_\eta)$. Therefore, the Lax-Milgram theorem gives that, for all $g \in L^2(\Gamma_\eta)$ and for all λ such that $\lambda < \frac{1}{2}\eta^2 - \|k_\eta\|_{L^2(\Gamma_\eta\times\Gamma_\eta)}$, the problem $(\tilde{P}_\eta)$ has a unique solution $u \in H_0^1(\Gamma_\eta)$ for all $v \in H_0^1(\Gamma_\eta)$. This solution also satisfy the problem (P_η). We conclude that (P_η) has unique solution, for all $\eta \in]0,1[$. this completes the proof.

□

4 Pseudo-Spectra and Spectra of L

In this section, we will establish the relationships concerning the spectrum and the pseudo-spectrum of the operators L and L_η. We need to show an important density lemma. We denote by $\mathcal{D}(\Gamma)$ the space of infinitly differentiable functions defined on Γ with compact support in it, and we denote by $\|\cdot\|_L$ the graph norm of the operator L, which is defined by $\|\cdot\|_L = \|L\cdot\|_{L^2(\Gamma)} + \|\cdot\|_{L^2(\Gamma)}$.

Lemma 4.1 *$\mathcal{D}(\Gamma)$ is dense in $D(L)$ with respect to the graph norm $\|.\|_L$.*

Proof. See [12].

We will show an important set equality, which gives us a relationship between the pseudo-spectrum of L and the pseudo-spectrum of L_η. The definition of the pseudo-spectrum of an unbounded linear operator A on a Hilbert space H, denoted $sp_\epsilon(A)$, is the set of $\lambda \in \mathbb{C}$ for which there exists a vector $x \in H$ with $\|x\| = 1$ such that

$$\|(A - \lambda I)x\| < \epsilon.$$

Formally, this is written as

$$sp_\epsilon(A) = \{\lambda \in \mathbb{C} : \exists x \in H, \|x\| = 1, \|(A - \lambda I)x\| < \epsilon\}.$$

This definition indicates that for each λ in the pseudo-spectrum, there exists a unit vector x such that the action of $A - \lambda I$ on x is very small. In other words, λ is almost an eigenvalue of A in the sense that A acts on x almost like multiplication by λ, see [10].

Theorem 4.1 *For all $\varepsilon > 0$, we have the relation*

$$sp_\varepsilon(L) = \bigcup_{0<\eta<1} sp_\varepsilon(L_\eta).$$

Proof. Let $\lambda \in \bigcup_{0<\eta<1} sp_\varepsilon(L_\eta)$. So, there exists $\eta_1 \in]0,1[$ such that $\lambda \in sp_\varepsilon(L_{\eta_1})$. But the operator L_{η_1} is self-adjoint, for that, there exists $u \in D(L_{\eta_1})$ with $\|\xi\|_{L^2(\Gamma_{\eta_1})} = 1$ such that

$$\|(L_{\eta_1} - \lambda I)\xi\|_{L^2(\Gamma_{\eta_1})} < \varepsilon.$$

On the other hand, we have $\mathcal{D}(\Gamma_{\eta_1})$ is dense in $D(L_{\eta_1})$ with respect to the graph norm (see Lemma 4.1). Then, there exists a sequence $(\xi_n)_{n\in\mathbb{N}}$ in $\mathcal{D}(\Gamma_{\eta_1})$ such that

$$\lim_{n\longrightarrow+\infty} \|\xi_n - \xi\|_{L_{\eta_1}} = 0,$$

where $\|\xi\|_{L_{\eta_1}} = \|\xi\|_{L^2(\Gamma_{\eta_1})} + \|L_{\eta_1}\xi\|_{L^2(\Gamma_{\eta_1})}$ is the graph norm.

As a result, $((L_{\eta_1} - \lambda I)\xi_n)_{n\in\mathbb{N}}$ converges to $(L_{\eta_1} - \lambda I)\xi$ in $L^2(\Gamma_{\eta_1})$. Now, for all $\theta > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have

$$\left| \frac{\|(L_{\eta_1} - \lambda I)\xi_n\|_{L^2(\Gamma_{\eta_1})}}{\|\xi_n\|_{L^2(\Gamma_{\eta_1})}} - \frac{\|(L_{\eta_1} - \lambda I)\xi\|_{L^2(\Gamma_{\eta_1})}}{\|\xi\|_{L^2(\Gamma_{\eta_1})}} \right| < \theta.$$

Let $\theta = \varepsilon - \|(L_{\eta_1} - \lambda I)\xi\|_{L^2(\Gamma_{\eta_1})} > 0$. Then there exists $n_0 \in \mathbb{N}$ such that

$$\frac{\|(L_{\eta_1} - \lambda I)\xi_{n_0}\|_{L^2(\Gamma_{\eta_1})}}{\|\xi_{n_0}\|_{L^2(\Gamma_{\eta_1})}} < \varepsilon,$$

we find $\xi_{n_0} \in \mathcal{D}(\Gamma_{\eta_1})$. Then we extend ξ_{n_0} by zero to Γ , we denote its extension by $\tilde{\xi}_{n_0}$. So, it is clear that $\tilde{\xi}_{n_0} \in \mathcal{D}(\Gamma) \subset D(L)$ and we have $\|\tilde{\xi}_{n_0}\|_{L^2(\Gamma_{\eta_1})} = \|\xi_{n_0}\|_{L^2(\Gamma_{\eta_1})}$. Let $v_{n_0} = \frac{\xi_{n_0}}{\|\xi_{n_0}\|} \in \mathcal{D}(\Gamma_{\eta_1})$ and its extension by zero to Γ is defined by $\tilde{v}_{n_0} = \frac{\tilde{\xi}_{n_0}}{\|\tilde{\xi}_{n_0}\|} \in \mathcal{D}(\Gamma)$. Then

$$\|(L - \lambda I)\tilde{v}_{n_0}\|_{L^2(\Gamma)} = \|(L_{\eta_1} - \lambda I)v_{n_0}\|_{L^2(\Gamma_{\eta_1})} = \frac{\|(L_{\eta_1} - \lambda I)\xi_{n_0}\|_{L^2(\Gamma_{\eta_1})}}{\|\xi_{n_0}\|_{L^2(\Gamma_{\eta_1})}} < \varepsilon.$$

So, we find that $\lambda \in sp_\varepsilon(L)$.

Reciprocally, let $\lambda \in sp_\varepsilon(L)$. There exists $\xi \in D(L)$ with $\|\xi\|_{L^2(\Gamma)} = 1$ such that

$$\|(L - \lambda I)\xi\|_{L^2(\Gamma)} < \varepsilon.$$

By Lemma 4.1, there exists a sequence $(\xi_n)_{n\in\mathbb{N}}$ in $\mathcal{D}(\Gamma)$ such that $\|\xi_n - \xi\|_L \to 0$. According to the same arguments as above, there exists $n_1 \in \mathbb{N}$ such that

$$\frac{\|(L - \lambda I)\xi_{n_1}\|_{L^2(\Gamma)}}{\|\xi_{n_1}\|_{L^2(\Gamma)}} < \varepsilon, \tag{8}$$

as $\operatorname{supp}\xi_{n_1} \subset \Gamma$, there exists $\eta_0 \in]0,1[$ such that $\operatorname{supp}\xi_{n_1} \subset \Gamma_{\eta_0}$. We get from (8) that

$$\|(L_{\eta_0} - \lambda I)v_{n_1}\|_{L^2(\Gamma_{\eta_0})} < \varepsilon, \text{ and } v_{n_1} \in D(L_{\eta_0}),$$

where $v_{n_1} = \frac{\xi_{n_1}}{\|\xi_{n_1}\|_{L^2(\Gamma_{\eta_0})}}$. We conclude that $\lambda \in sp_\varepsilon(L_{\eta_0})$. This completes the proof. □

Remark 4.1 It is clear that the operators $(L_\eta)_{\eta\in]0,1[}$ are normal operators with respect to $\langle\cdot,\cdot\rangle_\eta$.

The following theorems are the main focus of our results. The ε-neighborhood of a set S in $\mathbb{C}$ is denoted by $N_\varepsilon(S)$.

Theorem 4.2 *For all $\varepsilon > 0$, we obtain the relation*

$$\bigcup_{0<\eta<1} sp_\varepsilon(L_\eta) = N_\varepsilon(\bigcup_{0<\eta<1} sp(L_\eta)).$$

Proof. Let $\lambda \in \bigcup_{0<\eta<1} sp_\varepsilon(L_\eta)$. There exists $\eta_1 \in]0,1[$ such that

$$\lambda \in sp_\varepsilon(L_{\eta_1}) = N_\varepsilon(sp(L_{\eta_1})).$$

So, $\lambda = z + s$, where $s \in sp(L_{\eta_1})$ and $|z| < \varepsilon$. But $s \in \bigcup_{0<\eta<1} sp(L_\eta)$, this implies that

$$\lambda = z + s \in N_\varepsilon(\bigcup_{0<\eta<1} sp(L_\eta)).$$

Reciprocally, let $\lambda \in N_\varepsilon(\bigcup_{0<\eta<1} sp(L_\eta))$. Then $\lambda = z + s$, where $s \in \bigcup_{0<\eta<1} sp(L_\eta)$ and $|z| < \varepsilon$. There is $\eta_2 \in]0,1[$ such that $\lambda = z + s \in N_\varepsilon(sp(L_{\eta_2})) = sp_\varepsilon(L_{\eta_2})$. Then

$$\lambda \in \bigcup_{0<\eta<1} sp_\varepsilon(L_\eta).$$

□

The next theorem is our main result, characterising the spectrum of the operator L as a union of operators spectrum of L_η; $\eta \in]0,1[$, which allows to determine that the spectrum of the operator L is purely real.

Theorem 4.3 *The spectrum of L is localized in $\mathbb{R}$, where*

$$sp(L) = \bigcup_{0<\eta<1} sp(L_\eta).$$

Proof. We apply Theorem 4.1 and Theorem 4.2, we find

$$sp_\varepsilon(L) = \bigcup_{0<\eta<1} sp_\varepsilon(L_\eta) = N_\varepsilon(\bigcup_{0<\eta<1} sp(L_\eta)).$$

We use the propriety $\bigcap_{0<\varepsilon<1} N_\varepsilon(S) = S$, where S is a set in $\mathbb{C}$. Also, we use the fact that

$$sp(L) = \bigcap_{0<\varepsilon<1} sp_\varepsilon(L).$$

Then we conclude that

$$sp(L) = \bigcap_{0<\varepsilon<1} sp_\varepsilon(L) = \bigcap_{0<\varepsilon<1} N_\varepsilon(\bigcup_{0<\eta<1} sp(L_\eta)) = \bigcup_{0<\eta<1} sp(L_\eta).$$

This completes the proof. □

Acknowledgment

We extend our heartfelt appreciation to the reviewers and editors for their insightful feedback and valuable contributions to improving the quality of this manuscript. Their expertise and dedication greatly enhanced the final outcome of our work.

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Nonlinear Dynamics and Systems Theory, 24 (5) (2024) 442–459

A Priori Predictions for a Weak Solution to Time-Fractional Nonlinear Reaction-Diffusion Equations Incorporating an Integral Condition

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Received: February 3, 2024; Revised: September 21, 2024

Abstract: Within this paper, we lay out the necessary criteria that ensure a solution's presence and distinctiveness within a functionally weighted Sobolev space. This pertains to a specific group of initial-boundary value problems accompanied by an integral condition, all related to nonlinear partial fractional reaction-diffusion (RD) equations. Our findings are derived through the utilization of a priori estimates in Bouziani fractional spaces. By employing an iterative approach built upon outcomes from the linear counterpart, we successfully validate the existence and uniqueness of a weak generalized solution for the nonlinear conundrum.

Keywords: *fractional partial differential equation; existence; uniqueness; energy inequality.*

Mathematics Subject Classification (2010): 35R11, 35A01, 35A02, 70K75.

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1 Introduction

The nonlinear diffusion equation is a partial differential equation that describes the behavior of a diffusing quantity in a medium where the diffusion rate depends on the concentration or magnitude of the quantity itself, these are suggested as mathematical models of physical problems in many fields such as image processing, heat conduction in composite materials, reaction-diffusion systems, nonlinear diffusion in fluid mechanics and population dynamics [1–3].

Fractional differential equations (FDEs) generalize ordinary and partial differential equations by incorporating fractional derivatives instead of integer-order derivatives. FDEs have attracted considerable attention because they can describe complex phenomena characterized by long-range memory, anomalous diffusion, and fractal-like behavior, fractional differential equations (FDEs) are used to model a wide range of phenomena in various fields such as visco-elasticity, biological, electrical circuits, control systems, and geological systems [4–8]. These are just a few examples of the diverse range of applications where FDEs are treated. Fractional calculus and FDEs provide a powerful mathematical framework to capture complex dynamics, memory effects, and non-local interactions in various systems, as well as a variety of other physical phenomena. Recently, there has been a lot of progress in the study of fractional differential equations [9–13]. This is due to several recent studies in this field, see the monographs of Kilbas et al. [14], Miller and Ross [15], Samko et al. [16], and the papers of Agarwal et al. [17], Anguraj A. and Karthikeyan P. [18], Belmekki et al. [19], Daftardar-Gejji and Jafari [20, 21], Kaufmann and Mboumi [22], Kilbas and Marzan. [23], Yu and Gao [24], Oussaeif [25], and also the general references in Baleanu et al. [26], and the references therein.

However, many phenomena can better be described by integral boundary conditions, which are often used in problems where the system's physical or mathematical characteristics require considering the solution's cumulative behavior over a specific region. They can arise in various fields, including heat transfer, fluid mechanics, quantum mechanics, and population dynamics. Bouziani [24, 27] has extensively studied the topic and generated significant interest in various works. The recent surge in interest in nonlinear fractional reaction-diffusion (RD) equations [28, 29] can be attributed to their ability to exhibit self-organization phenomena and introduce the fractional index as a new parameter in the equation. Moreover, the analysis of these equations from both analytical and numerical perspectives has generated considerable attention. These equations provide a fertile, promising research area, offering rich mathematical insights. Despite efforts to investigate fractional RD equations under specific boundaries and initial conditions, explicit solutions are often elusive. This study delves into a more comprehensive examination of a generalized model for nonlinear time-fractional RD equations.

The aim of this paper is to expand the utilization of the energy inequality method to establish the existence and uniqueness of weak solutions in functionally weighted Sobolev spaces. Specifically, we focus on a class of initial-boundary value problems with a nonlocal condition referred to as the "integral condition" for a broader range of nonlinear partial fractional differential equations. To the best of our knowledge, this particular class of equations has not been previously investigated. Additionally, this work serves as an explanation and complement to our previous paper [24]. Furthermore, this research introduces novel theoretical concepts involving Bouziani fractional spaces.

2 Preliminaries

Let $\Omega = [0;T]$ be a finite interval of the real numbers R and $\Gamma(\cdot)$ denote the gamma function. For any $0 < \alpha < 1$ being a positive integer, the Caputo and Riemann-Liouville derivatives are, respectively, defined as follows:

- The left Caputo fractional derivative of order α is defined respectively by

$$ {}^{C}D_t^{\alpha}\mu(x,t) = \frac{1}{\Gamma(1-\alpha)} \int\limits_0^t \frac{\partial\mu(x,s)}{\partial s} \frac{1}{(t-s)^{\alpha}} ds. \tag{1} $$

- The right Caputo fractional derivatives of order α is defined respectively by

$$ {}_t^{C}D^{\alpha}\mu(x,t) = \frac{-1}{\Gamma(1-\alpha)} \int\limits_t^T \frac{\partial\mu(x,s)}{\partial s} \frac{1}{(s-t)^{\alpha}} ds. \tag{2} $$

- The left Riemann-Liouville fractional derivative of order α is defined respectively by

$$ {}^{R}D_t^{\alpha}\mu(x,t) = \frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial t} \int\limits_0^t \frac{\mu(x,s)}{(t-s)^{\alpha}} ds. \tag{3} $$

- The right Riemann-Liouville fractional derivative of order α is defined respectively by

$$ {}_t^{R}D^{\alpha}\mu(x,t) = \frac{-1}{\Gamma(1-\alpha)} \frac{\partial}{\partial t} \int\limits_t^T \frac{\mu(x,s)}{(s-t)^{\alpha}} ds. \tag{4} $$

Many authors think that Caputo's version is more natural because it makes the handling of homogeneous initial conditions easier. Then the two definitions (1) and (3) are linked by the following relationship, which can be verified by a direct calculation:

$$ {}_a^{R}D_t^{\alpha}\mu(x,t) = {}_a^{C}D_t^{\alpha}\mu(x,t) + \frac{\mu(x,0)}{\Gamma(1-\alpha)t^{\alpha}}. \tag{5} $$

Definition 2.1 $[30, 31]$ For any real $\theta > 0$ and finite interval $[a,b]$ of the real axis $\mathbb{R}$, we define the semi-norm

$$ |v|^2_{{}^{l}H^{\theta}(\Omega)} = \left\| {}^{R}D_t^{\theta} v \right\|^2_{L^2(\Omega)} $$

and the norm

$$ \|v\|^2_{{}^{l}H^{\theta}(\Omega)} = \|v\|^2_{L^2(\Omega)} + |v|^2_{{}^{l}H^{\theta}(\Omega)}. \tag{6} $$

Next, we define ${}^{l}H^{\theta}(\Omega)$ as the closure of $C_0^{\infty}(\Omega)$ with respect to the norm $\|\cdot\|_{{}^{l}H^{\theta}(\Omega)}$.

Definition 2.2 $[30, 31]$ For any real $\theta > 0$ and finite interval $[a,b]$ of the real axis $\mathbb{R}$, we define the semi-norm

$$ |v|^2_{{}^{r}H^{\theta}(\Omega)} = \left\| {}_t^{R}D^{\theta} v \right\|^2_{L^2(\Omega)} $$

and the norm

$$ \|v\|^2_{{}^{r}H^{\theta}(\Omega)} = \|v\|^2_{L^2(\Omega)} + |v|^2_{{}^{r}H^{\theta}(\Omega)}. \tag{7} $$

In what follows, we define ${}^{l}H^{\theta}(\Omega)$ as the closure of $C_0^{\infty}(\Omega)$ with respect to the norm $\|\cdot\|_{{}^{l}H^{\theta}(\Omega)}$.

Definition 2.3 For any real $\theta > 0$ and finite interval $[a, b]$ of the real axis $\mathbb{R}$, we define the semi-norm

$$|v|^2_{{}^{c}H^{\theta}(\Omega)} = \left| \frac{({}^{R}D_t^{\theta}v, {}_t^{R}D^{\theta}v)_{L^2(\Omega)}}{\cos(\theta\pi)} \right|$$

and the norm

$$\|v\|^2_{{}^{c}H^{\theta}(\Omega)} = \|v\|^2_{L^2(\Omega)} + |v|^2_{{}^{c}H^{\theta}(\Omega)}. \tag{8}$$

Lemma 2.1 *[30, 31] For any real $\theta \in \mathbb{R}_+$, if $u \in {}^{l}H^{\theta}(\Omega)$ and $v \in C_0^{\infty}(\Omega)$, then we have*

$$({}^{R}D_t^{\theta}u(t), v(t))_{L^2(\Omega)} = (u(t), {}_t^{R}D^{\theta}v(t))_{L^2(\Omega)}.$$

Lemma 2.2 *[30, 31] For $0 < \theta < 2$, $\theta \neq 1$ and $u \in H_0^{\frac{\theta}{2}}(\Omega)$, then*

$${}^{R}D_t^{\theta}u(t) = {}^{R}D_t^{\frac{\theta}{2}}u^{R}D_t^{\frac{\theta}{2}}u(t).$$

Lemma 2.3 *[30, 31] For any real $\theta \in \mathbb{R}_+$ and $\theta \neq n + \frac{1}{2}$, the semi norms $|\cdot|_{{}^{l}H^{\theta}(\Omega)}$, $|\cdot|_{{}^{r}H^{\theta}(\Omega)}$, and $|\cdot|_{{}^{c}H^{\theta}(\Omega)}$ are equivalent, then we pose*

$$|\cdot|_{{}^{l}H^{\theta}(\Omega)} \cong |\cdot|_{{}^{r}H^{\theta}(\Omega)} \cong |\cdot|_{{}^{c}H^{\theta}(\Omega)}.$$

Lemma 2.4 *[30, 31] For any real $\theta > 0$, the space ${}^{R}H_0^{\theta}(\Omega)$ with respect to the norm (7) is complete.*

Definition 2.4 We denote by $L_2(0, T, L_2(0, 1)) = L_2(Q)$ the space of functions which are square integrable in the Bochner sense with the scalar product

$$(u, w)_{L_2(0,T,L_2(0,1))} = \int_0^T ((u, \cdot), (w, \cdot))_{L_2(0,1)}\, dt. \tag{9}$$

Since the space $L_2(0, 1)$ is a Hilbert space, it can be shown that $L_2(0, T, L_2(0, 1))$ is a Hilbert space as well. Now, let $C^{\infty}(0, T)$ denote the space of infinitely differentiable functions on $(0, T)$ and $C_0^{\infty}(0, T)$ denote the space of infinitely differentiable functions with compact support in $(0, T)$.

3 Bouziani Functional Spaces

We introduce the function spaces needed in our investigation. Let $L^2(0, 1)$ and $L^2(0, T, L^2(0, 1))$ be the standard function spaces. Also, we denote by $C_0(0, 1)$ the vector space of continuous functions with compact support in (0.1). Since such functions are Lebesgue integrable with respect to dx, we can define it on $C_0(0, 1)$. The bilinear form is given by

$$(u.w) = \int_0^1 \Im_x u.\Im_x w dx, \tag{10}$$

where $\Im_x u = \int_0^x u(\zeta, \cdot)d\zeta$ and $\Im_x^* u = \int_x^1 u(\zeta, \cdot)d\zeta$. The previous bilinear form (22) is considered as a scalar product on $C_0(0, 1)$ for which $C_0(0, 1)$ is not complete.

Definition 3.1 We denote by $B^2(0,1)$ a completion of $C_0(0,1)$ for the scalar product (13), which is denoted by $(\cdot,\cdot)_{B^2(0,1)}$. It is also called the Bouziani space or the space of square-integrable primitive function on $(0,1)$. By the norm of the function u from $B^2(0,1)$, we can understand the non negative number

$$\|u\|_{B^2(0,1)} = \sqrt{(u,u)_{B^2(0,1)}} = \|\Im_x u\|_{L^2(0,1)}\,.$$

For $u \in L^2(0,1)$, we have the elementary inequality

$$\|u\|^2_{B^2(0,1)} \le \frac{1}{2}\,\|u\|^2_{L^2(0,1)}\,. \tag{11}$$

We denote by $L^2(0,T,B^2(0,1)) = B^2(\Omega)$ the space of functions, which are called the square integrable in the Bochner sense with the scalar product

$$(u.w)_{B^2(\Omega)} = \int_0^1 ((u,\cdot).(w,\cdot))_{B^2(0,1)}dx \tag{12}$$

for which $B^2(\Omega)$ is a Hilbert space.

4 Solvability of Solution of Diffusion Fractional Dirichlet Problems

4.1 Formulation of the problem

In this part, we assume $\Omega = (0;1)$ and $I = (0;T)$ with $0 \prec T \prec +\infty$. Also, we consider the following nonlinear fractional problem:

$$\begin{cases} {}^cD_t^\alpha u(x,t) - \frac{\partial}{\partial x}(a(x,t)\frac{\partial u(x,t)}{\partial x}) + bu(x,t) = f(x,t,u,\frac{\partial u}{\partial x}), & \forall (x,t) \in Q,\\ u(x,0) = 0 \qquad \forall x \in (0,1), & \\ \int_\Omega x^k u(x,t)dx = 0 \qquad \forall t \in (0,T) \text{ and } k = \{0,1\}, & \end{cases} \tag{P}$$

where $Q = \Omega \times I$ is an open bounded interval of $\mathbb{R}$ and a, b, f are known functions. Also, we suppose the following conditions:

- (A_1) We assume that $0 \prec a_0 \le a(x,t) \le a_1$ and $a_2 \le \frac{\partial^2 a(x,t)}{\partial x^2} \le a_3$ for all $(x,t) \in Q$.
- (A_2) We assume that the compatibility conditions

$$\int_\Omega x^k u(x,t)dx \quad = \quad 0 \qquad \forall t \in (0,T) \text{ and } k = \{0,1\}$$

  are verified.

4.2 The associated linear problem

In this part, we show the existence and uniqueness of the strong solution of the linear problem. The proof is based on an a priori estimate and the density of the set of values of the image of the operator generated by the problem

$$\begin{cases} {}^cD_t^\alpha u(x,t) - \frac{\partial}{\partial x}(a(x,t)\frac{\partial u(x,t)}{\partial x}) + bu(x,t) = f(x,t), \forall (x,t) \in Q,\\ u(x,0) = 0 \qquad \forall x \in (0,1),\\ \int_\Omega x^k u(x,t)dx = 0 \qquad \forall t \in (0,T) \text{ and } k = \{0,1\}, \end{cases} \tag{P_1}$$

whose diffusion problem is given as follows:

$$\pounds u =^{c} D_t^{\alpha} u(x,t) - \frac{\partial}{\partial x}(a(x,t)\frac{\partial u(x,t)}{\partial x}) + bu(x,t) = f(x,t) \tag{13}$$

with the initial condition

$$lu = u(x,0) = 0 \ \ \forall x \in [-1,1] \tag{14}$$

and the integral conditions

$$\int_{\Omega} x^k u(x,t) dx dt = 0 \qquad \forall t \in (0,T) \text{ and } k = \{0,1\}, \tag{15}$$

where $f(x,t)$ is a given function and α satisfies the assumption $0 \leq \alpha \leq 1$, for which $(x,t) \in \bar{Q}$.

4.3 A priori estimation

In this part, we aim to establish an a priori bound and prove the existence of a solution to the problems (13)-(15) with $Lu = F$, where $L = (\mathscr{L}, l)$ and $F = f$ is the operator equation corresponding to problems (13)-(15). To obtain a full overview about such an estimation, the reader may refer to [32]. The operator L acts from E to F, which is defined as follows. The Banach space E consists of all functions $u(x,t)$ with the finite norm

$$\|u\|_E^2 = \|\Im_x u\|_{L^2(Q)}^2 + \|u\|_{L^2(Q)}^2 . \tag{16}$$

The Hilbert space F consists of the vector-valued functions $F = f$ with the norm

$$\| \mathscr{F} \|_F^2 = \|f\|_{L^2_{(Q)}}^2 + \int_0^T I^{\alpha} \|f\|_{L^2_{(0,1)}}^2 dt + \|\Im_x \varphi\|_{L^2_{(0,1)}}^2 . \tag{17}$$

4.4 A priori bound

Theorem 4.1 *If the assumption (A_1) is satisfied, then for any function $u \in D(L)$, there exists a positive constant c independent of u such that*

$$\|\Im_x u\|_{L^2(Q)}^2 + \|u\|_{L^2(Q)}^2 \leq k \left(\|f\|_{L^2_{(Q)}}^2 + \int_0^T I^{\alpha} \|f\|_{L^2_{(0,1)}}^2 dt + \|\Im_x \varphi\|_{L^2_{(0,1)}}^2 \right) \tag{18}$$

for which $D(L)$ is the domain of definition of the operator L defined by

$$D(L) = \{u \in L^2(Q) \ / \ \Im_x u \in L^2(Q)\},$$

satisfying conditions (15).

Proof. By taking the scalar product in $L^2(Q)$ on (13) and the operator

$$Mu = \int_x^1 \left(\int_0^{\varsigma} u(\eta, t) \, d\eta \right) d\zeta,$$

where $Q^\tau = \Omega \times (0,T)$, we obtain

$$\begin{aligned}(\pounds u, Mu)_{L^2(Q^\tau)} &= \left({}^cD_t^\alpha u, \int_x^1 \left(\int_0^\varsigma u(\eta,t)\,d\eta\right) d\zeta\right)_{L^2(Q)} \\ &- \left(\frac{\partial}{\partial x}(a(x,t)\frac{\partial u(x,t)}{\partial x}) \int_x^1 \left(\int_0^\varsigma u(\eta,t)\,d\eta\right) d\zeta\right)_{L(Q)} \\ &+ \left(bu(x,t), \int_x^1 \left(\int_0^\varsigma u(\eta,t)\,d\eta\right) d\zeta\right)_{L^2(Q)} = \left(\tilde{f}, u\right)_{L^2(Q)}.\end{aligned} \tag{19}$$

The successive integration by parts of integrals on the right-hand side of (17) yields

$$\begin{aligned}\left({}^RD_t^\alpha u, Mu\right)_{L^2(Q)} &= \int_Q \left({}^cD_t^\alpha u(x,t) \,.\int_x^1 \left(\int_0^\varsigma u(\eta,t)\,d\eta\right) d\zeta\right) dx \\ &= \int_{Q^\tau} \left({}^cD_t^\alpha \int_0^x u(\zeta,t)\,d\zeta . \int_0^1 u(\zeta,t)\,d\zeta\right) dxdt \\ &= \int_{Q^\tau} \left({}^cD_t^{\frac{\alpha}{2}} \int_0^x u(\zeta,t)\,d\zeta . {}^cD_t^{\frac{\alpha}{2}} \int_0^x u(\zeta,t)\,d\zeta\right) dxdt \\ &= \left\| {}^cD_t^{\frac{\alpha}{2}} \Im_x u \right\|^2_{L^2(Q^\tau)}\end{aligned} \tag{20}$$

and

$$\begin{aligned}&-\left(\frac{\partial}{\partial x}(a\frac{\partial u}{\partial x}), Mu\right)_{L(Q)} = -\int_{Q^\tau} -\left(\frac{\partial}{\partial x}(a(x,t)\frac{\partial u(x,t)}{\partial x}) \int_x^1 \left(\int_0^\varsigma u(\eta,t)\,d\eta\right) d\zeta\right) dxdt \\ &= \int_{Q^\tau} \left(\left(a(x,t)\frac{\partial u(x,t)}{\partial x}\right)\left(\int_0^x u(\zeta,t)\,d\zeta\right)\right) dxdt \\ &= \int_{Q^\tau} a(x,t)\,(u(x,t))^2\,dxdt - \frac{1}{2}\int_{Q^\tau} \left(\frac{\partial^2 a(x,t)}{\partial x^2}\right)\left(\int_0^x u(\zeta,t)\,d\zeta\right) dxdt \\ &\geq a_0 \|u\|^2_{L^2(Q^\tau)} - \frac{a_3}{2}\|\Im_x u\|^2_{L^2(Q^\tau)}.\end{aligned} \tag{21}$$

Consequently, we have

$$\begin{aligned}(bu, Mu)_{L(Q)} &= \int_{Q^\tau} \left(b(x,t)\,u(x,t) \int_x^1 \left(\int_0^\varsigma u(\eta,t)\,d\eta\right) d\zeta\right) dxdt \\ &\geq b\int_{Q^\tau} \left(u(x,t) \int_x^1 \left(\int_0^\varsigma u(\eta,t)\,d\eta\right) d\zeta\right) dxdt \\ &\geq b\int_{Q^\tau} \left(\int_0^1 u(\zeta,t)\,d\zeta\right)\left(\int_0^1 u(\zeta,t)\,d\zeta\right) dxdt \\ &\geq b\int_{Q^\tau} (\Im_x u(x,t)^2\,dxdt \\ &\geq b\|\Im_x u\|^2_{L^2_{(0,1)}}.\end{aligned} \tag{22}$$

Substituting (20), (21) and (22) into (19) gives

$$\left\| {}^cD_t^{\frac{\alpha}{2}} \Im_x u \right\|^2_{L^2(Q^\tau)} + a_0\|u\|^2_{L^2(Q^\tau)} - \frac{a_3}{2}\|\Im_x u\|^2_{L^2(Q^\tau)} + b\|u\|^2_{L^2(Q^\tau)} \leq \left(\tilde{f}, Mu\right). \tag{23}$$

Now, we estimate the last term on the right-hand side of (23) by applying the Cauchy inequality $\left(|ab| \leq \frac{a^2}{2\varepsilon} + \frac{\varepsilon b^2}{2}\right)$ with ε. In other words, we have

$$\begin{aligned} &\left\| {}^cD_t^{\frac{\alpha}{2}} \Im_x u \right\|^2_{L^2(Q^\tau)} + a_0 \|u\|^2_{L^2(Q^\tau)} - \frac{a_3}{2} \|\Im_x u\|^2_{L^2(Q^\tau)} + b_0 \|u\|^2_{L^2(Q^\tau)} \\ \leq\ & \int_{Q^\tau} f(x,t) \left(\int_x^1 \left(\int_0^\varsigma u(\eta,t)\, d\eta \right) d\zeta \right) dxdt \\ \leq\ & \frac{1}{2\varepsilon} \int_{Q^\tau} \left(\int_0^x u(\zeta,t) d\zeta \right)^2 dxdt + \frac{\varepsilon}{2} \int_{Q^\tau} \left(\int_0^x f(\zeta,t) d\zeta \right)^2 dxdt. \end{aligned} \tag{24}$$

By using the Cauchy-Schwartz inequality, we obtain

$$\begin{aligned} & \frac{1}{2\varepsilon} \int_{Q^\tau} \left(\int_0^x u(\zeta,t) d\zeta \right)^2 dxdt + \frac{\varepsilon}{2} \int_{Q^\tau} \left(\int_0^x f(\zeta,t) d\zeta \right)^2 dxdt \\ \leq\ & \frac{1}{2\varepsilon} \int_{Q^\tau} (\Im_x u(x,t))^2\, dxdt + \frac{\varepsilon}{2} \int_{Q^\tau} (\Im_x f(x,t))^2\, dxdt \\ \leq\ & \frac{1}{4\varepsilon} \int_{Q^\tau} (u(x,t))^2\, dxdt + \frac{\varepsilon}{4} \int_{Q^\tau} (f(x,t))^2\, dxdt \\ =\ & \frac{1}{4\varepsilon} \|u\|^2_{L^2(Q^\tau)} + \frac{\varepsilon}{4} \|f\|^2_{L^2(Q^\tau)}. \end{aligned} \tag{25}$$

This, consequently, yields

$$\left\| {}^cD_t^{\frac{\alpha}{2}} \Im_x u \right\|^2_{L^2(Q^\tau)} + \left(a_0 + b - \frac{1}{4\varepsilon} \right) \|u\|^2_{L^2_{(Q^\tau)}} \leq \frac{\varepsilon}{4} \|f\|^2_{L^2(Q^\tau)}$$

and

$$\left\| {}^cD_t^{\frac{\alpha}{2}} \Im_x u \right\|^2_{L^2(Q^\tau)} + \|u\|^2_{L^2_{(Q^\tau)}} \leq \frac{\varepsilon}{4 \min\left\{1, \left(a_0 + b - \frac{1}{4\varepsilon}\right)\right\}} \|f\|^2_{L^2_{(Q^\tau)}}.$$

Now, we present

$$C = \frac{\varepsilon}{4 \min\left\{1, \left(a_0 + b - \frac{1}{4\varepsilon}\right)\right\}}$$

as

$$I^\alpha \left({}^cD_t^\alpha u(x,t)\right) = u(x,t) + \varphi(x).$$

This leads to

$$\|\Im_x u\|^2_{L^2(Q^\tau)} + \|u\|^2_{L^2_{(0,1)}} \leq C I^\alpha \|f\|^2_{L^2_{(0,1)}} + \|\Im_x \varphi\|^2_{L^2(0,1)}.$$

Consequently, we have

$$\|\Im_x u\|^2_{L^2(Q^\tau)} \leq C I^\alpha \|f\|^2_{L^2_{(Q^\tau)}} + \|\Im_x \varphi\|^2_{L^2(Q^\tau)}.$$

So, finally, we get

$$\|u\|^2_{L^2_{(Q^\tau)}} \leq C \|f\|^2_{L^2_{(Q^\tau)}},$$

where

$$C_1 = \max\{1, C\}$$

and

$$\|\Im_x u\|^2_{L^2(Q^\tau)} + \|u\|^2_{L^2_{(0,1)}} \le C_1 \left(I^\alpha \|f\|^2_{L^2_{(0,1)}} + \|f\|^2_{L^2_{(0,1)}} + \|\Im_x \varphi\|^2_{L^2(0,1)} \right).$$

With the use of successive integration $(0, T)$, we get

$$\|\Im_x u\|^2_{L^2_{(Q^\tau)}} + \|u\|^2_{L^2_{(Q^\tau)}} \le (C_1 \max\{1, T\}) \left(\int_0^T I^\alpha \|f\|^2_{L^2_{(Q^\tau)}} dt + \|f\|^2_{L^2_{(Q^\tau)}} + \|\Im_x \varphi\|^2_{L^2_{(Q^\tau)}} \right)$$

which implies

$$k = \quad (C_1 \max\{1, T\})^{\frac{1}{2}}$$

for which

$$\|u\|_E \le k \|Lu\|_F. \tag{26}$$

Let $R(L)$ be the range of the operator L. However, since we do not have any information about $R(L)$, except that $R(L) \subset F$, we must extend L so that (26) holds for the extension, and its range is the whole space F. For this purpose, we state the following proposition.

Proposition 4.1 *The operator $L : E \longrightarrow F$ has a closure.*

Proof. Let $(u_n)_{n \in \mathbb{N}} \subset D(L)$ be a sequence where

$$u_n \longrightarrow 0 \qquad \text{in } E$$

and

$$Lu_n \longrightarrow \left(\tilde{f}; 0\right) \qquad \text{in } F. \tag{27}$$

Herein, we must prove that

$$f \equiv 0.$$

The convergence of u_n to 0 in E leads to

$$u_n \longrightarrow 0 \qquad \text{in } D'(Q). \tag{28}$$

According to the continuity of the derivation of $D'(Q)$ in $D'(Q)$, the relation (28) involves

$$\mathcal{L} u_n \longrightarrow 0 \qquad \text{in } D'(Q). \tag{29}$$

Moreover, the convergence of $\mathcal{L} u_n$ to f in $L^2(Q)$ gives

$$\mathcal{L} u_n \longrightarrow f \qquad \text{in } D'(Q). \tag{30}$$

As we have the uniqueness of the limit in $D'(Q)$, we conclude from (29) and (30) that $f = 0$. Then, L is closable of this operator with the domain of definition $D(L)$.

Definition 4.1 A solution of the operator equation

$$\bar{L} u = F$$

is called a strong solution to problems (13)-(15). The a priori estimate (18) can be extended to strong solutions, i.e., we have the estimate

$$\|\Im_x u\|^2_{L^2(Q)} + \|u\|^2_{L^2(Q)} \le k \left(\|f\|^2_{L^2_{(Q)}} + \int_0^T I^\alpha \|f\|^2_{L^2_{(0,1)}} dt + \|\Im_x \varphi\|^2_{L^2_{(0,1)}} \right).$$

We deduce from the estimate (18) the subsequent result.

Corollary 4.1 *The range* $R(\bar{L})$ *of the operator* $\bar{L}$ *is closed in* F *and is equal to the closure* $\overline{R(L)}$ *of* $R(L)$*, that is,* $R(\bar{L}) = \overline{R(L)}$*.*

Proof. Let $z \in \overline{R(L)}$. Then, there is a Cauchy sequence $(z_n)_{n\in\mathbb{N}}$ in F constituting of the elements of the set $R(L)$ such that

$$\lim_{n\longrightarrow+\infty} z_n = z.$$

There is then a corresponding sequence $u_n \in D(L)$ such that

$$z_n = Lu_n.$$

With the use of estimate (18), we get

$$\|u_p - u_q\|_E \le C\,\|Lu_p - Lu_q\|_F \to 0,$$

where p, q tend towards infinity. We can deduce that $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in E. So, as E is a Banach space, there exists $u \in E$ such that

$$\lim_{n\longrightarrow+\infty} u_n = u \text{ in } E.$$

By virtue of the definition of $\bar{L}$ ($\lim_{n\longrightarrow+\infty} u_n = u$ in E, if $\lim_{n\longrightarrow+\infty} Lu_n = \lim_{n\longrightarrow+\infty} z_n = z$, then $\lim_{n\longrightarrow+\infty} \bar{L}u_n = z$ as $\bar{L}$ is closed, so $\bar{L}u = z$), the function u satisfies

$$v \in D\left(\bar{L}\right),\ \bar{L}v = z.$$

Then $z \in R(\bar{L})$, and so we have

$$\overline{R(L)} \subset R(\bar{L}).$$

Also, we conclude here that $R(\bar{L})$ is closed because it is Banach (any complete subspace of a metric space (not necessarily complete) is closed). Thus, it remains to show the reverse inclusion. To this aim, it should be noted that either $z \in R(\bar{L})$ and then there exists a Cauchy sequence $(z_n)_{n\in\mathbb{N}}$ in F constituting of the elements of the set $R(\bar{L})$ such that

$$\lim_{n\longrightarrow+\infty} z_n = z,$$

or $z \in R(\bar{L})$ because $R(\bar{L})$ is a closed subset of a completed F and so $R(\bar{L})$ is complete. There is then a corresponding sequence $u_n \in D(\bar{L})$ such that

$$\bar{L}u_n = z_n.$$

As a result, we get from (18) that

$$\|u_p - u_q\|_E \le C\left\|\bar{L}u_p - \bar{L}u_q\right\|_F \to 0,$$

where p and q tend towards infinity. We can then deduce that $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in E, and so as E is a Banach space, there exists $u \in E$ such that

$$\lim_{n\longrightarrow+\infty} u_n = u \ \text{ in } E.$$

Once again, there is a corresponding sequel $(Lu_n)_{n\in\mathbb{N}} \subset R(L)$ such that

$$\bar{L}u_n = Lu_n \text{ on } R(L), \ \forall n \in \mathbb{N}.$$

So, we obtain

$$\lim_{n\longrightarrow+\infty} Lu_n = z.$$

Consequently, we obtain $z \in \overline{R(L)}$, and then we conclude that

$$R(\bar{L}) \subset \overline{R(L)}.$$

4.5 Existence of solution

Theorem 4.2 *Let the assumptions* (A_1) *be satisfied. Then for all* $F = (f, 0) \in F$, *there exists a unique strong solution* $u = \bar{L}^{-1}\mathcal{F} = \overline{L^{-1}}\mathcal{F}$ *of the problem (2)-(4).*

Proof. We have

$$(Lu, W)_F = \int_Q \mathscr{L}u.wdxdt, \tag{31}$$

where

$$W = (w, 0).$$

So, for $w \in L^2(Q)$ and for all $u \in D_0(L) = \{u,\ u \in D(L) : \ell u = 0\}$, we have

$$\int_Q u.wdxdt = 0.$$

By putting $w = u$ and using the same estimate as previously, we obtain

$$\left\| {}^cD_t^{\frac{\alpha}{2}} \Im_x u \right\|^2_{L^2(0,1)} + \|u\|^2_{L^2_{(0,1)}} = 0,$$

which implies

$$\|u\| \leq 0 \Rightarrow u = 0.$$

So, we get $u = w = 0$.

Corollary 4.2 *If for any function* $u \in D(L)$, *we have the following estimate:*

$$\|u\|_E \leq C \|Lu\|_F,$$

then the solution of problem (P_1), *if it exists, is unique.*

Proof. Let u_1 and u_2 be two solutions to problem (P_1), i.e.,

$$\begin{cases} Lu_1 = \mathcal{F} \\ Lu_2 = \mathcal{F} \end{cases} \Longrightarrow Lu_1 - Lu_2 = 0,$$

where L is a linear operator. As a result, we obtain

$$L(u_1 - u_2) = 0. \tag{32}$$

Now, according to (32), we obtain

$$\|u_1 - u_2\|^2_E \leq c\|0\|^2_F = 0,$$

which, consequently, gives

$$u_1 = u_2.$$

5 Solvability of the Weak Solution of the Nonlinear Problem

This section is devoted to the proof of the existence and uniqueness of the solution of the nonlinear problem (Pr):

$$\begin{cases} {}^cD_t^\alpha u(x,t) - \frac{\partial}{\partial x}(a(x,t)\frac{\partial u(x,t)}{\partial x}) + bu(x,t) = f(x,t,u,\frac{\partial u}{\partial x}), \forall (x,t) \in Q, \\ u(x,0) = 0 \qquad \forall x \in (-1,1), \\ \int_\Omega x^k u(x,t)dx = 0 \qquad \forall t \in (0,T) \text{ and } k = \{0,1\}, \end{cases} \tag{P_2}$$

for which the function f is Lipchitzian. As a consequence, there is a positive constant k such that

$$\begin{aligned} &\left\| f\left(x,t,u_1,\frac{\partial u_1}{\partial x}\right) - f\left(x,t,u_2,\frac{\partial u_1}{\partial x}\right) \right\|_{L^2(Q)} \\ &\qquad \le k\left(\|u_1-u_2\|_{L^2(Q)} + \left\| \frac{\partial u_1}{\partial x} - \frac{\partial u_1}{\partial x} \right\|_{L^2(Q)} \right). \end{aligned} \tag{33}$$

Now, we shall prove that (P_2) has a unique weak solution. To this end, we let $u \in \widetilde{C^1}(Q)$ and $u \in C^1(Q)$. Also, we shall compute the integral $\int_Q (f\Im_x^* v)dxdt$. For this purpose, we assume $u,v \in \widetilde{C^1}(Q)$, $\int_Q x^k u(x,t)dx = 0$, and $\int_Q x^k u(x,t)dx = 0$ for all $k = \{0,1\}$. By using the condition on u and v, we have

$$\int_Q ({}^cD_t^\alpha u \cdot \Im_x^* v)dxdt = -\int_Q (v \cdot^c D_t^\alpha \Im_x^* u)dxdt,$$

$$-\int_Q (\frac{\partial}{\partial x}(a(x,t)\frac{\partial u}{\partial x}) \cdot \Im_x^* v)dxdt = \int_Q (v \cdot a(x,t)\frac{\partial u}{\partial x})dxdt,$$

and

$$b\int_Q (u \cdot \Im_x^* v)dxdt = -\int_Q (v \cdot \Im_x^* u)dxdt.$$

It then follows, from [24], that

$$A(u,v) = -\int_Q (v \cdot^c D_t^\alpha \Im_x^* u)dxdt + \int_Q (v \cdot a(x,t)\frac{\partial u}{\partial x})dxdt - \int_Q (v \cdot \Im_x^* u)dxdt.$$

Definition 5.1 A function u is called a weak solution of problem (P_1) and $u \in L^2(0,T,H^1(0,1))$.

By building a recurring sequence starting with $u^{(0)} = 0$, we can define the sequence $\left(u^{(n)}\right)_{n\in\mathbb{N}}$ as follows: given the element $u^{(n-1)}$, for $n = 1,2,3,\ldots$, we will solve the following problem:

$$\begin{cases} {}^cD_t^\alpha u^{(n)}(x,t) - \frac{\partial}{\partial x}(a(x,t)\frac{\partial u^{(n)}(x,t)}{\partial x}) + b(x,t)u^{(n)}(x,t) = f(x,t,u^{(n-1)},\frac{\partial u^{(n-1)}}{\partial x}), \\ \int_\Omega x^k u^{(n)}(x,t)dx = 0 \qquad \forall t \in (0,T) \text{ and } k = \{0,1\}. \end{cases} \tag{P_3}$$

Theorem 5.1 *According to the study of the previous linear problem and by fixing n, problem (P_3) admits a unique solution $u^{(n)}(x,t)$.*

Now, by supposing

$$z^{(n)}(x,t) = u^{(n+1)}(x,t) - u^{(n)}(x,t),$$

we might get a new problem, which has the form

$$\begin{cases} {}^{c}D_t^{\alpha} z^{(n)} - \left(a(x,t)\, z_x^{(n)}\right)_x + b(x,t) z^{(n)} = p^{(n-1)}(x,t), \\ z^{(n)}(x.0) = 0, \\ \int_\Omega x^k z^{(n)}(x,t)dx = 0, \qquad \forall t \in (0,T),\ k = \{0,1\}, \end{cases} \qquad (P_4)$$

where

$$p^{(n-1)}(x,t) = f\left(x,t,u^{(n)}, \frac{\partial u^{(n)}}{\partial x}\right) - f\left(x,t,u^{(n-1)}, \frac{\partial u^{(n-1)}}{\partial x}\right)$$

with the condition

$$z^{(n)}(x.0) = 0 \text{ and } \int_\Omega x^k z^{(n)}(x,t)dx = 0 \qquad \forall t \in (0,T),\ k = \{0,1\}. \qquad (34)$$

Lemma 5.1 *Assume that condition (34) holds, then for the linearized problem* (P_4)*, we have the following a priori estimate:*

$$\left\| Z^{(n)} \right\|_{L^2(0,1,H^1(0,1))} \le \lambda \left\| Z^{(n-1)} \right\|_{L^2(0,1,H^1(0,1))},$$

where λ *is a positive constant given by*

$$\lambda = \sqrt{\frac{5\phi^2}{2\varepsilon}},$$

for which $\varepsilon \prec\prec 1$.

Proof. Multiplying equation (P_4) by $\int_x^1 \left(\int_0^\varsigma z^n(\eta,t)\,d\eta\right) d\zeta$ and integrating the result over Q yield

$$\begin{aligned} & \int_Q \left(({}^{R}D_t^{\alpha} z^{(n)}(x,t) \cdot \int_x^1 \left(\int_0^\varsigma z^n(\eta,t)\,d\eta \right) d\zeta \right) dxdt \\ - & \int_Q \left(\frac{\partial}{\partial x}\left(a(x,t)\frac{\partial z^{(n)}(x,t)}{\partial x}\right) \cdot \int_x^1 \left(\int_0^\varsigma z^n(\eta,t)\,d\eta \right) d\zeta\right) dxdt \\ + & \int_Q b z^{(n)}(x,t) \cdot \int_x^1 \left(\int_0^\varsigma z^n(\eta,t)\,d\eta \right) d\zeta dxdt \\ = & \int_Q \left(p^{(n-1)}(x,t) \cdot \int_x^1 \left(\int_0^\varsigma z^n(\eta,t)\,d\eta \right) d\zeta \right) dxdt. \end{aligned} \qquad (35)$$

By using the standard integration by parts for each term in (P_4) coupled with condition (34), we obtain

$$\begin{aligned} & \int_{Q^\tau}^{c} \left({}^{c}D_t^{\alpha} \int_0^x z^n(\zeta,t)\,d\zeta \right)^2 dxdt + a_1 \int_{Q^\tau} \left(z^n(x,t)\right)^2 dxdt \\ \le & \int_{Q^\tau} \left(\frac{a_3}{2} + \frac{\varepsilon}{2} + \frac{b}{2} \right) \left(z^n(x,t)\right)^2 dxdt + \frac{1}{2\varepsilon} \int_{Q^\tau} \left(p^{(n-1)}\right)^2 dxdt. \end{aligned} \qquad (36)$$

On the other hand, by applying the operator $\Im_x^*$ to equation (36), we get

$$ {}^cD_t^{\alpha}\left(\Im_x^* Z^n\right) + a(x,t)\frac{\partial Z^{(n)}(x,t)}{\partial x} = \Im_x^*\left(p^{(n-1)}\right). \tag{37} $$

Consequently, by taking into account condition (37), multiplying the obtained equality by $\frac{\partial Z^n}{\partial x}$, and then integrating the result over $\Omega^\tau = (0,1)\times(0,\tau)$, where $0 \le \tau \le T$, we obtain

$$ \begin{aligned} \int_{Q^\tau} \left({}^cD_t^{\alpha} Z^n\right)\cdot Z^n dxdt + \int_{Q^\tau} a(x,t)\left(\frac{\partial Z^{(n)}(x,t)}{\partial x}\right)^2 dxdt \\ = \int_{Q^\tau} \Im_x^*\left(p^{(n-1)}(x,t)\right)\frac{\partial Z^n}{\partial x} dxdt. \end{aligned} \tag{38} $$

Now, by using Lemmas 2.2, 2.3, 2.4, and 5.1 coupled with the Cauchy inequality with ε, we obtain

$$ \begin{aligned} & \int_{Q^\tau} \left({}^cD_t^{\frac{\alpha}{2}} Z^n\right)^2 dxdt + \int_{Q^\tau} a(x,t)\left(\frac{\partial Z^{(n)}(x,t)}{\partial x}\right)^2 dxdt \\ \le\ & \frac{1}{2\varepsilon}\int_{Q^\tau} \Im_x^*\left(p^{(n-1)}(x,t)\right)^2 dxdt + \frac{\varepsilon}{2}\int_{Q^\tau}\left(\frac{\partial Z^n}{\partial x}\right)^2 dxdt. \end{aligned} \tag{39} $$

Combining the last two inequalities (38) and (39) gives

$$ \begin{aligned} & \int_{Q^\tau}^c \left({}^cD_t^{\alpha}\int_0^x z^n(\zeta,t)\,d\zeta\right)^2 dxdt + \int_{Q^\tau}\left({}^cD_t^{\frac{\alpha}{2}} Z^n\right)^2 dxdt \\ + \ & a_1\int_{Q^\tau} \left(z^n(x,t)\right)^2 dxdt + \int_{Q^\tau} a(x,t)\left(\frac{\partial Z^{(n)}(x,t)}{\partial x}\right)^2 dxdt \\ \le\ & \frac{1}{2\varepsilon}\int_{Q^\tau}\left(p^{(n-1)}\right)^2 dxdt + \frac{1}{2\varepsilon}\int_{Q^\tau}\Im_x^*\left(p^{(n-1)}(x,t)\right)^2 dxdt \\ + \ & \int_{Q^\tau}\left(\frac{a_3}{2}+\frac{\varepsilon}{2}+\frac{b}{2}\right)\left(z^n(x,t)\right)^2 dxdt + \frac{\varepsilon}{2}\int_{Q^\tau}\left(\frac{\partial Z^n}{\partial x}\right)^2 dxdt. \end{aligned} \tag{40} $$

By eliminating two first integrals on the left-hand-side of inequality (40) and using the Cauchy inequality with ε, we get

$$ \begin{aligned} & \left(a(x,t)-\frac{a_3}{2}-\frac{\varepsilon}{2}-\frac{b}{2}\right)\left\{\int_{Q^\tau}\left(\left(z^n(x,t)\right)^2 + \left(\frac{\partial Z^{(n)}(x,t)}{\partial x}\right)^2\right)dxdt\right\} \\ \le\ & \frac{1}{2\varepsilon}\left(\int_{Q^\tau}\left(p^{(n-1)}\right)^2 dxdt + \int_{Q^\tau}\Im_x^*\left(p^{(n-1)}(x,t)\right)^2 dxdt\right). \end{aligned} \tag{41} $$

Therefore, we have the estimate

$$ \begin{aligned} \left\|\Im_x^* p^{(n-1)}\right\|_{L^2(0,1)}^2 &\le \frac{1}{4}(1-0)^2\left\|p^{(n-1)}\right\|_{L^2(0,1)}^2 \\ &\le \frac{1}{4}\left\|p^{(n-1)}\right\|_{L^2(0,1)}^2 \end{aligned} \tag{42} $$

and

$$\begin{aligned}
&\int\limits_{Q^\tau} \left(p^{(n-1)}\right)^2 dxdt \\
\leq\ & \phi^2 \int\limits_{Q^\tau} \left(\left|z^{(n-1)}(x,t)\right| + \left|\frac{\partial Z^{(n-1)}(x,t)}{\partial x}\right|\right)^2 dxdt \\
\leq\ & \phi^2 \int_0^\tau \left\|z^{(n-1)}(\cdot,t)\right\|^2_{L^2(0,1)} + \left\|\frac{\partial Z^{(n-1)}(\cdot,t)}{\partial x}\right\|^2_{L^2(0,1)} dtdx. \qquad (43)
\end{aligned}$$

Now, substituting (41) and (42) into (44) yields

$$\begin{aligned}
&\left(a(x,t) - \frac{a_3}{2} - \frac{\varepsilon}{2} - \frac{b}{2}\right) \int_0^\tau \left\{\left\|z^{(n)}(\cdot,t)\right\|^2_{L^2(0,1)} + \left\|\frac{\partial Z^{(n)}(\cdot,t)}{\partial x}\right\|^2_{L^2(0,1)}\right\} dtdx \\
\leq\ & \frac{5\phi^2}{2\varepsilon} \int_0^\tau \left\{\left\|z^{(n-1)}(\cdot,t)\right\|^2_{L^2(0,1)} + \left\|\frac{\partial Z^{(n-1)}(\cdot,t)}{\partial x}\right\|^2_{L^2(0,1)}\right\} dtdx. \qquad (44)
\end{aligned}$$

Since $a(x,t) - \frac{a_3}{2} - \frac{\varepsilon}{2} - \frac{b}{2} \geq 0$, we find

$$\begin{aligned}
&\int_0^\tau \left\{\left\|z^{(n)}(\cdot,t)\right\|^2_{L^2(0,1)} + \left\|\frac{\partial Z^{(n)}(\cdot,t)}{\partial x}\right\|^2_{L^2(0,1)}\right\} dt \\
&\frac{5\phi^2}{2\varepsilon} \int_0^\tau \left\{\left\|z^{(n-1)}(\cdot,t)\right\|^2_{L^2(0,1)} + \left\|\frac{\partial Z^{(n-1)}(\cdot,t)}{\partial x}\right\|^2_{L^2(0,1)}\right\} dtdx.
\end{aligned}$$

The right-hand side here is independent of τ, and hence we shall replace the left-hand side by the upper bound with respect to τ to obtain the desired inequality, i.e.,

$$\left\|Z^{(n)}\right\|^2_{L^2(0,1,H^1(0,1))} \leq \frac{5\phi^2}{2\varepsilon} \left\|Z^{(n-1)}\right\|^2_{L^2(0,1,H^1(0,1))}.$$

This forms the criteria of convergence of the series $\sum\limits_{n=1}^{\infty} z^{(n)}$, which converges if $\frac{5\phi^2}{2\varepsilon} < 1$, that is, if $\phi = \sqrt{\frac{2\varepsilon}{5}}$. Since $Z^{(n)}(x,t) = u^{(n+1)}(x,t) - u^n(x,t)$, it follows that the sequence $(u^n)_{n\in N}$ will be defined by

$$u^{(n)}(x,t) = \sum_{n=1}^{n-1} z^{(i)} + u^{(0)}(x,t),$$

which converges to an element $u \in L^2\left(0,1,H^1(0,1)\right)$.

Therefore, we have established the following result.

Theorem 5.2 *Under the condition (34), the solution for problem* (P_2) *is unique.*

Proof. Suppose that u_1 and u_2 in $L^2\left(0,1,H^1(0,1)\right)$ are two solutions of (P_4), then $Z = u_1 - u_2$ satisfies $Z \in L^2\left(0,1,H^1(0,1)\right)$. As a result, we have

$${}^cD_t^\alpha u(x,t) - \frac{\partial}{\partial x}(a(x,t)\frac{\partial u(x,t)}{\partial x}) + bu(x,t) = \vartheta(x,t)$$

and

$$\int_{\Omega} x^k z^{(n)}(x,t)dx = 0 \qquad \forall t \in (0,T)\,,\ k = \{0,1\},$$

where

$$\vartheta(x,t) = f\left(x,t,u_1,\frac{\partial u_1}{\partial x}\right) - f\left(x,t,u_2,\frac{\partial u_1}{\partial x}\right).$$

Following the same procedure as in establishing the proof of Lemma 5.1, we get

$$\left\|Z^{(n)}\right\|_{L^2(0,1,H^1(0,1))} \leq \lambda \left\|Z^{(n-1)}\right\|_{L^2(0,1,H^1(0,1))},$$

where λ is the same constant as in Lemma 5.1. Since $\lambda < 1$, then we obtain

$$(1-\lambda)\left\|Z^{(n-1)}\right\|_{L^2(0,1,H^1(0,1))},$$

form which we conclude that $u_1 = u_2$ in $L^2\left(0,1,H^1\left(0,1\right)\right)$.

6 Conclusion

This paper has explained the important factors needed to ensure a solution stands out and fits well within a certain type of mathematical space. This is particularly relevant to a set of problems involving equations with fractions and reactions that change over time. We have figured out these factors by using certain mathematical estimates and techniques. By building upon previous work and using a step-by-step method, we have confirmed that there is indeed a unique solution to these tricky equations. There will be future research and applications on fractional partial differential equations.

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Numerical Solution of Fractional Hopfield Neural Networks Using Reproducing Kernel Hilbert Space Method

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Received: May 24, 2024; Revised: September 25, 2024

Abstract: Artificial neural networks (ANN) consist of a group of the virtual neurons that are designed by computerized programs which use a variety of mathematical fractional equations. In this paper, we introduce the Reproducing Kernel Hilbert Space (RKHS) method for solving some certain fractional differential systems in the artificial neural networks field, which is the Hopfield network, using the conformable derivative.

Keywords: *Reproducing Kernel Hilbert Space Method (RKHSM), fractional derivative, artificial neural networks, differential systems, chaotic attractors.*

Mathematics Subject Classification (2010): 46E22, 26A33, 92B20, 34A30, 70K55.

1 Introduction

Artificial Neural Networks (ANN) is a recently emerging powerful computer-aided design (CAD) technology for modeling devices and circuits. These networks consist of a set of virtual neurons that are generated by computer programs that use a number of fractional mathematical equations to process the data that come from the neurons. The Hopfield network is a variety of recurrent artificial neural networks. Its idea arose from the behavior of particles in a magnetic field such that each particle is communicated (completely linked) with another particle by magnetic forces. This is referred to as activation in the

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case of neurons. As a result, both particles or neurons spin, encouraging one another to continue this rotation. The Hopfield neural networks come in two versions: binary and continuous. In the binary form, all neurons are linked to each other, there is no connection from a neuron to itself, in the continuous version, all connections, including self-connections, are allowed. The Hopfield network neurons will allow each other to rotate while they are in a spinning state. The movement of the particles is to process information, so they will be in the activation situation. For example, if two particles are in a rotating state to process information, this is known as binary activation in the Hopfield neural networks [1].

For obtaining the solution of the Hopfield neural network equation systems, we propose the reproducing kernel Hilbert space method which was used for the first time at the beginning of the 20th century by S. Zaremba for the harmonic and biharmonic functions to find solutions for boundary value problems (BVPs). The RKHS approach is a valuable framework for creating numerical solutions in applied sciences. This theory has been successfully extended to a variety of important applications in numerical analysis, computational mathematics, image processing, machine learning, probability and statistics, and finance [2–5], and it has been shown to be very effective in different fields of integrative equations [6,7], integrative differential equations [8–12] and partial differential equations [13, 14], and others [15–19], especially when the derivative order is fractional.

Recently, a new definition in the fractional calculus has been introduced concerning the conformable fractional derivative. This concept is a natural extension of the first-order derivative, and it satisfies some of the properties which are lost in the other fractional definitions such as those of the derivative of product and quotient of two functions formulas, and the chain rule.

This paper is organized as follows. In Section 2, some basic definitions and concepts are presented. We construct the reproducing kernel Hilbert spaces and present the structure of the analytical and approximate solutions in Section 3. In Sections 4 and 5, the convergence and error estimator are discussed to provide a number of numerical results to demonstrate the efficiency and accuracy of the reproducing kernel Hilbert space method. Finally, in Section 6, a short conclusion is provided.

2 Preliminaries and Backgrounds

In this section, we present some concepts and meanings of the conformable fractional derivative and the critical RKHS materials that will be used in this study.

Definition 2.1 Let the function $f : [0, \infty) \longrightarrow \mathbb{R}$, then the conformable fractional derivative of f of order $0 < \alpha \leq 1$ is given by

$$(\mathcal{D}^{\alpha} f)(x) = \lim_{\epsilon \to 0} \frac{f(x + \epsilon x^{1-\alpha}) - f(x)}{\epsilon}.$$

Moreover, if f is α-differentiable in some $(0, \alpha)$ and $\lim_{x\to 0} f^{(\alpha)}(x)$ exists, then $f^{(\alpha)}(0) = \lim_{x\to 0^+} f^{(\alpha)}(x)$.

Definition 2.2 Let $a \in (0, 1)$ and $f : [0, \infty) \longrightarrow \mathbb{R}$ be an $\alpha-$fractional integral function, then the "conformable fractional integral" of f is given by

$$\mathcal{I}_a^{\alpha}(f)(x) = \mathcal{I}_a^{1}(x^{\alpha-1} f) = \int_a^x \frac{f(t)}{t^{1-\alpha}} dt,$$

where the integral is the usual Riemann integral.

Theorem 2.1 *For $x \geq a$ and f being any continuous function in the domain of $\mathcal{I}^\alpha$, we have*

$$\mathcal{D}^\alpha(\mathcal{I}_a^\alpha f)(x) = f(x).$$

Proof. See [20].

Theorem 2.2 *If f is differentiable and $\alpha \in (0,1]$, then for $\forall x > a$, we have*

$$\mathcal{I}_a^\alpha \mathcal{D}_a^\alpha f(x) = f(x) - f(a).$$

Definition 2.3 Let $\mathcal{X}$ be a nonempty set, then the function $\mathcal{K} : \mathcal{X} \times \mathcal{X} \longrightarrow \mathbb{C}$ is a reproducing kernel of the Hilbert space $\mathcal{H}$ if and only if

1. $\mathcal{K}(.,x) \in \mathcal{X}, \forall x \in \mathcal{X}$,
2. $\forall x \in \mathcal{X}, \forall u \in \mathcal{H} : \langle u(.), \mathcal{K}(.,x)\rangle = u(x)$.

Here, the second condition is "the reproducing property"; the value of the function u at the point x is reproduced by the inner product of u with $\mathcal{K}(.,x)$. For instance, the reproducing kernel is unique, symmetric and positive definite.

Definition 2.4 The space $\Pi_2^m[a,b]$ is defined by

$$\Pi_2^m[a,b] = \left\{u \mid u^{(i)} \text{ are absolutely continuous, } i = 1,2,...,m-1 \text{ and } u^{(m)} \in L^2[a,b]\right\}.$$

The inner product and the norm of $\Pi_2^m[a,b]$ are given by

$$\langle u, v\rangle_{\Pi_2^m} = \textstyle\sum_{i=0}^{m-1} u^{(i)}(a)v^{(i)}(a) + \int_a^b u^{(m)}(t)v^{(m)}(t)dt$$

with

$$\|u\|_{\Pi_2^m} = \sqrt{\langle u, u\rangle_{\Pi_2^m}}.$$

3 Statement and Solution of the Problem

Consider the general HNN problem as follows:

$$\begin{cases} \mathcal{D}^\alpha X(\tau) &= -X(\tau) + WF(\tau, X), \\ X(0) &= A_0, \end{cases} \tag{1}$$

where $X = (x_1(\tau), x_2(\tau), \cdots, x_n(\tau))^T \in \mathbb{R}^n$, $W = (w_{ij}) \in M_{n\times n}$, $F = (f_1(X), f_2(X), \cdots, f_n(X))$, and $A_0 = (a_1, a_2, \cdots, a_n)$ such that a^η, $\eta = 1, ..., n$, are constants.

$\mathcal{D}^\alpha$ represents the conformable fractional derivative, and $\tau \in T = [a,b]$, f_η are non-linear (generally non-linear) continuous functions. To find the approximate solutions of the problem (1), we utilize the RKHS algorithm over the long interval $T = [a,b]$.
Consider the spaces $\Pi_2^1[a,b]$, $\Pi_2^2[a,b]$ which are defined, respectively, by

$$\Pi_2^1[a,b] = \left\{u \mid u \text{ is absolutely continuous, and } u' \in L^2[a,b]\right\},$$

$$\Pi_2^2[a,b] = \left\{u \mid u,\ u' \text{ are absolutely continuous, and } u,\ u', u'' \in L^2[a,b],\ u(a) = 0\right\}.$$

We give the inner product and the norm in the above spaces, respectively, as follows:

$$\begin{cases} \langle u,v\rangle_{\Pi_2^1} = u(a)v(a) + \int_a^b u'(s)v'(s)ds, \\ \|u\|_{\Pi_2^1} = \sqrt{\langle u,u\rangle_{\Pi_2^1}}, \\ \langle u,v\rangle_{\Pi_2^2} = u(a)v(a) + u'(a)v'(a) + \int_a^b u''(s)v''(s)ds, \\ \|u\|_{\Pi_2^2} = \sqrt{\langle u,u\rangle_{\Pi_2^2}}. \end{cases}$$

Suppose that the interval $T = [a,b]$ is divided into M subintervals $[\gamma^{m-1},\gamma^m],\ m = 1,...,M$, of equal step size $z = \frac{b-a}{M}$, by using the nodes $\gamma^m = mz$. Firstly, we apply the RKHS approach over the interval $[a,\gamma^1]$ to find the numerical solution. For the subintervals $[\gamma^{m-1},\gamma^m],\ m \geq 2$, we apply the RKHS method directly after using the initial conditions obtained over $[a,\gamma^1]$. Repeate the process and then generate a sequence of approximate solutions over $\{[a,\gamma^1],[\gamma^1,\gamma^2],...[\gamma^{M-1},\gamma^M]\}$.

3.1 Implementation of the process

The method steps can be summarized in the following points:

- Homogenize the initial conditions, and construct the space Π_2^2 in which each function satisfies the homogenous initial conditions of (1), then use the space Π_2^1.

- Take $\mathcal{K}$ and $\mathcal{R}$ as the reproducing kernel functions of the spaces Π_2^2 and Π_2^1, respectively, which are defined by

$$\mathcal{R}_\tau(t) = \frac{1}{\sinh(b-a)}[\cosh(t+\tau-b-a) + \cosh|t-\tau| - b - a],$$

  and

$$\mathcal{K}_\tau(t) = \frac{1}{6}\begin{cases} (\tau-a)(2a^2-\tau^2+3t(2+\tau)-a(6+3t+\tau))\ \tau \leq t, \\ (t-a)(2a^2-t^2+3\tau(2+t)-a(6+3\tau+t))\ \tau > t, \end{cases}$$

  and define the linear bounded operator $L : \Pi_2^2 \to \Pi_2^1$ such that $LX(\tau) = \mathcal{D}^\alpha X(\tau)$.

- Apply the conformable fractional integral to both sides and using $X(a) = 0$, we get

$$\begin{cases} X(\tau) = \mathcal{F}(\tau, X), \\ X(a) = 0. \end{cases} \tag{2}$$

- Choose a countable dense set $\{t_i\}_{i=1}^\infty$ from $[a,b]$ for the space $\Pi_2^2[a,b]$, and so define the complete system $\psi_i(\tau) = L^*\phi_i(\tau)$, where $\phi_i(\tau) = \mathcal{R}_{t_i}(\tau)$, and L^* is the adjoint operator of L.

- Derive an orthonormal functions system $\{\overline{\psi}_i(\tau)\}_{i=1}^\infty$ of the space $\Pi_2^2[a,b]$ from the Gram-Schmidt orthogonalization process of $\{\psi_i(\tau)\}_{i=1}^\infty$ as follows:

$$\overline{\psi}_i(\tau) = \sum_{k=1}^{i} B_{ik}\psi_k(\tau),\ i = 1,2,\cdots,$$

such that B_{ij} are positive orthogonalization coefficents which are given by

$$B_{11} = \frac{1}{\|\Psi_1\|},\ B_{ii} = \frac{1}{\sqrt{\|\Psi_i\|^2 - \sum_{k=1}^{i-1}(C_{ik})^2}},\ B_{ij} = \frac{-\sum_{k=1}^{i-1} C_{ik}B_{kj}}{\sqrt{\|\Psi_i\|^2 - \sum_{k=1}^{i-1}(C_{ik})^2}},\ j < i,$$

where $C_{ik} = \langle \Psi_i, \Psi_k \rangle_{\mathcal{W}_2^2}$.

Lemma 3.1 *$\psi_i(\tau)$ can be written as follows:*

$$\psi_i(\tau) = L_t\mathcal{K}_t(\tau) \mid_{\tau=t_i}.$$

Proof. See [19].

Theorem 3.1 *If L is an invertible operator, and if $\{t_i\}_{i=1}^{\infty}$ is dense on $[a, b]$, then $\{\psi_i(\tau)\}_{i=1}^{\infty}$ is the complete function system of the space $\Pi_2^2[a, b]$.*

Proof. See [19].

Theorem 3.2 *For every $X(\tau) \in \Pi_2^2[a, b]$, the series $\sum_{i=0}^{\infty} \left\langle X(\tau), \overline{\psi}_i(\tau) \right\rangle_{\pi_2^2} \overline{\psi}_i(\tau)$ are convergent in the sense of $\|.\|_{\Pi_2^2[a,b]}$. And if $\{t_i\}_{i=1}^{\infty}$ is a dense subset on $[a, b]$, then the solutions of (1) are given by*

$$X(\tau) = \sum_{i=1}^{\infty} \sum_{k=1}^{i} B_{ik}\mathcal{F}(\tau_k, X(\tau_k))\overline{\psi}_i(\tau). \tag{3}$$

Proof. The steps of the proof are detailed in [19]. The next algorithm explains the implementation of the procedure for solving a system of differential equations in numerical form in terms of their network nodes based on the RKHS method.

3.2 Existence and uniqueness

Let the continuous function $F = (f_1, f_2, ..., f_n)$ and the Banach space $E = \{(\tau, X) \in \mathbb{R} \times \mathbb{R}^n | \tau \in J,\ X \in B\}$, where

$$\begin{array}{rcl} J & = & [0, T], \\ B & = & \{X \in \mathbb{R}^n |\ \|X - X_0\| \le b\}. \end{array}$$

Suppose that F satisfies the Lipchitz condition, i.e., $\forall\ (\tau, X),\ (\tau, Y) \in E,\ \exists K \ge 0$ such that

$$\|F(\tau, X) - F(\tau, Y)\|_2 \le K \|(\tau, X) - (\tau, Y)\|_2 .$$

Theorem 3.3 *If F satisfies the following conditions:*

1. *F satisfies the Lipchitz condition.*
2. *$\|F(\tau, X(\tau))\|_2 \le C$.*
3. *$T^{\alpha} \le \min\left\{a^{\alpha} + \frac{\alpha b}{b+C\|W\|_2}, a^{\alpha} + \frac{\alpha}{2b(1+K\|W\|_2)}\right\}$,*

then the system (1) has a unique solution on $[0, T]$.

Proof. For all $(\tau, X) \in E$, and by applying the conformable fractional integral to both sides of system (1), we get

$$X(\tau) = \int_a^\tau \frac{-X(s) + WF(s, X(s))}{s^{1-\alpha}} ds = \varphi(\tau, X(\tau)).$$

We need to show that $\varphi(\tau, X(\tau))$ is a map from E to E. Let $(\tau, X(\tau)) \in E$, for all $\tau \leq T$, we have

$$\begin{aligned} \|\varphi(\tau, X(\tau)) - \varphi(\tau_0, X_0(\tau))\|_2 &= \|\varphi(\tau, X(\tau))\|_2 = \left\| \int_a^\tau \frac{-X(s)+WF(s,X(s))}{s^{1-\alpha}} ds \right\|_2 \\ &\leq [\|X\|_2 + \|W\|_2 . \|F\|_2] \left(\frac{T^\alpha - a^\alpha}{\alpha}\right) \\ &\leq [\|X\|_2 + C\|W\|_2] \left(\frac{T^\alpha - a^\alpha}{\alpha}\right) \end{aligned}$$

since $T^\alpha \leq a^\alpha + \frac{\alpha b}{b + C\|W\|_2}$ and $\|F\|_2 \leq C$, so $\|\varphi(\tau, X(\tau))\|_2 \leq b$. Then $\varphi(\tau, X(\tau))$ is a map from E to E.

Now, we prove that $\varphi(\tau, X(\tau))$ is a contraction. Let $(\tau, X(\tau)),\ (\tau, Y(\tau)) \in E$, we have

$$\begin{aligned} \|\varphi(\tau, X(\tau)) - \varphi(\tau, Y(\tau))\|_2 &= \left(\frac{T^\alpha - a^\alpha}{\alpha}\right) \|X - Y\|_2 + \left(\frac{T^\alpha - a^\alpha}{\alpha}\right) \|W\|_2 \|F(\tau, X(\tau)) - F(\tau, Y(\tau))\|_2 \\ &\leq \left(\frac{T^\alpha - a^\alpha}{\alpha}\right) \|X - Y\|_2 + K\left(\frac{T^\alpha - a^\alpha}{\alpha}\right) \|W\|_2 \|X - Y\|_2 \\ &= \left(\frac{T^\alpha - a^\alpha}{\alpha}\right) \|X - Y\|_2 \left[1 + K\|W\|_2\right] \end{aligned}$$

since $T^\alpha \leq \frac{\alpha}{2b(1+K\|W\|_2)} + a^\alpha$, then $\left(\frac{T^\alpha - a^\alpha}{\alpha}\right)[1 + K\|W\|_2] \leq \frac{1}{2}$, so $\varphi(\tau, X(\tau))$ is a contraction. Thus, by the Banach fixed point theorem, there is a unique fixed point, then the solution is unique on $[0, T]$.

Algorithm 3.1 *Use the following stages to approximate the solutions of the problem (1) based on the RKHS method.*
Input*: The endpoints of* $[a, b]$*, the unit truth interval* $[a, b]$*, the integers* n *and* m*, the kernel function* $\mathcal{K}_t(\tau)$*, the differential operator* L *, the initial condition* A_0 *, and the function* F*.*
Output*: Approximate solution* $X_n(\tau)$*.*

- ***Stage A:*** *Fixed* $\tau \in [a, b]$ *and set* $t \in [a, b]$
  for $i = 1, ..., n$ *do*
  - ***stage 1:*** *set* $t_i = a + \frac{i-1}{n-1}$;
  - ***stage 1:*** *if* $t \leq \tau$ *let*
  $$\mathcal{K}_\tau(t) = \frac{1}{6}(t - a)(2a^2 - t^2 + 6\tau + 3t\tau - a(6 + t + 3\tau));$$
    else let
  $$\mathcal{K}_\tau(t) = \frac{1}{6}(\tau - a)(2a^2 - \tau^2 + 6t + 3\tau t - a(6 + \tau + 3t)).$$
  - ***stage 2:*** *For* $i = 1, ..., n$ *do*
    set

$$\psi_i(\tau) = L_t\mathcal{K}_\tau(t)|_{\tau=t_i}.$$

Output the orthogonal functions system $\psi_i(\tau)$.

- ***Stage B:*** *Obtain the orthogonalization coefficients B_{ij} as follows:*

  For $i = 1, ..., n$;
  For $j = 1, ..., i$ *set* $C_{ik} = \langle\psi_i, \psi_j\rangle_{\Pi_2^2}$ *and* $B_{11} = \frac{1}{Sqrt(C_{11})}$.
  Output C_{ij} and B_{11}.

- ***Stage C:*** *For* $i = 1, ..., n$, *set* $B_{ii} = (\|\psi_i\|^2_{\Pi_2^2} - \sum_{k=1}^{i-1}(C_{ik})^2)^{\frac{-1}{2}}$;
  else if $j \neq i$ *set* $B_{ij} = -(\sum_{k=1}^{i-1} C_{ik}B_{kj}).(\|\psi_i\|^2_{\Pi_2^2} - \sum_{k=1}^{i-1}(C_{ik})^2)^{\frac{-1}{2}}$.
  Output the orthogonalization coefficients B_{ij}.

- ***Stage D:*** *For* $i = 1, ..., n$ *set* $\overline{\psi}_i(t) = \sum_{k=1}^{i} B_{ik}\psi_i(t)$.
  Output the orthonormal functions system $\overline{\psi}_i(t)$.

- ***Stage E:*** *Set* $\tau_1 = 0$ *and choose* $X_0(\tau_1) = 0$;
  For $i = 1, ..., n$; *set*

  $$\lambda_i = \sum_{k=1}^{i} B_{nk}\mathcal{F}(\tau_k, X_{k-1}(\tau_k))$$

  set

  $$X_n(\tau) = \sum_{i=1}^{n} \lambda_i\overline{\psi}_i(\tau).$$

  Outcome the numerical solutions $X_n(\tau)$.

4 Numerical Experiments

Example 4.1 Consider the following HNN system:

$$\begin{cases} \mathcal{D}^\alpha X &= -X + WF(X), \\ X(0) &= [-0.109, -0.832, 1.721]^T, \end{cases} \tag{4}$$

where

$$W = \begin{pmatrix} 2 & -1.2 & 0 \\ 2 & 1.7 & 1.15 \\ -4.75 & 0 & 1.1 \end{pmatrix}, \quad \text{and } F = \tanh(X).$$

Using the RKHS method for solving the system (4) and taking $n = 500$, we get the following results.

Example 4.2 Consider the following system:

$$\begin{cases} \mathcal{D}^\alpha X &= -X + WF(X), \\ X(0) &= [1.048, -0.2233, -0.3150]^T, \end{cases} \tag{5}$$

where

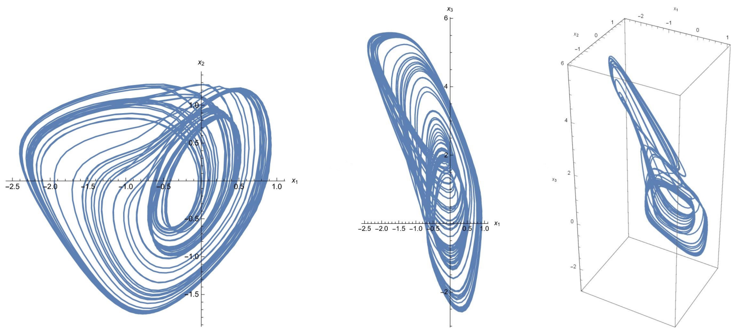

Figure 1: Chaotic attractors for $\alpha = 1$.

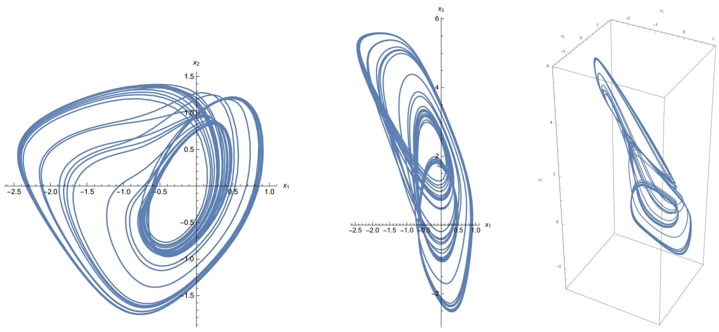

Figure 2: Chaotic attractors for $\alpha = 0.9$.

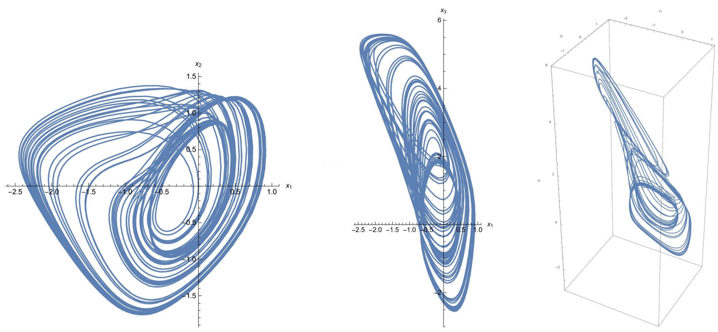

Figure 3: Chaotic attractors for $\alpha = 0.95$.

$$W = \begin{pmatrix} -1.4 & 1.2 & -7 \\ 1.1 & 0 & 2.8 \\ -k & -2 & 4 \end{pmatrix}, \quad \text{and } F = \tanh(X).$$

Using the RKHS method for solving the system (5) and taking $n = 250$, we get the following results.

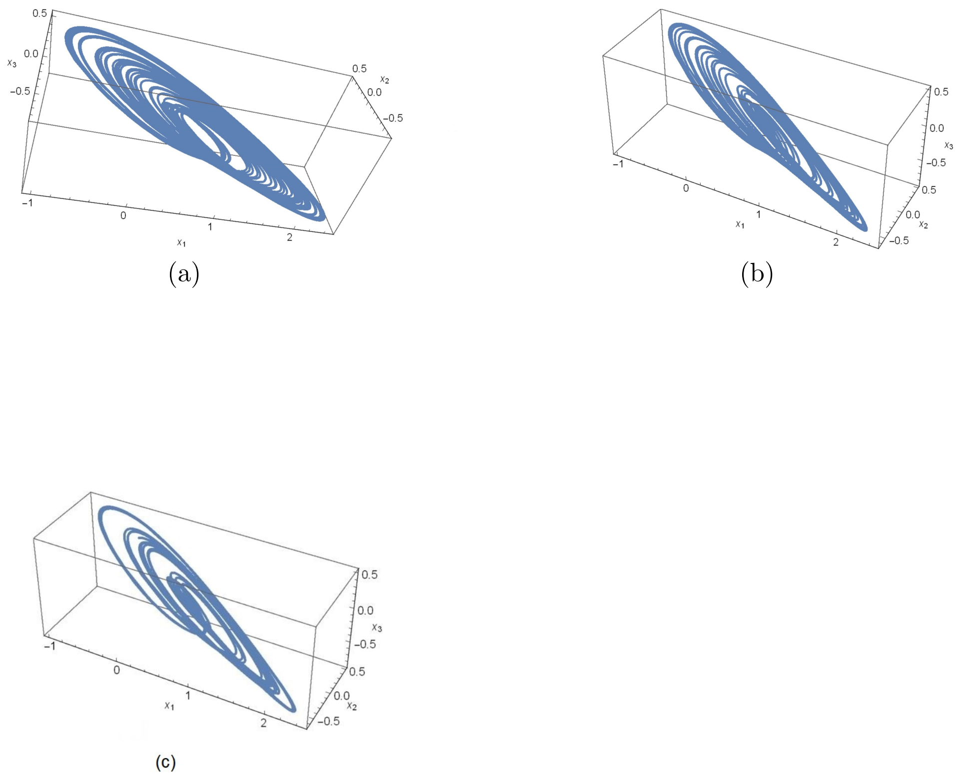

Figure 4: Chaotic attractors of HNN for Example 4.2 such that: (a): $\alpha = 0.95,\ k = 0.95/$ (b): $\alpha = 0.85,\ k = 0.95/$ and (c): $\alpha = 0.75,\ k = 0.95$.

Example 4.3 Consider the following HNN system:

$$\begin{cases} \mathcal{D}^{\alpha} X &= -X + WF(X), \\ X(0) &= [0.0225, 0.1788, -4.831]^T, \end{cases} \tag{6}$$

where

$$W = \begin{pmatrix} 3.4 & -1.6 & 0.7 \\ 2.5 & 0 & 0.95 \\ k & 0.5 & 0 \end{pmatrix}, \quad \text{and } F = \tanh(X).$$

Using the RKHS method for solving the system (6) and taking $n = 300$, we get the following results.

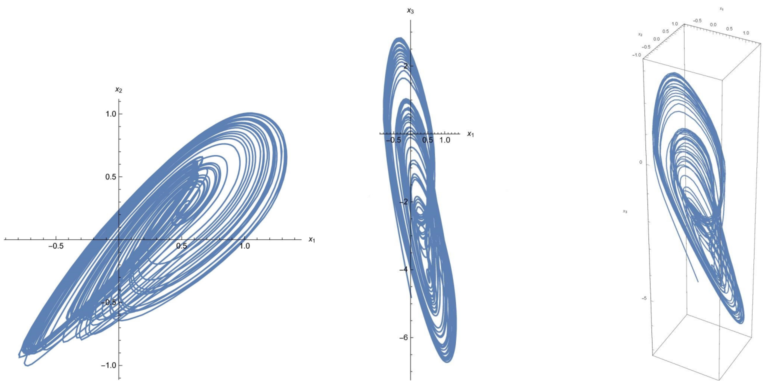

Figure 5: Chaotic attractors for $\alpha = 1$ and $k = -9$.

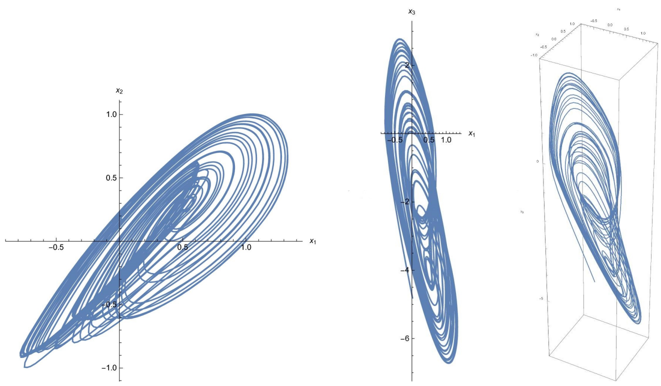

Figure 6: Chaotic attractors for $\alpha = 0.95$ and $k = -9$.

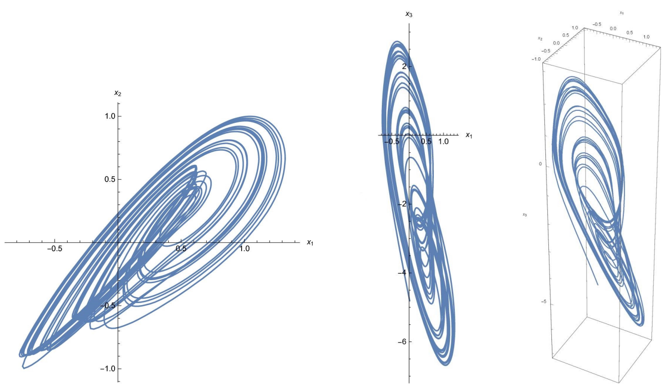

Figure 7: Chaotic attractors for $\alpha = 0.9$ and $k = -9$.

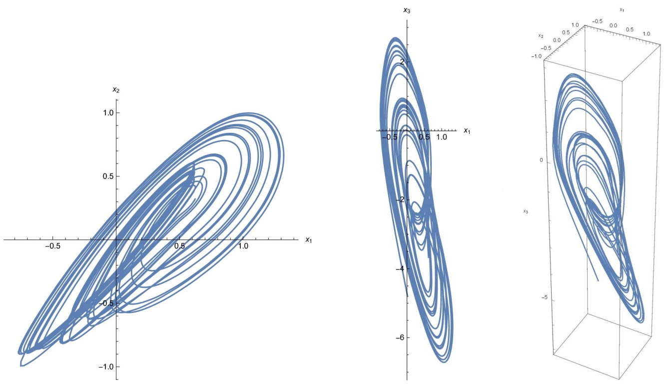

Figure 8: Chaotic attractors for $\alpha = 0.85$ and $k = -9$.

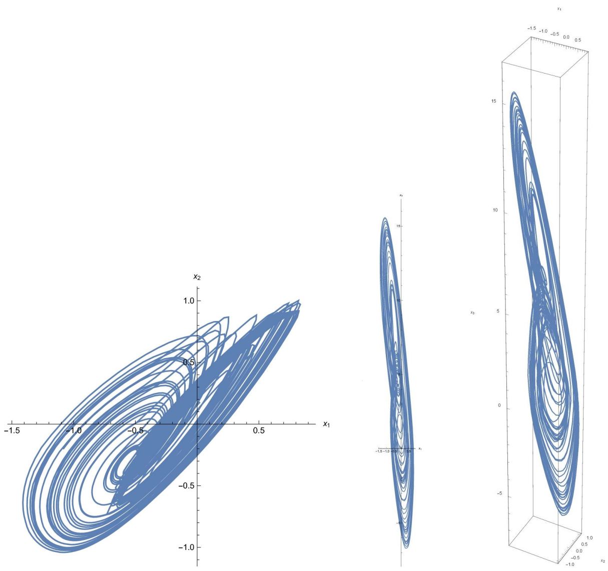

Figure 9: Chaotic attractors for $\alpha = 1$ and $k = -20$.

5 Conclusion

The Fractional Hopfield Neural Networks enhance complex nonlinear dynamics modeling by incorporating fractional calculus concepts, capturing long-term dependencies and memory effects more accurately, which is crucial for the investigation of real-world phenomena.

In this paper, we effectively used the RKHS method to solve numerical equation systems for the Hopfield Neural Network Equation Systems. We have demonstrated the accuracy and efficiency of the method in solving the systems of Hopfield neural network equations through the results we got in the numerical examples. In the future, we recommend that more research be done on the RKHS method for neural network systems. We also expect good results by solving the systems of Hopfield neural network equations with the Caputo and Atangana-Baleanu fractional derivatives.

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A Dynamic Problem with Wear Involving Thermoviscoelastic Materials with a Long Memory

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Received: February 2, 2024; Revised: October 9, 2024

Abstract: We consider a dynamic contact problem with friction in thermo-viscoelasticity with long memory body. The body is in contact with an obstacle. The contact is frictional and bilateral with a moving rigid foundation which results in the wear of the contacting surface. The problem is formulated as a coupled system of an elliptic variational inequality for the displacement and the heat equation for the temperature. We establish a variational formulation for the model and we prove the existence of a unique weak solution to the problem. The proof is based on a classical existence and uniqueness result for parabolic inequalities, differential equations and fixed point arguments.

Keywords: *frictional contact; thermo-visco-elastic; fixed point; dynamic process; variational inequality; wear.*

Mathematics Subject Classification (2010): 74M10, 74M15, 74F15, 49J40, 70k75, 93-10.

1 Introduction

Scientific research and recent papers in mechanics are articulated around two main components, one devoted to the laws of behavior and the other devoted to boundary conditions imposed on the body. The boundary conditions reflect the binding of the body with the outside world. Recent researches use coupled laws of behavior between mechanical and electric effects or between mechanical and thermal effects. For the case of coupled laws of behavior between mechanical and electric effects, general models can be found in [5,6]. For the case of coupled laws of behavior between mechanical and thermal effects, the transmission problem in thermo-viscoplasticity is studied in [3], the contact problem

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with adhesion for thermo-viscoplasticity is considered in [1]. Situations of contact between deformable bodies are very common in the industry and everyday life. Contact of braking pads with wheels, tires with roads, pistons with skirts or the complex metal forming processes are just a few examples. The constitutive laws with internal variables have been used in various publications in order to model the effect of internal variables in the behavior of real bodies like metals, rocks, polymers and so on, for which the rate of deformation depends on the internal variables. Some of the internal state variables considered by many authors are the spatial display of dislocation, the work-hardening of materials. In this paper, we consider a general model for the dynamic process of bilateral frictional contact between a deformable body and an obstacle, which results in the wear of the contacting surface. The material obeys a thermo-viscoelasticity constitutive law with long memory body. We derive a variational formulation of the problem which includes a variational second order evolution inequality. We establish the existence of a unique weak solution of the problem. The idea is to reduce the second order nonlinear evolution inequality of the system to the first order evolution inequality. After this, we use classical results for first order nonlinear evolution inequalities and equation, a parabolic variational inequality and the fixed point arguments.

The paper is structured as follows. In Section 2, we present the thermo-visco-elastic contact model with friction and provide comments on the contact boundary conditions. In Section 3, we list the assumptions on the data and derive the variational formulation. In Section 4, we present our main existence and uniqueness results.

2 Problem Statement

Problem P: Find a displacement field $u : \Omega\times[0.T] \to \mathbb{R}^d$, a stress field $\sigma : \Omega\times[0.T] \to \mathbb{S}^d$, the temperature $\theta : \Omega \times [0.T] \to \mathbb{R}_+$ and the wear $\omega : \Gamma_3 \times [0.T] \to \mathbb{R}_+$ such that

$$\sigma = \mathcal{A}(\varepsilon(u(t))) + \mathcal{G}(\varepsilon(\dot{u}(t))) + \int_0^t \mathcal{B}(t-s)\varepsilon(u(s))ds - \theta(t)\mathcal{M}, \quad \text{in } \Omega \times [0.T], \tag{1}$$

$$\rho\ddot{u} = Div\ \sigma + f_0, \quad \text{in } \Omega \times [0.T], \tag{2}$$

$$\dot{\theta} - div(K\nabla\theta) = -\mathcal{M}.\nabla\dot{u} + q, \ \text{in } \Omega \times [0.T], \tag{3}$$

$$u = 0, \text{on } \Gamma_1 \times [0.T], \tag{4}$$

$$\sigma\nu = h, \text{on } \Gamma_2 \ \times [0.T], \tag{5}$$

$$\begin{cases} \sigma_\nu = -\alpha\left|\dot{u}_\nu\right|, \ \ |\sigma_\tau| = -\mu\sigma_\nu, \\ \sigma_\tau = -\lambda\left(\dot{u}_\tau - v^*\right), \ \lambda \geq 0, \ \dot{\omega} = -kv^*\sigma_\nu, \ k > 0. \text{ on } \Gamma_3 \ \times [0.T], \end{cases} \tag{6}$$

$$-k_{ij}\frac{\partial\theta}{\partial\nu}\nu_j = k_e(\theta - \theta_R) - h_\tau(|\dot{u}_\tau|), \ \text{on} \Gamma_3 \times [0.T], \tag{7}$$

$$\theta = 0, \ \text{in } \Gamma_1 \cup \ \Gamma_2 \times [0.T], \tag{8}$$

$$u(0) = u_0, \dot{u}(0) = u_1, \theta(0) = \theta_0, \omega(0) = \omega_0, \ \text{in } \Omega, \tag{9}$$

where (1) is the thermo-visco-elastic constitutive law with long memory, we denote $\varepsilon(u)$ (respectively, $\mathcal{A}, \mathcal{G}, \xi, \xi^*$) the linearized strain tensor (respectively, the elasticity tensor, the viscosity nonlinear tensor, the third order piezoelectric tensor and its transpose), (2) represents the equation of motion, where ρ represents the mass density, we mention that $Div\sigma$ is the divergence operator, (3) represents the evolution equation of the heat field, (4) and (5) are the displacement and traction boundary conditions, (6) describes the bilateral frictional contact with wear described above on the potential contact surface Γ_3, (7) is the pointwise heat exchange condition on the contact surface, where k_{ij} are the components of the thermal conductivity tensor, v_j are the normal components of the outward unit normal v, k_e is the heat exchange coefficient, θ_R is the known temperature of the foundation. (8) represents the temperature boundary conditions. Finally, (9) represents the initial conditions.

3 Variational Formulation and Preliminaries

For a weak formulation of the problem, first, we introduce some notation. The indices i, j, k, l range from 1 to d and summation over repeated indices is implied. An index that follows the comma represents the partial derivative with respect to the corresponding component of the spatial variable, e.g., $u_{i.j} = \frac{\partial u_i}{\partial x_j}$. We also use the following notations:

$$\begin{gathered} H = \mathbb{L}^2(\Omega)^d = \{u = (u_i)/u_i \in \mathbb{L}^2(\Omega)\}, \\ \mathcal{H} = \sigma = (\sigma_{ij})/\sigma_{ij} = \sigma_{ji} \in \mathbb{L}^2(\Omega), \\ H_1 = u = (u_i)/\varepsilon(u) \in \mathcal{H} = H^1(\Omega)^d \\ \mathcal{H}_1 = \sigma \in \mathcal{H}/Div\sigma \in H. \end{gathered}$$

The operators of deformation ε and divergence Div are defined by

$$\varepsilon(u) = (\varepsilon_{ij}(u)), \varepsilon_{ij}(u) = \tfrac{1}{2}(u_{i,j} + u_{j,i}), Div\sigma = (\sigma_{ij,j}).$$

The spaces $H, \mathcal{H}, H_1$ and $\mathcal{H}_1$ are real Hilbert spaces endowed with the canonical inner products given by

$$\begin{gathered} (u, v)_H = \textstyle\int_\Omega u_i v_i dx, \forall u, v \in H, \\ (\sigma, \tau)_{\mathcal{H}} = \textstyle\int_\Omega \sigma_{ij}\tau_{ij} dx, \forall \sigma, \tau \in \mathcal{H}, \\ (u, v)_{H_1} = (u, v)_H + (\varepsilon(u), \varepsilon(v))_{\mathcal{H}}, \forall u, v \in H_1, \\ (\sigma, \tau)_{\mathcal{H}_1} = (\sigma, \tau)_{\mathcal{H}} + (Div\sigma, Div\tau)_H, \sigma, \tau \in \mathcal{H}_1. \end{gathered}$$

We denote by $|\cdot|_H$ (respectively, $|\cdot|_{\mathcal{H}}$, $|\cdot|_{H_1}$and $|\cdot|_{\mathcal{H}1}$) the associated norm on the space H (respectively, $\mathcal{H}$, H_1 and $\mathcal{H}_1$).

Let $H_\Gamma = (H^{1/2}(\Gamma))^d$ and $\gamma : H^1(\Gamma)^d \to H_\Gamma$ be the trace map. For every element $v \in (H^1(\Gamma))^d$, we also use the notation v to denote the trace map γv of v on Γ, and we denote by v_ν and v_τ the normal and tangential components of v on Γ given by

$$v_\nu = v.\nu, v_\tau = v - v_\nu.$$

Similarly, for a regular (say $\mathcal{C}^1$) tensor field $\sigma : \Omega \to \mathbb{S}^d$, we define its normal and tangential components by

$$\sigma_\nu = (\sigma\nu).\nu, \sigma_\tau = \sigma\nu - \sigma_\nu.$$

We use standard notation for the $\mathbb{L}^p$ and the Sobolev spaces associated with Ω and Γ and, for a function $\psi \in H^1(\Omega)$, we still write ψ to denote its trace on Γ. We recall that the summation convention applies to a repeated index.

When σ is a regular function, the following Green's type formula holds:

$$(\sigma, \varepsilon(v))_{\mathcal{H}} + (Div\sigma, v)_H = \int_\Gamma \sigma\nu.vda \quad \forall v \in H_1. \tag{10}$$

Next, we define the space

$$V = \{u \in H_1 / \ u = 0 \ \text{ on } \Gamma_1\}.$$

Since $meas$ $(\Gamma_1) > 0$, the following Korn's inequality holds:

$$|\varepsilon(u)|_{\mathcal{H}} \geq c_K \, |v|_{H_1} \, \forall v \in V, \tag{11}$$

where $c_K > 0$ is a constant which depends only on Ω and Γ_1. On the space V, we use the inner product

$$(u, v)_V = (\varepsilon(u), \varepsilon(v))_{\mathcal{H}}, \tag{12}$$

let $|.|_V$ be the associated norm. It follows by (12) that the norms $|.|_{H_1}$ and $|.|_V$ are equivalent norms on V and therefore, $(V, |.|_V)$ is a real Hilbert space. Moreover, by the Sobolev trace theorem, there exists a constant c_0 depending only on the domain Ω, Γ_1 and Γ_3 such that

$$|v|_{\mathbb{L}^2(\Gamma_3)^d} \leq c_0 \, |v|_V \, \forall v \in V. \tag{13}$$

Finally, for a real Banach space $(X, \ |.|_X)$, we use the usual notation for the space $\mathbb{L}^p\,(0.T, X)$ and $W^{k.p}\,(0.T, X)$, where $1 \leq p \leq \infty, k = 1, 2......$; we also denote by $C\,(0.T, X)$ and $C^1\,(0.T, X)$ the spaces of continuous and continuously differentiable function on $[0.T]$ with values in X, with the respective norms:

$$|x|_{C(0.T,X)} = \max_{t\in[0.T]} |x\,(t)|_X \,,$$

$$|x|_{C^1(0.T,X)} = \max_{t\in[0.T]} |x\,(t)|_X + \max_{t\in[0.T]} \left|\dot{x}\,(t)\right|_X .$$

In what follows, we assume the following assumptions on the problem P. The elasticity operator $\mathcal{A} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d$ satisfies

$$\begin{cases} (a) \, \exists \ L_{\mathcal{A}} > 0 \text{ such that } : |\mathcal{A}\,(x, \varepsilon_1) - \mathcal{A}\,(x, \varepsilon_2)| \leq L_{\mathcal{A}}\,|\varepsilon_1 - \varepsilon_2| \\ \forall \ \varepsilon_1 \,, \varepsilon_2 \in \mathbb{S}^d, \ a.\ e.\ x \in \Omega, \\ (c) \ \text{ The mapping } x \to \mathcal{A}\,(x, \varepsilon) \ \text{ is Lebesgue measurable in } \Omega \text{ for all } \varepsilon \in \mathbb{S}^d, \\ (d) \ \text{ The mapping } x \to \mathcal{A}\,(x, 0) \in \mathcal{H}. \end{cases} \tag{14}$$

The viscosity operator $\mathcal{G} : \Omega \times \mathbb{S}^d \ \times \mathbb{S}^d \to \mathbb{S}^d$ satisfies

$$\begin{cases} (a) \ \exists \ L_{\mathcal{G}} > 0 : |\mathcal{G}(x, \varepsilon_1) - \mathcal{G}(x, \varepsilon_2)| \leq L_{\mathcal{G}}|\varepsilon_1 - \varepsilon_1|, \forall \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d, \ a.e. \ x \in \Omega, \\ (b) \ \exists m_{\mathcal{G}} > 0 : (\mathcal{G}\,(x, \varepsilon_1) - \mathcal{G}\,(x, \varepsilon_2)\,, \varepsilon_1 - \varepsilon_2) \geq m_{\mathcal{G}}\,|\varepsilon_1 - \varepsilon_2|^2 \,, \forall \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d, \\ (c) \text{ the mapping } x \to \mathcal{G}\,(x, \varepsilon) \ \text{ is Lebesgue measurable in } \Omega \text{ fo rall } \varepsilon \in \mathbb{S}^d, \\ (d) \text{ the mapping } x \mapsto \mathcal{G}(x, 0) \in \ \mathcal{H}. \end{cases} \tag{15}$$

The relaxation tensor $\mathcal{B} : [0.T] \times \Omega \times \mathbb{S}^d \to \mathbb{S}^d$ such that $(t, x, \tau) \mapsto (\mathcal{B}_{ijkh}\,(t, x)\,\tau_{kh})$ satisfies

$$\begin{cases} (a) \mathcal{B}_{ijkh} \in W^{1.\infty}(0.T, \mathbb{L}^\infty(\Omega)), \\ (b) \mathcal{B}\,(t)\,\sigma \cdot \tau = \sigma \cdot \mathcal{B}\,(t)\,\tau, \forall \sigma, \tau \in \mathbb{S}^d, \ p.p.t \in [0.T]\,, \ a.e.\text{in } \Omega. \end{cases} \tag{16}$$

The function $h_\tau : \Gamma_3 \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ satisfies

$$\begin{cases} (a) \ \exists \ L_\tau > 0 \ : \ |h_\tau(x, r_1) - \ h_\tau(x, r_2)| \leq L_h|r_1 - r_2| \quad \forall r_1, r_2 \in \mathbb{R}_+, \ \ a.e.x \in \Gamma_3, \\ (b) \ x \longrightarrow \ h_\tau\,(x, r) \in \mathbb{L}^2(\Gamma_3) \ \text{ is Lebesgue measurable in} \Gamma_3, \forall r \in \mathbb{R}_+. \end{cases} \tag{17}$$

The mass density ρ satisfies

$$\rho \in \mathbb{L}^\infty(\Omega) \text{ there exists } \rho^* > 0 \text{ such that } \rho(x) \geq \rho^*, a.e.x \in \Omega. \tag{18}$$

The body forces, surface tractions, the densities of electric charges, and the functions α and μ satisfy

$$\begin{cases} f_0 \in \mathbb{L}^2(0.T, H), h \in \mathbb{L}^2(0.T, \mathbb{L}^2(\Gamma_2)^d), \\ q \in W^{1.\infty}(0.T, \mathbb{L}^2(\Omega)), \theta_R \in W^{1.\infty}(0.T, \mathbb{L}^2(\Gamma_3)), k_e \in \mathbb{L}^\infty(\Omega, \mathbb{R}_+), \\ \mathcal{M} = (m_{ij}), m_{ij} = m_{ji} \in \mathbb{L}^\infty(\Omega), \\ \begin{cases} K = (k_{i,j});\ k_{ij} = k_{ji} \in \mathbb{L}^\infty(\Omega), \\ \forall c_k > 0, \forall (\xi_i) \in \mathbb{R}^d, k_{ij}\xi_i\xi_j \geq c_k \xi_i \xi_i. \end{cases} \\ \alpha \in \mathbb{L}^\infty(\Gamma_3), \alpha(x) \geq \alpha^* > 0, a.e. \text{on } \Gamma_3, \\ \mu \in \mathbb{L}^\infty(\Gamma_3), \mu(x) > 0, a.e. \text{on } \Gamma_3. \end{cases} \tag{19}$$

The initial data satisfies

$$u_0 \in V, u_1 \in \mathbb{L}^2(\Omega), \theta_0 \in \mathbb{L}^2(\Omega), \omega_0 \in \mathbb{L}^\infty(\Gamma_3). \tag{20}$$

We use a modified inner product on $H = \mathbb{L}^2(\Omega)^d$ given by

$$((u, v)) = (\rho u, v)_{\mathbb{L}^2(\Omega)^d}, \forall u.v \in H.$$

That is, it is weighted with ρ. We let H be with the associated norm

$$\|v\|_H = (\rho v, v)^{\frac{1}{2}}_{\mathbb{L}^2(\Omega)^d},\ \forall v \in H.$$

We use the notation $(.,.)_{V' \times V}$ to represent the duality pairing between V' and V . Then we have

$$(u, v)_{V' \times V} = ((u, v)), \forall u \in H, \forall v \in V.$$

It follows from assumption (18) that $\|.\|_H$ and $|.|_H$ are equivalent norms on H, and also, the inclusion mapping of $(V, |.|_V)$ into $(H, \|.\|_H)$ is continuous and dense. We denote by V' the dual space of V. Identifying H with its own dual, we can write the Gelfand triple $V \subset H = H' \subset V'$.

We define the space

$$E = \{\gamma \in H^1(\Omega)/\gamma = 0 \text{ on } \Gamma_1 \cup \Gamma_2\}. \tag{21}$$

We define the function $f(t) \in V$ by

$$(f(t), v)_V = \int_\Omega f_0(t) v dx + \int_{\Gamma_2} h(t) v da, \forall v \in V, t \in [0.T],$$

for all u, $v \in V$, $\psi \in W$ and $t \in [0.T]$, and note that condition (19) implies that

$$f \in \mathbb{L}^2(0.T, V'). \tag{22}$$

We consider the wear functional $j : V \times V \to \mathbb{R}$,

$$j(u, v) = \int_{\Gamma_3} \alpha |u_\nu| (\mu |v_\tau - v^*|) da. \tag{23}$$

Finally, we consider $\phi : V \times V \to \mathbb{R}$,

$$\phi(u,v) = \int_{\Gamma_3} \alpha \left| u_\nu \right| v_\nu da, \forall v \in V. \tag{24}$$

We define for all $\varepsilon > 0$,

$$j_\varepsilon(g,v) = \int_{\Gamma_3} \alpha \left| g_\nu \right| (\mu\sqrt{|v_\tau - v^*|^2 + \varepsilon^2}) da, \forall v \in V.$$

We define $Q : [0,T] \to E^{'}$; $K : E \to E^{'}$ and $R : V \to E^{'}$ by

$$(Q(t),\mu)_{E'\times E} = \int_{\Gamma_3} k_e \theta_R(t)\mu ds + \int_\Omega q\mu dx, \forall \mu \in E, \tag{25}$$

$$(K\tau,\mu)_{E'\times E} = \sum_{i,j=1}^{d} \int_\Omega k_{ij} \frac{\partial \tau}{\partial x_j} \frac{\partial \mu}{\partial x_i} dx + \int_{\Gamma_3} k_e \tau \mu ds, \forall \mu \in E, \tag{26}$$

$$(Rv,\mu)_{E'\times E} = \int_{\Gamma_3} h_\tau(|v_\tau|)\mu dx - \int_\Omega (\mathcal{M}.\nabla v)\mu dx, \forall v \in V, \tau, \mu \in E. \tag{27}$$

Using the above notation and Green's formula, we derive the following variational formulation of mechanical problem P.

Problem PV : Find a displacement field $u : \Omega \times [0.T] \to V$, a stress field $\sigma : \Omega \times [0.T] \to \mathbb{S}^d$, the temperature $\theta : \Omega \times [0.T] \to \mathbb{R}_+$ and the wear $\omega : \Gamma_3 \times [0.T] \to \mathbb{R}_+$ such that

$$\begin{aligned} &(\ddot{u}(t), w - \dot{u}(t))_{V'\times V} + (\sigma(t), \varepsilon(w - \dot{u}(t)))_{\mathcal{H}} + j(\dot{u}, w) - j(\dot{u}, \dot{u}(t)) + \\ &\phi(\dot{u}, w) - \phi(\dot{u}, \dot{u}(t)) \geq \left(f(t), w - \dot{u}(t)\right), \ \forall u, w \in V, \end{aligned} \tag{28}$$

$$\dot{\theta}(t) + K\theta(t) = R\dot{u}(t) + Q(t), \text{ on } E' \tag{29}$$

$$\dot{\omega} = -kv^*\sigma_\nu. \tag{30}$$

4 Existence and Uniqueness Result

Our main result which states the unique solvability of Problem PV is the following.

Theorem 4.1 *Let the assumptions* (14) − (20) *hold. Then Problem PV has a unique solution* $(u, \sigma, \theta, \omega)$ *which satisfies*

$$u \in C^1(0.T, H) \cap W^{1.2}(0.T, V) \cap W^{2.2}(0.T, V'), \tag{31}$$

$$\sigma \in \mathbb{L}^2(0.T, \mathcal{H}_1), Div\sigma \in \mathbb{L}^2(0.T, V'), \tag{32}$$

$$\theta \in W^{1.2}(0.T, E') \cap \mathbb{L}^2(0.T, E) \cap C(0.T, \mathbb{L}^2(\Omega)), \tag{33}$$

$$\omega \in C^1(0.T, \mathbb{L}^2(\Gamma_3)). \tag{34}$$

We conclude that under the assumptions (14) − (20), the mechanical problem (1) − (9) has a unique weak solution with the regularity (31) − (34). The proof of this theorem will be carried out in several steps. It is based on the arguments of first order nonlinear evolution inequalities, evolution equations, a parabolic variational inequality, and fixed point arguments.

First step: Let $g \in \mathbb{L}^2(0.T, V)$ and $\eta \in \mathbb{L}^2(0.T, V')$ be given, we deduce a variational formulation of Problem PV.

Problem $PV_{g\eta}$: Find a displacement field $u_{g\eta} : [0.T] \to V$ such that

$$\begin{cases} (\ddot{u}_{g\eta}(t), w - \dot{u}_{g\eta}(t))_{V'\times V} + (\mathcal{G}\varepsilon(\dot{u}_{g\eta}(t)), w - \dot{u}_{g\eta}(t))_{V'\times V} + (\eta(t), w - \dot{u}_{g\eta})_{V'\times V} \\ j(g, w) - j(g, \dot{u}_{g\eta}(t)) \geq (f(t), w - \dot{u}_{g\eta}(t))_{V'\times V} \ , \ \forall \ w \in V, t \in [0.T], \\ u_{g\eta}(0) = u_0, \dot{u}_{g\eta}(0) = u_1. \end{cases} \tag{35}$$

We define $f_\eta(t) \in V$ for $a.e.t \in [0.T]$ by

$$(f_\eta(t), w)_{V' \times V} = (f(t) - \eta(t), w)_{V' \times V}, \forall w \in V. \tag{36}$$

From (22), we deduce that

$$f_\eta \in \mathbb{L}^2(0.T, V'). \tag{37}$$

Let now $u_{g\eta} : [0.T] \to V$ be the function defined by

$$u_{g\eta}(t) = \int_0^t v_{g\eta}(s)ds + u_0, \forall t \in [0.T]. \tag{38}$$

We define the operator $G : V' \to V$ by

$$(Gv, w)_{V' \times V} = (\mathcal{G}\varepsilon(v(t)), \varepsilon(w))_{\mathcal{H}}, \forall v, w \in V. \tag{39}$$

Lemma 4.1 *For all $g \in \mathbb{L}^2(0.T, V)$ and $\eta \in \mathbb{L}^2(0.T, V')$, $PV_{g\eta}$ has a unique solution with the regularity*

$$v_{g\eta} \in C(0.T, H) \cap \mathbb{L}^2(0.T, V) \text{ and } \quad \dot{v}_{g\eta} \in \mathbb{L}^2(0.T, V'). \tag{40}$$

Proof. The proof from (35) nonlinear second order evolution inequalities is given in [2, 4, 7, 8].

In the second step, we use the displacement field $u_{g\eta}$ to consider the following variational problem.

Second step: We use the displacement field $u_{g\eta}$ to consider the following variational problem.

Problem $P_{\theta g\eta}$: Find $\theta_{g\eta} \in E$ such that

$$\dot{\theta}_{g\eta}(t) + K\theta_{g\eta}(t) = R\dot{u}_{g\eta}(t) + Q(t), \text{ on } E'. \tag{41}$$

Lemma 4.2 *Under the assumptions* $(14) - (20)$, *the problem $P_{\theta g\eta}$ has a unique solution*

$$\theta_{g\eta} \in W^{1.2}(0.T, E') \cap \mathbb{L}^2(0.T, E) \cap C(0.T, \mathbb{L}^2(\Omega)).$$

Proof. Since we have the Gelfand triple $E \subset \mathbb{L}^2(\Omega) \subset E'$, we use a classical result for first order evolution equations given in [9] to prove the unique solvability of (41). Now, we have $\theta_0 \in \mathbb{L}^2(\Omega)$. The operator K is linear and continuous, so $a(\tau, \mu) = (K\tau, \mu)_{E' \times E}$ is bilinear, continuous and coercive, we use the continuity of $a(.,.)$ and from (19), we deduce that

$$a(\tau, \mu) = (K\tau, \mu)_{E' \times E} \leq |k|_{\mathbb{L}^\infty(\Omega)^{d \times d}} |\nabla\tau|_E |\nabla\mu|_E + |k_e|_{\mathbb{L}^\infty(\Gamma_3)} |\tau|_{\mathbb{L}^2(\Gamma_3)} |\mu|_{\mathbb{L}^2(\Gamma_3)}$$
$$\leq C |\tau|_E |\mu|_E .$$

We have

$$a(\tau, \tau) = (K\tau, \tau)_{E' \times E} = \sum_{i,j=1}^{d} \int_\Omega k_{ij} \frac{\partial\tau}{\partial x_j} \frac{\partial\tau}{\partial x_i} dx + \int_{\Gamma_3} k_e \tau^2 ds.$$

By(19), there exists a constant $C > 0$ such that

$$(K\tau, \tau)_{E' \times E} \geq C\, |\tau|_E^2 .$$

We have $\theta_0 \in \mathbb{L}^2(\Omega)$. Let

$$F(t) \in E' : \ \ (F(t), \tau)_{E' \times E} = (R\dot{u}_{g\eta}(t) + Q(t), \tau) \quad \forall \tau \in E.$$

Under the assumptions (17), (19), we have

$$\textstyle\int_0^T |R\dot{u}|_{E'}^2\, dt < \infty, \quad \int_0^T |Q(t)|_{\mathbb{E}'}^2\, dt < \infty, \quad \int_0^T |F|_{E'}^2\, dt < \infty.$$

We find

$$F \in \mathbb{L}^2(0.T, E^{'}).$$

By a classical result for first order evolution equations,

$$\exists ! \theta_{g\eta} \in W^{1.2}(0.T, E^{'}) \cap \mathbb{L}^2(0.T, E) \cap C(0.T, \mathbb{L}^2(\Omega)).$$

Consider the operator

$$\begin{aligned}
&\Lambda : \mathbb{L}^2(0.T, V \times V') \to \mathbb{L}^2(0.T, V \times V'),\\
&\Lambda(g, \eta) = (\Lambda_1(g), \Lambda_2(\eta)), \forall g \in \mathbb{L}^2(0.T, V), \forall \eta \in \mathbb{L}^2(0.T, V'),\\
&\Lambda_1(g) = v_{g\eta},\\
&(\Lambda_2(\eta), w)_{V' \times V} = (\mathcal{A}(\varepsilon(u(t))), \varepsilon(w))_{\mathcal{H}} + (\textstyle\int_0^t \mathcal{B}(t-s)\varepsilon(u(s))ds - \theta(t)\mathcal{M}, \varepsilon(w))_{\mathcal{H}}\\
&+\phi(g, w),\\
&|\Lambda(g_2, \eta_2) - \Lambda(g_1, \eta_2)|_{\mathbb{L}^2(0.T;V \times V')}^2 = |(\Lambda_1(g_2), \Lambda_2(\eta_2)) - (\Lambda_1(g_1), \Lambda_2(\eta_1))|_{\mathbb{L}^2(0.T;V \times V')}^2 ,\\
&= |\Lambda_1(g_2) - \Lambda_1(g_1)|_{\mathbb{L}^2(0.T;V \times V')}^2 + |\Lambda_2(\eta_2) - \Lambda_2(\eta_1)|_{\mathbb{L}^2(0.T;V \times V')}^2 .
\end{aligned} \tag{42}$$

We have the following result.

Lemma 4.3 *The mapping* $\Lambda : \mathbb{L}^2(0.T, V \times V') \to \mathbb{L}^2(0.T, V \times V')$ *has a unique element* $(g^*, \eta^*) \in \mathbb{L}^2(0.T, V \times V')$ *such that*

$$\Lambda(g^*, \eta^*) = (g^*, \eta^*). \tag{43}$$

Proof. Let $(g_i, \eta_i) \in \mathbb{L}^2(0.T, V \times V')$. We use the notation (u_i, φ_i). For $(g, \eta) = (g_i, \eta_i), i = 1.2$, let $t \in [0.T]$. We have

$$\Lambda_1(g) = v_{g\eta}. \tag{44}$$

So

$$|g_1(t) - g_2(t)|_V^2 \leq |v_1(t) - v_2(t)|_V^2 . \tag{45}$$

It follows that

$$\begin{aligned}
&(\dot{v}_1(t) - \dot{v}_2(t), v_1(t) - v_2(t)) + (\mathcal{G}\varepsilon(v_1(t)) - \mathcal{G}\varepsilon(v_2(t)), \varepsilon(v_1(t)) - \varepsilon(v_2(t)))+\\
&(\eta_1(t) - \eta_2(t), v_1(t) - v_2(t)) + j(g_1, v_1(t)) - j(g_1, v_2(t)) - j(g_2, v_1(t)) + j(g_2, v_2(t)) \leq 0.
\end{aligned} \tag{46}$$

From the definition of the functional j given by (23), and using (13), (19), we have

$$j(g_2, v_2(t)) - j(g_2, v_1(t)) - j(g_1, v_2(t)) + j(g_1, v_1(t)) \leq C\, |g_1 - g_2|_V\, |v_1 - v_2|_V . \tag{47}$$

Integrating inequality (46) with respect to time, using the initial conditions $v_2(0) = v_1(0) = v_0$, using (13), (15), using Cauchy-Schwartz's inequality and the inequalities $2ab \leq \frac{C}{m_{\mathcal{G}}}a^2 + \frac{m_{\mathcal{G}}}{C}b^2$ and $2ab \leq \frac{1}{m_{\mathcal{G}}}a^2 + m_{\mathcal{G}}b^2$, by Gronwall's inequality, we find

$$|v_1(t) - v_2(t)|_V^2 \leq C(\int_0^t |g_1(s) - g_2(s)|_V^2 \, ds + \int_0^t |\eta_1(s) - \eta_2(s)|_{V'}^2 \, ds). \tag{48}$$

So

$$|g_1 - g_2|_V^2 \leq C(\int_0^t |g_1(s) - g_2(s)|_V^2 \, ds + \int_0^t |\eta_1(s) - \eta_2(s)|_{V'}^2 \, ds). \tag{49}$$

And we have

$$(\Lambda_2(\eta), w)_{V' \times V} = (\mathcal{A}(\varepsilon(u(t))), \varepsilon(w))_{\mathcal{H}} + (\int_0^t \mathcal{B}(t-s)\varepsilon(u(s))ds - \theta(t)\mathcal{M}, \varepsilon(w))_{\mathcal{H}} + \phi(g, w). \tag{50}$$

From the definition of the functional ϕ given by (24), and using (13), (19), we have

$$\phi(g_1, v_2(t)) - \phi(g_1, v_1(t)) - \phi(g_2, v_2(t)) + \phi(g_2, v_1(t)) \leq C\,|g_1 - g_2|_V\,|v_1 - v_2|_V\,. \tag{51}$$

So

$$\begin{gathered} |\eta_1(t) - \eta_2(t)|_{V'}^2 \leq C(|u_1(t) - u_2(t)|_V^2 + \textstyle\int_0^t |u_1(s) - u_2(s)|_V^2 \, ds + \\ |\theta_1(t) - \theta_2(t)|_{\mathbb{L}^2(\Omega)}^2 + |g_1(t) - g_2(t)|_V^2). \end{gathered} \tag{52}$$

By (46), using the inequalities $2ab \leq \frac{2C}{m_{\mathcal{G}}}a^2 + \frac{m_{\mathcal{G}}}{2C}b^2$ and $2ab \leq \frac{2}{m_{\mathcal{G}}}a^2 + \frac{m_{\mathcal{G}}}{2}b^2$, we find

$$\begin{aligned} &\tfrac{1}{2}\,|v_1(t) - v_2(t)|_V^2 + m_{\mathcal{G}} \textstyle\int_0^t |v_1(s) - v_2(s)|_V^2 \, ds \leq \frac{1}{m_{\mathcal{G}}} \int_0^t |\eta_1(s) - \eta_2(s)|_{V'}^2 \, ds + \\ &+ \tfrac{m_{\mathcal{G}}}{4} \textstyle\int_0^t |v_1(s) - v_2(s)|_V^2 \, ds + C \times \frac{C}{m_{\mathcal{G}}} \int_0^t |g_1(s) - g_2(s)|_V^2 \, ds + \\ &C \times \tfrac{m_{\mathcal{G}}}{4C} \textstyle\int_0^t |v_1(s) - v_2(s)|_V^2 \, ds. \end{aligned} \tag{53}$$

So

$$\int_0^t |v_1(s) - v_2(s)|_V^2 \, ds \leq C(\int_0^t |\eta_1(s) - \eta_2(s)|_{V'}^2 \, ds + \int_0^t |g_1(s) - g_2(s)|_V^2 \, ds). \tag{54}$$

By (41), we find

$$\begin{aligned} &\left(\dot{\theta}_1(t) - \dot{\theta}_2(t), \theta_1(t) - \theta_2(t)\right)_{E' \times E} + (K(\theta_1) - K(\theta_2), \theta_1(t) - \theta_2(t))_{E' \times E} \\ &= (R(v_1) - R(v_2), \theta_1(t) - \theta_2(t))_{E' \times E}\,. \end{aligned} \tag{55}$$

We integrate (55) over [0.T], we use the initial conditions $\theta_1(0) = \theta_2(0) = \theta_0$, and we use the coercive of K and the Lipschitz continuity of R to deduce that

$$\begin{gathered} \tfrac{1}{2}\,|\theta_1(t) - \theta_2(t)|_{\mathbb{L}^2(\Omega)}^2 + C \textstyle\int_0^t |\theta_1(s) - \theta_2(s)|_{\mathbb{L}^2(\Omega)}^2 \, ds \leq \\ C\left(\textstyle\int_0^t \, |v_1(s) - v_2(s)|_V \, |\theta_1(s) - \theta_2(s)|_{\mathbb{L}^2(\Omega)} \, ds\right). \end{gathered}$$

Using the inequality $2ab \leq \frac{1}{2}a^2 + 2b^2$, we find

$$\begin{gathered} \tfrac{1}{2}\,|\theta_1(t) - \theta_2(t)|_{\mathbb{L}^2(\Omega)}^2 + C \textstyle\int_0^t |\theta_1(s) - \theta_2(s)|_{\mathbb{L}^2(\Omega)}^2 \, ds \leq \\ \tfrac{C}{4} \textstyle\int_0^t \, |v_1(s) - v_2(s)|_V \, ds + C\,|\theta_1(s) - \theta_2(s)|_{\mathbb{L}^2(\Omega)} \, ds. \end{gathered}$$

Also,

$$|\theta_1(t) - \theta_2(t)|^2_{\mathbb{L}^2(\Omega)} \leq C \int_0^t |v_1(s) - v_2(s)|^2_V \, ds. \tag{56}$$

By (54) , we find

$$|\theta_1(t) - \theta_2(t)|^2_{\mathbb{L}^2(\Omega)} \leq C \left(\int_0^t |\eta_1(s) - \eta_2(s)|^2_{V'} \, ds + \int_0^t |g_1(s) - g_2(s)|^2_V \, ds \right). \tag{57}$$

So,

$$|\eta_1(t) - \eta_2(t)|^2_{V'} \leq C \left(\int_0^t |g_1(s) - g_2(s)|^2_V \, ds + \int_0^t |\eta_1(s) - \eta_2(s)|^2_{V'} \, ds \right). \tag{58}$$

Also,

$$|u_1(t) - u_2(t)|^2_V + \int_0^t |u_1(s) - u_2(s)|^2_V \, ds \leq C(\int_0^t |v_1(s) - v_2(s)|^2_V + \int_0^t |u_1(s) - u_2(s)|^2_V) ds. \tag{59}$$

And

$$|u_1(t) - u_2(t)|^2_V \geq 0.$$

$$\textstyle\int_0^t \int_0^s |u_1(r) - u_2(r)|^2_V \, dr ds \geq 0.$$

So,

$$|u_1 - u_2|^2_V + \int_0^t |u_1 - u_2|^2_V \, ds \leq C \int_0^t (|v_1(s) - v_2(s)|^2_V + |u_1(s) - u_2(s)|^2_V + \\ + \int_0^s |u_1(r) - u_2(r)|^2_V \, dr) ds.$$

By Gronwall's inequality, and using (54), we have

$$|u_1 - u_2|^2_V + \int_0^t |u_1 - u_2|^2_V \, ds \leq C(\int_0^t |\eta_1(s) - \eta_2(s)|^2_{V'} \, ds + \int_0^t |g_1(s) - g_2(s)|^2_V \, ds). \tag{60}$$

And using (49) and (58), we find

$$|\Lambda(g_1, \eta_1) - \Lambda(g_2, \eta_2)|^2_{\mathbb{L}^2(0.T;V \times V')} \leq C \int_0^t |(g_1, \eta_1) - (g_2, \eta_2)|^2_{V \times V'} \, ds. \tag{61}$$

Thus, for m sufficiently large, Λ^m is a contraction on $\mathbb{L}^2(0.T, V \times V')$ and so Λ has a unique fixed point in this Banach space. We consider the operator $\mathcal{L} : C(0.T, \mathbb{L}^2(\Gamma_3)) \to C(0.T, \mathbb{L}^2(\Gamma_3))$,

$$\mathcal{L}\omega(t) = -kv^* \int_0^t \sigma_\nu(s) ds, \forall t \in [0.T]. \tag{62}$$

Lemma 4.4 *The operator $\mathcal{L} : C(0.T, \mathbb{L}^2(\Gamma_3)) \to C(0.T, \mathbb{L}^2(\Gamma_3))$ has a unique element $\omega^* \in C(0.T, \mathbb{L}^2(\Gamma_3))$ such that*

$$\mathcal{L}\omega^* = \omega^*.$$

Proof. Using (62), we have

$$|\mathcal{L}\omega_1(t) - \mathcal{L}\omega_2(t)|^2_{\mathbb{L}^2(\Gamma_3)} \leq kv^* \int_0^t |\sigma_1(s) - \sigma_2(s)|^2_{\mathcal{H}} \, ds. \tag{63}$$

From (1), we have

$$\begin{aligned} |\mathcal{L}\omega_1(t) - \mathcal{L}\omega_2(t)|^2_{\mathbb{L}^2(\Gamma_3)} \leq C \textstyle\int_0^t (|u_1(t) - u_2(t)|^2_V + \int_0^t |u_1(s) - u_2(s)|^2_V \, ds + \\ + |\theta_1(t) - \theta_2(t)|^2_{\mathbb{L}^2(\Omega)}) ds. \end{aligned} \tag{64}$$

By (56) and (59), we find

$$|u_1 - u_2|^2_V + \int_0^t |u_1 - u_2|^2_V \, ds + |\theta_1(t) - \theta_2(t)|^2_{\mathbb{L}^2(\Omega)} \leq \int_0^t |v_1(s) - v_2(s)|^2_V \, ds. \tag{65}$$

So,

$$\begin{aligned} |u_1 - u_2|^2_V + \textstyle\int_0^t |u_1 - u_2|^2_V \, ds + |\theta_1(t) - \theta_2(t)|^2_{\mathbb{L}^2(\Omega)} \leq C(\int_0^t |v_1(s) - v_2(s)|^2_V \, ds + \\ + |\omega_1(t) - \omega_2(t)|^2_{\mathbb{L}^2(\Gamma_3)}). \end{aligned}$$

So, we have

$$|u_1 - u_2|^2_V + \int_0^t |u_1 - u_2|^2_V \, ds + |\theta_1(t) - \theta_2(t)|^2_{\mathbb{L}^2(\Omega)} \leq C \, |\omega_1(t) - \omega_2(t)|^2_{\mathbb{L}^2(\Gamma_3)} . \tag{66}$$

By (64), we find $|\mathcal{L}\omega_1(t) - \mathcal{L}\omega_2(t)|_{\mathbb{L}^2(\Gamma_3)} \leq C \int_0^t |\omega_1(s) - \omega_2(s)|_{\mathbb{L}^2(\Gamma_3)} \, ds$. Thus, for m sufficiently large, $\mathcal{L}^m$ is a contraction on $C(0.T, \mathbb{L}^2(\Gamma_3))$ and so $\mathcal{L}$ has a unique fixed point in this Banach space. Now, we have all the ingredients to prove Theorem 4.1.

Existence. Let $(g^*, \eta^*) \in \mathbb{L}^2(0.T, V \times V')$ be the fixed point of Λ defined by (42), let $\omega^* \in C(0.T, \mathbb{L}^2(\Gamma_3))$ be the fixed point of $\mathcal{L}\omega^*$ defined by (62), and let $(u, \theta) = (u_{g^*\eta^*}, \theta_{g^*\eta^*})$ be the solutions of Problems $PV_{g^*\eta^*}$ and $P_{\theta g \eta}$. It results from (35), (41) that $(u_{g^*\eta^*}, \theta_{g^*\eta^*})$ is the solution of Problem PV. Properties (31) – (34) follow from Lemmas 4.1 and 4.2.

Uniqueness. The uniqueness of the solution is a consequence of the uniqueness of the fixed point of the operators $\Lambda, \mathcal{L}$ defined by (42), (62), and the unique solvability of the Problems $PV_{g\eta}$ and $P_{\theta g \eta}$. This completes the proof.

5 Concluding Remarks and Perspectives

This paper studies electromechanical and thermomechanical contact problems with or without friction within the framework of the mechanics of continuous media. This involves extending the previous results on the existence and uniqueness of the solution by new techniques. We will also try to obtain the properties of the solution for different boundary conditions (traction displacement, contact with or without friction, contact with adhesion, etc.) In the study of this type of problem, there is also a question of developing numerical methods for the resolution of the nonlinear equations concerned.

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Nonlinear Dynamics and Systems Theory, **24** (5) (2024) 485–494

Relationship between Persymmetric Solutions and Minimal Persymmetric Solutions of $AXA^{(*)} = B$

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Received: January 26, 2024; Revised: September 8, 2024

Abstract: The minimal rank persymmetric solution of the quaternion matrix equation $AXA^{(*)} = B$ is defined as the matrix X that minimizes the rank of the difference $AXA^{(*)} - B$ or, equivalently, $r\big(AXA^{(*)} - B\big) = \min$, where B is persymmetric. In this paper, we focus on the quaternion matrix equation $AXA^{(*)} = B$. Our aim is to investigate the inclusion relationships between two sets, Ω_1 and Ω_2, where Ω_1, Ω_2 are, respectively, the set of persymmetric solutions and the set of minimal rank persymmetric solutions of the quaternion matrix equation $AXA^{(*)} = B$. Then, we deduce the necessary and sufficient conditions for the following relations to hold: $\Omega_1 \cap \Omega_2 \neq \varnothing$, $\Omega_1 \subseteq \Omega_2$ and $\Omega_1 \supseteq \Omega_2$.

Keywords: *linear system; persymmetric solution; Moore-Penrose inverse; rank.*

Mathematics Subject Classification (2010): 15A24; 15A09; 15A03; 93B30; 93B25.

1 Introduction

Throughout this paper, $\mathbb{R}$ and $\mathbb{C}$ stand for the real number field and the complex number field, respectively. Let $\mathbb{H}^{m\times n}$ be the set of $m\times n$ matrices over the real quaternion algebra:

$$\mathbb{H} = \left\{a_0 + a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k} \mid \boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{ijk} = -1,\ a_0, a_1, a_2, a_3 \in \mathbb{R}\right\}.$$

The symbols A^* and $r(A)$ stand for the conjugate transpose and the rank of A, respectively. I_m denotes the identity matrix of order m. The Moore-Penrose generalized

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inverse of a given matrix $A \in \mathbb{H}^{m\times n}$ is defined to be the unique matrix $A^{+} \in \mathbb{H}^{n\times m}$ which satisfies the following four matrix equations:

$$(a)\ AXA = A,\ (b)\ XAX = X,\ (c)\ (AX)^{*} = AX,\ (d)\ (XA)^{*} = XA.$$

The Moore-Penrose inverse has been the subject of many studies (see [1], [8]).

Furthermore, F_A and E_A stand for the two projectors $F_A = I_n - A^{+}A$ and $E_A = I_m - AA^{+}$ induced by $A \in \mathbb{H}^{m\times n}$. We denote the $n \times n$ permutation matrix whose elements along the southwest–northeast diagonal are ones and whose remaining elements are zeros by V_n.

In 1843, Irish mathematician Sir William Rowan Hamilton made a significant contribution when he introduced quaternions. However, it is important to note that the quaternion algebra H is an associative noncommutative division algebra over the real number field $\mathbb{R}$. It has applications in diverse fields such as computer science, orbital mechanics, signal and color image processing, and control theory, see e.g. [7], [10], [16], [19].

Consider the quaternion matrix equation

$$AXA^{(*)} = B, \tag{1}$$

where $A \in \mathbb{H}^{m\times n}$, $B = B^{(*)} \in \mathbb{H}^{m\times m}$ are given matrices, and $X \in \mathbb{H}^{n\times n}$ is an unknown matrix.

The two main objectives of matrix theory, which has focused on matrix equations, are first to establish the conditions in which a solution exists and then to provide a general solution for the problems. The efforts are focused on defining the behaviors of the solutions based on the identification of the general solution. Numerous criteria are included in this, namely, ranks, ranges, norms, definiteness, etc.

In information theory, engineering systems, linear estimation theory, linear system theory, and other fields, persymmetric and perskew-symmetric matrices are just as helpful as symmetric and skew-symmetric matrices. For example, in [14], Wang et al. provided symmetric, persymmetric, and centrosymmetric solutions for certain systems of quaternion matrix equations, in [17], Xie and Sheng presented the problem of generating a matrix A with specified eigenpair, where A is an anti-symmetric and persymmetric matrix, in [15], Wang et al. derived the expressions of the least squares Hermitian persymmetric and bisymmetric solution for the quaternion matrix equation $AXB = C$ by using the semi-tensor product of matrices and the real vector representation of the quaternion matrix. For further related works, one may refer to [9], [4], [10].

Wang et al. in [13] gave the following definition.

Definition 1.1 Let $A = (a_{ij}) \in \mathbb{H}^{m\times n}$, $A^{*} = (\overline{a}_{ji}) \in \mathbb{H}^{n\times m}$and $A^{(*)} = (\overline{a}_{m-j+1,n-i+1}) \in \mathbb{H}^{n\times m}$, where $\overline{a}_{ji}$ is the conjugate of the quaternion a_{ji}. Then $A^{(*)} = V_n A^{*} V_m$. We say that the matrix $A = (a_{ij}) \in \mathbb{H}^{n\times n}$ is symmetric if $A = A^{*}$, the matrix $A = (a_{ij}) \in \mathbb{H}^{n\times n}$ is persymmetric if $A = A^{(*)}$ and the matrix $A = (a_{ij}) \in \mathbb{H}^{n\times n}$ is perskew-symmetric if $A = -A^{(*)}$.

The rank of a matrix is a quantity that plays an important role in characterizing the algebraic properties of matrices, this concept has been the subject of research by many authors, Tian proposed the notion of least-rank solutions to matrix equations in [11,12] based on the minimal rank of the linear matrix function $A - BXC$ over the field $\mathbb{C}$. In [2], Guerarra studied positive and negative definite submatrices in an Hermitian least-rank solution of the matrix equation $AXA^* = B$. Further, in [3], she investigated necessary and sufficient conditions for the matrix equation $AXB = C$ to have a Hermitian Re-positive or Re-negative definite solution. In [18], Xu et al. used the Moore-Penrose inverse to deduce the necessary and sufficient conditions for the existence of Hermitian (skew-Hermitian), Re-nonnegative (Re-positive) definite, and Re-nonnegative (Re-positive) definite least-rank solutions to $AXB = C$ and presented explicit representations of the general solutions in cases for which the solvability conditions were satisfied.

Motivated by the works mentioned above, and in view of the applications of and interest in quaternion matrices, in this study, we investigate the inclusion relationships between two sets, as well as the set of persymmetric solutions and the set of minimal rank persymmetric solutions of the quaternion matrix equation (1).

The following lemma is due to Matsagalia and Styan [8], and can be easily generalized to $\mathbb{H}$.

Lemma 1.1 *Let $A \in \mathbb{H}^{s\times r}$, $B \in \mathbb{H}^{s\times k}$, $C \in \mathbb{H}^{l\times r}$, $D \in \mathbb{H}^{l\times k}$. Then*

$$r\begin{bmatrix} A, & B \end{bmatrix} = r(A) + r(E_A B) = r(B) + r(E_B A), \tag{2}$$

$$r\begin{bmatrix} A \\ C \end{bmatrix} = r(A) + r(CF_A) = r(C) + r(AF_C), \tag{3}$$

$$r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = r(B) + r(C) + r(E_B A F_C). \tag{4}$$

The following formulas are derived from (2), (3), and (4):

$$r\begin{bmatrix} A & BF_N \\ E_R C & 0 \end{bmatrix} = r\begin{bmatrix} A & B & 0 \\ C & 0 & R \\ 0 & N & 0 \end{bmatrix} - r(N) - r(R).$$

$$r\begin{bmatrix} M & L \\ E_R A & E_R B \end{bmatrix} = r\begin{bmatrix} M & L & 0 \\ A & B & R \end{bmatrix} - r(R),$$

$$r\begin{bmatrix} M & AF_N \\ L & BF_N \end{bmatrix} = r\begin{bmatrix} M & A \\ L & B \\ O & N \end{bmatrix} - r(N).$$

2 Relationship between Persymmetric Solutions and Minimal Persymmetric Solutions of $AXA^{(*)} = B$

It is well known that the persymmetric solution and the minimal persymmetric solution of the quaternion equation (1) are not necessarely unique, throughout this section, we adopt the following notations:

$$\Omega_1 = \left\{ X \in \mathbb{H}^{n\times n} \ / \ AXA^{(*)} = B \right\}, \tag{5}$$

$$\Omega_2 = \left\{ X_m \in \mathbb{H}^{n\times n} \ / \ r\left(B - AXA^{(*)}\right) = \min \right\}. \tag{6}$$

Notably, the two sets equations in (5) and (6) may not necessarily be equal. In this section, we focus on the necessary and sufficient conditions for the following relations to hold:

$$\Omega_1 \cap \Omega_2 \neq \emptyset, \ \ \Omega_1 \subseteq \Omega_2 \text{ and } \Omega_1 \supseteq \Omega_2.$$

We need the following lemmas.

Lemma 2.1 *[5] Let $A \in \mathbb{H}^{m\times n}$, $B = B^{(*)} \in \mathbb{H}^{m\times m}$. Then the matrix equation (1) has a persymmetric solution if and only if $\mathcal{R}(B) \subseteq \mathcal{R}(A)$. In this case, the general persymmetric solution can be expressed as*

$$X = A^{+}BA^{+^{(*)}} + F_A V + V^{(*)}F_A^{(*)},$$

where $V \in \mathbb{H}^{n\times n}$ is arbitrary.

Lemma 2.2 *[5] Let $A \in \mathbb{H}^{m\times n}$, $B = B^{(*)} \in \mathbb{H}^{m\times m}$ be given. Then the expression of general minimal rank persymmetric solution of (1) is given by*

$$X = -S^{(*)}M^{+}S + S_1^{(*)}U^{(*)} + US_1, \tag{7}$$

where

$$M = \begin{bmatrix} A & B \\ 0 & A^{(*)} \end{bmatrix},\ S = \begin{bmatrix} 0 \\ I_n \end{bmatrix},\ S_1 = E_M S,$$

and $U \in \mathbb{H}^{n\times n}$ is arbitrary.

Khan et al. in [5] [Theorem 2.4], derived the minimal rank formula of the quaternion matrix expression

$$p(X,Y) = G - LX - (LX)^{(*)} - CYC^{(*)}$$

with respect to the pair of matrices X and $Y = Y^{(*)}$, where $G \in \mathbb{H}^{m\times m}$, $L \in \mathbb{H}^{m\times n}$ and $C \in \mathbb{H}^{m\times m}$ are given, $X \in \mathbb{H}^{n\times m}$ and $Y = Y^{(*)} \in \mathbb{H}^{m\times m}$ are variable matrices, and then derived the following result.

Lemma 2.3 *[5] Let $G = G^{(*)} \in \mathbb{H}^{m\times m}$, $L \in \mathbb{H}^{m\times n}$ be given, and $X \in \mathbb{H}^{n\times m}$ be a variable matrix. Then*

$$\min_{X\in\mathbb{H}^{n\times m}} r\left[G - LX - (LX)^{(*)}\right] = r\begin{bmatrix} L & G \\ 0 & L^{(*)} \end{bmatrix} - 2r(L). \tag{8}$$

The following lemma is due to Liu and Tian in [6], and can be easily generalized to $\mathbb{H}$.

Lemma 2.4 *Let $D \in \mathbb{H}^{s\times r}$, $N \in \mathbb{H}^{s\times l}$ and $K \in \mathbb{H}^{k\times r}$ be matrices such that $\mathcal{R}\left(K^{(*)}\right) \subset \mathcal{R}(N)$. Then*

$$\max_{X\in\mathbb{C}^{l\times k}} r\left(D - NXK - (NXK)^{(*)}\right) = \min\left\{r\begin{bmatrix} D & N \end{bmatrix},\ r\begin{bmatrix} K & D \\ 0 & K^{(*)} \end{bmatrix}\right\}, \tag{9}$$

$$\min_{X\in\mathbb{C}^{l\times k}} r\left(D - NXK - (NXK)^{(*)}\right) = 2r\begin{bmatrix} D & N \end{bmatrix} + r\begin{bmatrix} K & D \\ 0 & K^{(*)} \end{bmatrix} - 2r\begin{bmatrix} D & N \\ K & 0 \end{bmatrix}. \tag{10}$$

The main result of this work is the following theorem.

Theorem 2.1 *Let $A \in \mathbb{H}^{m\times n}$ and $B = B^{(*)} \in \mathbb{H}^{m\times m}$ be given, and suppose that X, X_m are persymmetric solutions and minimal persymmetric solution of Eq. (1), respectively, and let Ω_1 and Ω_2 be as given in (5) and (6), respectively. Then the following hold:*
(a) Eq. (1) has persymmetric solutions and minimal rank solutions, that is, $\Omega_1 \cap \Omega_2 \neq \emptyset$ if and only if

$$r\begin{bmatrix} A & B & 0 & 0 & 0 \\ 0 & A^{(*)} & 0 & 0 & 0 \\ A & 0 & 0 & 0 & B \\ 0 & 0 & B & A & 0 \\ 0 & 0 & A^{(*)} & 0 & A^{(*)} \end{bmatrix} = 2r\left(A\right) + 2r\begin{bmatrix} B \\ A^{(*)} \end{bmatrix}.$$

(b) All the persymmetric solutions of Eq. (1) are minimal rank persymmetric solutions of Eq. (1), that is, $\Omega_1 \subseteq \Omega_2$ if and only if

$$r\begin{bmatrix} A & B & 0 & 0 \\ 0 & A^{(*)} & 0 & A^{(*)} \\ A & 0 & 0 & -B \\ 0 & 0 & A & B \end{bmatrix} = r\left(A\right) + 2r\begin{bmatrix} A & B \end{bmatrix} \quad or \quad r\begin{bmatrix} A & B \end{bmatrix} = r\begin{bmatrix} A & B \\ 0 & A^{(*)} \end{bmatrix}.$$

(c) All the minimal rank persymmetric solutions of Eq. (1) are persymmetric solutions of Eq. (1), that is, $\Omega_1 \supseteq \Omega_2$ if and only if

$$r\begin{bmatrix} A & B & 0 & 0 & 0 \\ 0 & A^{(*)} & 0 & 0 & 0 \\ A & 0 & 0 & 0 & B \\ 0 & 0 & A & B & 0 \\ 0 & 0 & 0 & A^{(*)} & 0 \end{bmatrix} = 2r\begin{bmatrix} A & B \\ 0 & A^{(*)} \end{bmatrix} \quad or\ A = 0.$$

Proof. (a) The intersection $\Omega_1 \cap \Omega_2 \neq \emptyset$ means that Eq. (1) has minimal rank persymmetric solutions and persymmetric solutions, which implies the minimum rank of the matrix expression $X - X_m$ is zero, that is,

$$\min_{X\in\Omega_1,\ X_m\in\Omega_2} r\left(X - X_m\right) = 0. \tag{11}$$

According to Lemmas 2.1 and 2.2, the expressions for the general persymmetric solution and the minimal-rank solution of the matrix equation (1) can be written as follows:

$$X = A^{+}BA^{+^{(*)}} + F_AV + V^{(*)}F_A^{(*)},$$
$$X_m = -S^{(*)}M^{+}S + S_1^{(*)}U^{(*)} + US_1.$$

We can write the expression $X - X_m$ as follows:

$$\begin{aligned} X - X_m &= A^{+}BA^{+^{(*)}} + F_AV + V^{(*)}F_A^{(*)} + S^{(*)}M^{+}S - S_1^{(*)}U^{(*)} - US_1 \\ &= A^{+}BA^{+^{(*)}} + S^{(*)}M^{+}S + \begin{bmatrix} F_A & -S_1^{(*)} \end{bmatrix}\begin{bmatrix} V \\ U^{(*)} \end{bmatrix} \\ &+ \left(\begin{bmatrix} F_A & -S_1^{(*)} \end{bmatrix}\begin{bmatrix} V \\ U^{(*)} \end{bmatrix}\right)^{(*)} \\ &= G + HW + (HW)^{(*)}, \end{aligned} \tag{12}$$

where $G = A^{+}BA^{+^{(*)}} + S^{(*)}M^{+}S$, $H = \begin{bmatrix} F_A & -S_1^{(*)} \end{bmatrix}$ and $W = \begin{bmatrix} V \\ U^{(*)} \end{bmatrix}$ is arbitrary with appropriate size.

Applying (8) in Lemma 2.3 to (12) yields

$$\min_{X\in\Omega_1,\ X_m\in\Omega_2} r\left(X - X_m\right) = \min_{W} r\left(G + HW + (HW)^{(*)}\right) = r\begin{bmatrix} H & G \\ 0 & H^{(*)} \end{bmatrix} - 2r\left(H\right).$$

By applying Lemma 1.1 and three elementary block matrix operations and simplifying by $AA^{+}B = B$, we obtain

$$r\begin{bmatrix} H & G \\ 0 & H^{(*)} \end{bmatrix} = r\begin{bmatrix} F_A & -S_1^{(*)} & A^{+}BA^{+^{(*)}} + S^{(*)}M^{+}S \\ 0 & 0 & F_A^{(*)} \\ 0 & 0 & -S_1 \end{bmatrix}$$

$$= r\begin{bmatrix} A^{+}BA^{+^{(*)}} + S^{(*)}M^{+}S & F_A & S^{(*)}F_{M^{(*)}} \\ E_{A^{(*)}} & 0 & 0 \\ E_M S & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & I_n & S^{(*)} & 0 & 0 \\ I_n & 0 & 0 & A^{(*)} & 0 \\ S & 0 & 0 & 0 & M \\ AA^{+}BA^{+^{(*)}} & A & 0 & 0 & 0 \\ MM^{+}S & 0 & M & 0 & 0 \end{bmatrix} - 2r\left(A\right) - 2r\left(M\right)$$

$$= \begin{bmatrix} 0 & I_n & S^{(*)} & 0 & 0 \\ I_n & 0 & 0 & A^{(*)} & 0 \\ S & 0 & 0 & 0 & M \\ 0 & A & 0 & B & 0 \\ 0 & 0 & M & 0 & M \end{bmatrix} - 2r\left(A\right) - 2r\left(M\right)$$

$$= r\begin{bmatrix} 0 & I_n & I_n & 0 & 0 & 0 & 0 \\ I_n & 0 & 0 & 0 & A^{(*)} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & A & B \\ I_n & 0 & 0 & 0 & 0 & 0 & A^{(*)} \\ 0 & A & 0 & 0 & B & 0 & 0 \\ 0 & 0 & A & B & 0 & A & B \\ 0 & 0 & 0 & A^{(*)} & 0 & 0 & A^{(*)} \end{bmatrix} - 2r\left(A\right) - 2r\left(M\right)$$

$$= r\begin{bmatrix} 0 & I_n & 0 & 0 & 0 & 0 & 0 \\ I_n & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & A & B \\ 0 & 0 & 0 & 0 & -A^{(*)} & 0 & A^{(*)} \\ 0 & 0 & A & 0 & B & 0 & 0 \\ 0 & 0 & -A & B & 0 & A & B \\ 0 & 0 & 0 & A^{(*)} & 0 & 0 & A^{(*)} \end{bmatrix} - 2r\left(A\right) - 2r\left(M\right)$$

$$= 2n + \begin{bmatrix} A & B & 0 & 0 & 0 \\ 0 & A^{(*)} & 0 & 0 & 0 \\ A & 0 & 0 & 0 & B \\ 0 & 0 & B & A & 0 \\ 0 & 0 & A^{(*)} & 0 & A^{(*)} \end{bmatrix} - 2r\left(A\right) - 2r\left(M\right), \tag{13}$$

and

$$r(H) = r\begin{bmatrix} F_A & S_1^{(*)} \end{bmatrix} = r\begin{bmatrix} I_n & S^{(*)} \\ A & 0 \\ 0 & M \end{bmatrix} - r(A) - r(M) = n + r(A) + r\begin{bmatrix} B \\ A^{(*)} \end{bmatrix} - r(A) - r(M) = n + r\begin{bmatrix} B \\ A^{(*)} \end{bmatrix} - r(M). \tag{14}$$

By substituting (13) and (14) into (11), we obtain (a).

b) Note that $\Omega_1 \subseteq \Omega_2$ means that all the persymmetric solutions of Eq. (1) are minimal rank persymmetric solutions, hence the inclusion $\Omega_1 \subseteq \Omega_2$ is equivalent to

$$\max_{X\in\Omega_1}\min_{X_m\in\Omega_2} r(X - X_m) = 0. \tag{15}$$

Applying (8) in Lemma 2.3 to the matrix expression $X - X_m$ yields

$$\min_{X_m\in\Omega_2} r(X - X_m) = \min_{X_m\in\Omega_2} r\left(X + S^{(*)}M^+S - S_1^{(*)}U^{(*)} - US_1\right) = r\begin{bmatrix} S_1^{(*)} & X + S^{(*)}M^+S \\ 0 & S_1 \end{bmatrix} - 2r(S_1). \tag{16}$$

The 2×2 block matrix in (16) can be written as

$$\begin{bmatrix} S_1^{(*)} & X + S^{(*)}M^+S \\ 0 & S_1 \end{bmatrix} = \begin{bmatrix} S_1^{(*)} & A^+BA^{+^{(*)}} + F_AV + V^{(*)}F_A^{(*)} + S^{(*)}M^+S \\ 0 & S_1 \end{bmatrix}$$

$$= \left(\begin{matrix} \begin{bmatrix} S_1^{(*)} & A^+BA^{+^{(*)}} + S^{(*)}M^+S \\ 0 & S_1 \end{bmatrix} + \begin{bmatrix} F_A \\ 0 \end{bmatrix} V \begin{bmatrix} 0 & I_n \end{bmatrix} \\ + \left(\begin{bmatrix} F_A \\ 0 \end{bmatrix} V \begin{bmatrix} 0 & I_n \end{bmatrix}\right)^{(*)} \end{matrix} \right). \tag{17}$$

Applying (9) to (17) yields

$$\max_{X\in\Omega_1} \begin{bmatrix} S_1^{(*)} & X + S^{(*)}M^+S \\ 0 & S_1 \end{bmatrix}$$

$$= \min\left\{ \begin{matrix} r\begin{bmatrix} S_1^{(*)} & A^+BA^{+^{(*)}} + S^{(*)}M^+S & F_A \\ 0 & S_1 & 0 \end{bmatrix}, \\ r\begin{bmatrix} I_n & S_1^{(*)} & A^+BA^{+^{(*)}} + S^{(*)}M^+S \\ 0 & 0 & S_1 \\ 0 & 0 & I_n \end{bmatrix} \end{matrix} \right\}$$

$$= \min\left\{ 2n + r\begin{bmatrix} A & B & 0 & 0 \\ 0 & A^{(*)} & 0 & A^{(*)} \\ A & 0 & 0 & -B \\ 0 & 0 & A & B \end{bmatrix} - r(A) - 2r(M),\ 2n \right\}. \tag{18}$$

Furthermore we have

$$\begin{aligned} r\left(S_1\right) &= r\left(E_M S\right) \\ &= \begin{bmatrix} S & M \end{bmatrix} - r\left(M\right) \\ &= \begin{bmatrix} 0 & A & B \\ I_n & 0 & A^{(*)} \end{bmatrix} - r\left(M\right) \\ &= n + r\begin{bmatrix} A & B \end{bmatrix} - r\left(M\right). \end{aligned} \tag{19}$$

Substituting (19) into (16) and combining (18) and (16) yield

$$\begin{aligned} &\max_{X\in\Omega_1}\min_{X_m\in\Omega_2} r\left(X - X_m\right) \\ &= \min\left\{ \begin{array}{c} r\begin{bmatrix} A & B & 0 & 0 \\ 0 & A^{(*)} & 0 & A^{(*)} \\ A & 0 & 0 & -B \\ 0 & 0 & A & B \end{bmatrix} - r\left(A\right) - 2r\begin{bmatrix} A & B \end{bmatrix}, \\ -2r\begin{bmatrix} A & B \end{bmatrix} + 2r\left(M\right) \end{array} \right\}. \end{aligned} \tag{20}$$

Finally, by substituting (20) into (15), we obtain the desired results in (b).

c) The inclusion $\Omega_1 \supseteq \Omega_2$ means that all the minimal rank persymmetric solutions of Eq. (1) are persymmetric solutions, then the inclusion $\Omega_1 \supseteq \Omega_2$ is equivalent to

$$\max_{X_m\in\Omega_2}\min_{X\in\Omega_1} r\left(X - X_m\right) = 0. \tag{21}$$

Then we have

$$\min_{X\in\Omega_1} r\left(X - X_m\right) = \min_{V}\left(A^{+}BA^{+^{(*)}} - X_m + F_A V + V^{(*)}F_A^{(*)}\right). \tag{22}$$

Applying (8) to (22) yields

$$\min_{X\in\Omega_1} r\left(X - X_m\right) = r\begin{bmatrix} F_A & A^{+}BA^{+^{(*)}} - X_m \\ 0 & F_A^{(*)} \end{bmatrix} - 2r\left(F_A\right). \tag{23}$$

From Lemma 1.1, we have

$$r\left(F_A\right) = r\begin{bmatrix} I_n \\ A \end{bmatrix} - r\left(A\right) = n - r\left(A\right). \tag{24}$$

The 2×2 block matrix on the right-hand side of (23) can be rewritten as

$$\begin{aligned} r\begin{bmatrix} F_A & A^{+}BA^{+^{(*)}} - X_m \\ 0 & F_A^{(*)} \end{bmatrix} &= r\begin{bmatrix} F_A & A^{+}BA^{+^{(*)}} + S^{(*)}M^{+}S - S_1^{(*)}U^{(*)} - US_1 \\ 0 & F_A^{(*)} \end{bmatrix} \\ &= r\begin{bmatrix} F_A & A^{+}BA^{+^{(*)}} + S^{(*)}M^{+}S \\ 0 & F_A^{(*)} \end{bmatrix} + \begin{bmatrix} -S_1^{(*)} \\ 0 \end{bmatrix} U^{(*)}\begin{bmatrix} 0 & I_n \end{bmatrix} \\ &+ \left(\begin{bmatrix} -S_1^{(*)} \\ 0 \end{bmatrix} U^{(*)}\begin{bmatrix} 0 & I_n \end{bmatrix}\right)^{(*)}. \end{aligned} \tag{25}$$

Hence, by applying (9) to (25), we obtain

$$\max_{X_m\in\Omega_2} r\begin{bmatrix} F_A & A^+BA^{+^{(*)}}-X_m \\ 0 & F_A^{(*)} \end{bmatrix}$$
$$=\min\left\{\begin{array}{l} r\begin{bmatrix} F_A & A^+BA^{+^{(*)}}+S^{(*)}M^+S & -S_1^{(*)} \\ 0 & F_A^{(*)} & 0 \end{bmatrix}, \\ \begin{bmatrix} I_n & F_A & A^+BA^{+^{(*)}}+S^{(*)}M^+S \\ 0 & 0 & F_A^{(*)} \\ 0 & 0 & I_n \end{bmatrix} \end{array}\right\}$$
$$=\min\left\{2n+r\begin{bmatrix} A & B & 0 & 0 & 0 \\ 0 & A^{(*)} & 0 & 0 & 0 \\ A & 0 & 0 & 0 & B \\ 0 & 0 & A & B & 0 \\ 0 & 0 & 0 & A^{(*)} & 0 \end{bmatrix} - 2r\left(A\right) - 2r\left(M\right),\, 2n\right\}. \quad (26)$$

Substituting (24) into (23) and combining (26) and (23) yield

$$\max_{X_m\in\Omega_2}\min_{X\in\Omega_1} r\left(X-X_m\right)$$
$$=\min\left\{r\begin{bmatrix} A & B & 0 & 0 & 0 \\ 0 & A^{(*)} & 0 & 0 & 0 \\ A & 0 & 0 & 0 & B \\ 0 & 0 & A & B & 0 \\ 0 & 0 & 0 & A^{(*)} & 0 \end{bmatrix} - 2r\left(M\right),\, r\left(A\right)\right\}. \quad (27)$$

By substituting (27) into (21), we obtain the desired results in (c).

3 Conclusion

In the previous sections, we have studied some algebraic characterizations of relationships between two sets consisting of two types of solutions for the same quaternion matrix equation $AXA^{(*)} = B$. These two types are, respectively, the persymmetric solutions and the minimal rank persymmetric solutions. Further, the persymmetric solutions exist under the solvability conditions where the matrix equations will be consistent, otherwise, the minimal rank persymmetric solutions always exist. Also, the results obtained clearly illustrate the fundamental characteristics and attributes of some prominent linear matrix equations and their relationships when we have used the matrix rank method, which is considered a useful and effective approach for solving various matrix equality and matrix set inclusion problems.

Acknowledgment

The author is grateful for the detailed comments from the referees, which significantly improved the quality of the paper.

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Nonlinear Dynamics and Systems Theory, **24** (5) (2024) 495–504

Conformable Fractional Khalouta Transform and Its Applications to Fractional Differential Equations

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Received: February 24, 2024; Revised: September 30, 2024

Abstract: In 2023, the author [6] introduced a new integral transform called the Khalouta transform which is a generalization of many well-known integral transforms. In this paper, our aim is to generalize the formula of the Khalouta transform to the conformable fractional order. Moreover, we present and prove some main properties and theorems related to the conformable fractional Khalouta transform. In order to illustrate the validity, efficiency, and applicability of the proposed technique, we apply the conformable fractional Khalouta transform to solve some fractional differential equations. Finally, the results show that our new technique is powerful, effective, and applicable for the both conformable fractional problems.

Keywords: *fractional differential equations; Khalouta transform; conformable fractional derivative; exact solutions.*

Mathematics Subject Classification (2010): 34A08, 26A33, 70K75, 93-06.

1 Introduction

Fractional differential equations are a very important mathematial tool for modeling many applications in real life sciences and engineering such as fluid dynamics, mathematical biology, electrical circuits, optics, quantum mechanics, biophysics, wave theory, polymers, continuum mechanics, etc. [1, 4, 5, 7, 11–13]. There are many definitions of fractional derivatives and integrals used in many applications and natural phenomena such as Riemann–Liouville [10], Liouville–Caputo [9], Caputo–Fabrizio [3], Atangana–Baleanu [2] derivatives and so on. In 2014, Khalil et al. [8] introduced a new definition of the fractional derivative which is called the conformable fractional derivative, and it

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is very easily computed compared with other previous definitions. In recent years, many mathematics researchers have been interested in solving fractional differential equations using different fractional integral transform. The main objective of this paper is to extend the definition of the Khalouta transform [6] to a fractional order in the sense of the conformalble derivative and to give new interesting results for solving various types of conformalble fractional differential equations.

2 Basic Notions and Preliminaries

Definition 2.1 [8] The conformable fractional derivative of order α is defined for a function $u : [0, +\infty) \longrightarrow \mathbb{R}$ by

$$\mathcal{C}^{(n\alpha)}u(t) = \frac{d^{n\alpha}}{dt^{n\alpha}}u(t) = \lim_{\varepsilon\to 0}\frac{u^{[\alpha]-1}(t+\varepsilon t^{[\alpha]-\alpha}) - u^{[\alpha]-1}(t)}{\varepsilon}, t > 0,$$

where $n - 1 < n\alpha \leq n, n \in \mathbb{N}$ and $[\alpha]$ is the smallest integer greater than or equal to α, provided $\mathcal{C}^{(n\alpha)}u(0) = \lim_{t\to 0^+}\mathcal{C}^{(n\alpha)}u(t)$ is $n-$differentiable and $\lim_{t\to 0^+}\mathcal{C}^{(n\alpha)}u(t)$ exists.

As a special case, if $0 < \alpha \leq 1$, then we have

$$\mathcal{C}^{(\alpha)}u(t) = \frac{d^{\alpha}}{dt^{\alpha}}u(t) = \lim_{\varepsilon\to 0}\frac{u(t+\varepsilon t^{1-\alpha}) - u(t)}{\varepsilon}, t > 0,$$

provided $\mathcal{C}^{(\alpha)}u(0) = \lim_{t\to 0^+}\mathcal{C}^{(\alpha)}u(t)$ is $\alpha-$differentiable and $\lim_{t\to 0^+}\mathcal{C}^{(\alpha)}u(t)$ exists.

The most important and useful rule is that: If $u(t)$ is an $n-$differentiable function at $t > 0$ and $n - 1 < n\alpha \leq n$, $n \in \mathbb{N}$, then

$$\mathcal{C}^{(n\alpha)}u(t) = t^{[\alpha]-\alpha}u^{[\alpha]}(t).$$

Definition 2.2 [8] The conformable fractional integral of order α is defined for a function $u : [0, +\infty) \longrightarrow \mathbb{R}$ by

$$\mathcal{I}^{(\alpha)}u(t) = \int_0^t u(t)d_\alpha t = \int_0^t u(t)t^{\alpha-1}dt, 0 < \alpha \leq 1, t > 0.$$

If $u : [0, +\infty) \longrightarrow$ is an $\alpha-$differentiable function and $0 < \alpha \leq 1$, then

$$\mathcal{C}^{(\alpha)}u(t) = t^{1-\alpha}u'(t), \tag{1}$$

and

$$\mathcal{C}^{(\alpha)}\mathcal{I}^{(\alpha)}u(t) = u(t).$$

3 Conformable Fractional Khalouta Transform

Definition 3.1 The conformable fractional Khalouta transform of a piecewise continuous function $u : [0, +\infty) \longrightarrow \mathbb{R}$ of exponential order is defined on the set

$$\mathcal{S}_\alpha = \left\{u(t) : \exists K, \vartheta_1, \vartheta_2 > 0, |u(t)| < K\exp\left(\alpha\vartheta_j\left|t^\alpha\right|\right), \text{ if } t^\alpha \in (-1)^j \times [0,\infty)\right\}$$

by the following integral:

$$\begin{aligned}\mathbb{KH}_\alpha[u(t)] &= \mathcal{K}_\alpha(s,\gamma,\eta) = \frac{s}{\gamma\eta}\int_0^\infty \exp\left(-\frac{st^\alpha}{\gamma\eta\alpha}\right)u(t)t^{\alpha-1}dt \\ &= \lim_{\sigma\to\infty}\frac{s}{\gamma\eta}\int_0^\sigma \exp\left(-\frac{st^\alpha}{\gamma\eta\alpha}\right)u(t)t^{\alpha-1}dt,\end{aligned}$$

where $s>0,\gamma>0$ and $\eta>0$ are the Khalouta transform variables, σ is a real number and the integral is taken along the line $t=\sigma$.

Theorem 3.1 *Let $u:[0,+\infty)\longrightarrow\mathbb{R}$ be a real value function such that*

$$\mathbb{KH}_\alpha[u(t)] = \mathcal{K}_\alpha(s,\gamma,\eta),$$

and $0<\alpha\le 1$, then

$$\mathcal{K}_\alpha(s,\gamma,\eta) = \mathbb{KH}\left[u\left((\alpha t)^{\frac{1}{\alpha}}\right)\right].$$

Proof. Using Definition 3.1, we get

$$\begin{aligned}\mathcal{K}_\alpha(s,\gamma,\eta) &= \frac{s}{\gamma\eta}\int_0^\infty \exp\left(-\frac{st^\alpha}{\gamma\eta\alpha}\right)u(t)t^{\alpha-1}dt \\ x &= \frac{t^\alpha}{\alpha} \Longrightarrow dx = t^{\alpha-1}dt \text{ and } t=(\alpha x)^{\frac{1}{\alpha}} \\ &= \frac{s}{\gamma\eta}\int_0^\infty \exp\left(-\frac{sx}{\gamma\eta}\right)u((\alpha x)^{\frac{1}{\alpha}})dx \\ &= \mathbb{KH}\left[u\left((\alpha t)^{\frac{1}{\alpha}}\right)\right].\end{aligned}$$

Theorem 3.2 *Let $u:[0,+\infty)\longrightarrow\mathbb{R}$ be an $\alpha-$differentiable function and $0<\alpha\le 1$, then*

$$\mathbb{KH}_\alpha\left[\mathcal{C}^{(\alpha)}u(t)\right] = \frac{s}{\gamma\eta}\mathbb{KH}_\alpha[u(t)] - \frac{s}{\gamma\eta}u(0).$$

Proof. Using Definition 3.1 and equation (1), we have

$$\begin{aligned}\mathbb{KH}_\alpha\left[\mathcal{C}^{(\alpha)}u(t)\right] &= \frac{s}{\gamma\eta}\int_0^\infty \exp\left(-\frac{st^\alpha}{\gamma\eta\alpha}\right)\mathcal{C}^{(\alpha)}u(t)t^{\alpha-1}dt \\ &= \frac{s}{\gamma\eta}\int_0^\infty \exp\left(-\frac{st^\alpha}{\gamma\eta\alpha}\right)t^{1-\alpha}u'(t)t^{\alpha-1}dt \\ &= \frac{s}{\gamma\eta}\int_0^\infty \exp\left(-\frac{st^\alpha}{\gamma\eta\alpha}\right)u'(t)dt.\end{aligned}$$

With integration by parts, we get

$$\begin{aligned}\mathbb{KH}_\alpha\left[\mathcal{C}^{(\alpha)}u(t)\right] &= \frac{s}{\gamma\eta}\left(\lim_{\sigma\to\infty}\left[\exp\left(-\frac{st^\alpha}{\gamma\eta\alpha}\right)u(t)\right]_0^\sigma \right. \\ &\quad \left. +\frac{s}{\gamma\eta}\int_0^\infty \exp\left(-\frac{st^\alpha}{\gamma\eta\alpha}\right)u(t)t^{\alpha-1}dt\right) \\ &= \frac{s}{\gamma\eta}\left(-u(0)+\mathbb{KH}_\alpha[u(t)]\right) \\ &= \frac{s}{\gamma\eta}\mathbb{KH}_\alpha[u(t)] - \frac{s}{\gamma\eta}u(0).\end{aligned}$$

Theorem 3.3 *Let $u : [0, +\infty) \longrightarrow \mathbb{R}$ be an $n-$differentiable function and $0 < \alpha \leq 1$, then*

$$\mathbb{KH}_{\alpha}\left[\mathcal{C}^{(n\alpha)}u(t)\right] = \frac{s^n}{\gamma^n\eta^n}\mathbb{KH}_{\alpha}\left[u(t)\right] - \frac{s^n}{\gamma^n\eta^n}u(0).$$

Proof. The proof follows using the induction process on n and Theorem 3.2.

4 Conformable Fractional Khalouta Transform for Some Functions

Theorem 4.1 *Let $a, b, c \in \mathbb{R}$ and $0 < \alpha \leq 1$, then*

$$1)\ \mathbb{KH}_{\alpha}\left[b\right] = b.$$

$$2)\ \mathbb{KH}_{\alpha}\left[t^c\right] = \left(\frac{\alpha\gamma\eta}{s}\right)^{\frac{c}{\alpha}}\Gamma\left(\frac{c}{\alpha}+1\right).$$

$$3)\ \mathbb{KH}_{\alpha}\left[\frac{t^{n\alpha}}{\alpha^n}\right] = \left(\frac{\gamma\eta}{s}\right)^{n}\Gamma\left(n+1\right).$$

$$4)\ \mathbb{KH}_{\alpha}\left[\exp\left(a\frac{t^{\alpha}}{\alpha}\right)\right] = \frac{s}{s-a\gamma\eta}.$$

$$5)\ \mathbb{KH}_{\alpha}\left[\sin\left(a\frac{t^{\alpha}}{\alpha}\right)\right] = \frac{as\gamma\eta}{s^2+a^2\gamma^2\eta^2}.$$

$$6)\ \mathbb{KH}_{\alpha}\left[\sinh\left(a\frac{t^{\alpha}}{\alpha}\right)\right] = \frac{as\gamma\eta}{s^2-a^2\gamma^2\eta^2}.$$

$$7)\ \mathbb{KH}_{\alpha}\left[\cos\left(a\frac{t^{\alpha}}{\alpha}\right)\right] = \frac{s^2}{s^2+a^2\gamma^2\eta^2}.$$

$$8)\ \mathbb{KH}_{\alpha}\left[\cosh\left(a\frac{t^{\alpha}}{\alpha}\right)\right] = \frac{s^2}{s^2-a^2\gamma^2\eta^2}.$$

Proof. Using Theorem 3.1, we get

1)

$$\mathbb{KH}_{\alpha}\left[b\right] = \mathbb{KH}\left[b\right] = b.$$

2)

$$\begin{aligned}\mathbb{KH}_{\alpha}\left[t^c\right] &= \mathbb{KH}\left[(\alpha t)^{\frac{c}{\alpha}}\right] = \alpha^{\frac{c}{\alpha}}\mathbb{KH}\left[t^{\frac{c}{\alpha}}\right] \\ &= \left(\frac{\alpha\gamma\eta}{s}\right)^{\frac{c}{\alpha}}\Gamma\left(\frac{c}{\alpha}+1\right).\end{aligned}$$

3) If we put $c = n\alpha$, then

$$\mathbb{KH}_{\alpha}\left[t^{n\alpha}\right] = \mathbb{KH}\left[(\alpha t)^{\frac{n\alpha}{\alpha}}\right] = \alpha^n\mathbb{KH}\left[t^n\right].$$

So

$$\mathbb{KH}_{\alpha}\left[\frac{t^{n\alpha}}{\alpha^n}\right] = \mathbb{KH}\left[t^n\right] = \left(\frac{\gamma\eta}{s}\right)^{n}\Gamma\left(n+1\right).$$

4)

$$\begin{aligned}\mathbb{KH}_\alpha\left[\exp\left(a\frac{t^\alpha}{\alpha}\right)\right] &= \mathbb{KH}\left[\exp\left(a\frac{\left((\alpha t)^{\frac{1}{\alpha}}\right)^\alpha}{\alpha}\right)\right] \\ &= \mathbb{KH}\left[\exp\left(at\right)\right] = \frac{s}{s-a\gamma\eta}.\end{aligned}$$

5)

$$\begin{aligned}\mathbb{KH}_\alpha\left[\sin\left(a\frac{t^\alpha}{\alpha}\right)\right] &= \mathbb{KH}\left[\sin\left(a\frac{\left((\alpha t)^{\frac{1}{\alpha}}\right)^\alpha}{\alpha}\right)\right] \\ &= \mathbb{KH}\left[\sin\left(at\right)\right] = \frac{as\gamma\eta}{s^2+a^2\gamma^2\eta^2}.\end{aligned}$$

6)

$$\begin{aligned}\mathbb{KH}_\alpha\left[\sinh\left(a\frac{t^\alpha}{\alpha}\right)\right] &= \mathbb{KH}\left[\sinh\left(a\frac{\left((\alpha t)^{\frac{1}{\alpha}}\right)^\alpha}{\alpha}\right)\right] \\ &= \mathbb{KH}\left[\sinh\left(at\right)\right] = \frac{as\gamma\eta}{s^2-a^2\gamma^2\eta^2}.\end{aligned}$$

7)

$$\begin{aligned}\mathbb{KH}_\alpha\left[\cos\left(a\frac{t^\alpha}{\alpha}\right)\right] &= \mathbb{KH}\left[\cos\left(a\frac{\left((\alpha t)^{\frac{1}{\alpha}}\right)^\alpha}{\alpha}\right)\right] \\ &= \mathbb{KH}\left[\cos\left(at\right)\right] = \frac{s^2}{s^2+a^2\gamma^2\eta^2}.\end{aligned}$$

8)

$$\begin{aligned}\mathbb{KH}_\alpha\left[\cosh\left(a\frac{t^\alpha}{\alpha}\right)\right] &= \mathbb{KH}\left[\cosh\left(a\frac{\left((\alpha t)^{\frac{1}{\alpha}}\right)^\alpha}{\alpha}\right)\right] \\ &= \mathbb{KH}\left[\cosh\left(at\right)\right] = \frac{s^2}{s^2-a^2\gamma^2\eta^2}.\end{aligned}$$

5 Properties of Conformable Fractional Khalouta Transform

Theorem 5.1 *Let $u, v : [0, +\infty) \longrightarrow \mathbb{R}$ be given functions such that*

$$\mathbb{KH}_\alpha\left[u(t)\right] = \mathcal{K}_\alpha(s, \gamma, \eta),$$

and

$$\mathbb{KH}_\alpha\left[v(t)\right] = \mathcal{H}_\alpha(s, \gamma, \eta),$$

and let $\lambda, \mu \in \mathbb{R}$ and $0 < \alpha \leq 1$, then we have

1) *Linear property*

$$\mathbb{KH}_\alpha\left[\lambda u(t)\pm\mu v(t)\right]=\lambda\mathbb{KH}_\alpha\left[u(t)\right]\pm\mu\mathbb{KH}_\alpha\left[v(t)\right].$$

2) *Shifting property*

$$\mathbb{KH}_\alpha\left[\exp\left(-a\frac{t^\alpha}{\alpha}\right)u(t)\right]=\frac{s}{s+a\gamma\eta}\mathcal{K}_\alpha(s,\frac{s}{s+a\gamma\eta},\eta).$$

3) *Integral property*

$$\mathbb{KH}_\alpha\left[\mathcal{I}^{(\alpha)}u(t)\right]=\frac{\gamma\eta}{s}\mathbb{KH}_\alpha\left[u(t)\right].$$

4) *Convolution property*

$$\mathbb{KH}_\alpha\left[\left(u*v\right)(t)\right]=\frac{\gamma\eta}{s}\mathcal{K}_\alpha(s,\gamma,\eta)\mathcal{H}_\alpha(s,\gamma,\eta),$$

where $u*v$ is a convolution of two functions defined by

$$\left(u*v\right)(t)=\int_0^t u(\tau)v(t-\tau)d\tau=\int_0^t u(t-\tau)v(\tau)d\tau.$$

Proof. 1) Using Definition 3.1, we get

$$\begin{aligned}
\mathbb{KH}_\alpha\left[\lambda u(t)\pm\mu v(t)\right] &= \frac{s}{\gamma\eta}\int_0^\infty \exp\left(-\frac{st}{\gamma\eta}\right)(\lambda u(t)\pm\mu v(t))t^{\alpha-1}dt \\
&= \frac{s}{\gamma\eta}\int_0^\infty \exp\left(-\frac{st}{\gamma\eta}\right)\lambda u(t)t^{\alpha-1}dt \\
&\quad \pm\frac{s}{\gamma\eta}\int_0^\infty \exp\left(-\frac{st}{\gamma\eta}\right)\mu v(t))t^{\alpha-1}dt \\
&= \lambda\left(\frac{s}{\gamma\eta}\int_0^\infty \exp\left(-\frac{st}{\gamma\eta}\right)u(t)t^{\alpha-1}dt\right) \\
&\quad \pm\mu\left(\frac{s}{\gamma\eta}\int_0^\infty \exp\left(-\frac{st}{\gamma\eta}\right)v(t))t^{\alpha-1}dt\right) \\
&= \lambda\mathbb{KH}_\alpha\left[u(t)\right]\pm\mu\mathbb{KH}_\alpha\left[v(t)\right].
\end{aligned}$$

2) Using Theorem 3.1, we get

$$\begin{aligned}
\mathbb{KH}_\alpha\left[\exp\left(-a\frac{t^\alpha}{\alpha}\right)u(t)\right] &= \mathbb{KH}\left[\exp\left(-a\frac{\left((\alpha t)^{\frac{1}{\alpha}}\right)^\alpha}{\alpha}\right)u((\alpha t)^{\frac{1}{\alpha}})\right] \\
&= \mathbb{KH}\left[\exp\left(-at\right)u((\alpha t)^{\frac{1}{\alpha}})\right] \\
&= \frac{s}{\gamma\eta}\int_0^\infty \exp\left(-\frac{st}{\gamma\eta}\right)\exp\left(-at\right)u((\alpha t)^{\frac{1}{\alpha}})dt \\
&= \frac{s}{\gamma\eta}\int_0^\infty \exp\left(-\left(\frac{s}{\gamma\eta}+a\right)t\right)u((\alpha t)^{\frac{1}{\alpha}})dt \\
&= \frac{s}{\gamma\eta}\int_0^\infty \exp\left(-\left(\frac{s+a\gamma\eta}{\gamma\eta}\right)t\right)u((\alpha t)^{\frac{1}{\alpha}})dt. \qquad (2)
\end{aligned}$$

If $(s+a\gamma\eta)\,t = sx$ and $t = \frac{sx}{s+a\gamma\eta}$, then we have $dt = \frac{s}{s+a\gamma\eta}dx$, so equation (2) becomes

$$\begin{aligned}\mathbb{KH}_\alpha\left[\exp\left(-a\frac{t^\alpha}{\alpha}\right)u(t)\right] &= \frac{s}{\gamma\eta}\int_0^\infty \exp\left(-\frac{sx}{\gamma\eta}\right)u\left(\left(\frac{\alpha sx}{s+a\gamma\eta}\right)^{\frac{1}{\alpha}}\right)\frac{s}{s+a\gamma\eta}dx \\ &= \frac{s}{s+a\gamma\eta}\left(\frac{s}{\gamma\eta}\int_0^\infty \exp\left(-\frac{sx}{\gamma\eta}\right)u\left(\left(\frac{\alpha sx}{s+a\gamma\eta}\right)^{\frac{1}{\alpha}}\right)dx\right) \\ &= \frac{s}{s+a\gamma\eta}\mathcal{K}_\alpha(s,\frac{s}{s+a\gamma\eta},\eta).\end{aligned}$$

3) Using Theorem 3.2, we get

$$\mathbb{KH}_\alpha\left[\mathcal{C}^{(\alpha)}\mathcal{I}^{(\alpha)}u(t)\right] = \frac{s}{\gamma\eta}\mathbb{KH}_\alpha\left[\mathcal{I}^{(\alpha)}u(t)\right] - \frac{s}{\gamma\eta}\mathcal{I}^{(\alpha)}u(0).$$

But $\mathcal{C}^{(\alpha)}\mathcal{I}^{(\alpha)}u(t) = u(t)$ and $\mathcal{I}^{(\alpha)}u(0) = 0$, so we have

$$\mathbb{KH}_\alpha\left[\mathcal{I}^{(\alpha)}u(t)\right] = \frac{\gamma\eta}{s}\mathbb{KH}_\alpha\left[u(t)\right].$$

4) Using Theorem 3.1 and the convolution definition, we get

$$\begin{aligned}\mathbb{KH}_\alpha\left[(u*v)(t)\right] &= \mathbb{KH}\left[(u*v)\left((\alpha t)^{\frac{1}{\alpha}}\right)\right] \\ &= \frac{s}{\gamma\eta}\int_0^\infty \exp\left(-\frac{st}{\gamma\eta}\right)(u*v)\left((\alpha t)^{\frac{1}{\alpha}}\right)dt \qquad (3) \\ &= \frac{s}{\gamma\eta}\int_0^\infty \exp\left(-\frac{st}{\gamma\eta}\right)\left(\int_0^t u\left(\alpha\,(t-\tau)^{\frac{1}{\alpha}}\right)v\left((\alpha\tau)^{\frac{1}{\alpha}}\right)d\tau\right)dt.\end{aligned}$$

From the region $R = \{(\tau,t)\in\mathbb{R}^2 : 0\le\tau\le t \text{ and } 0\le t\le+\infty\}$, we can change the order of integration, i.e., equation (3) becomes

$$\mathbb{KH}_\alpha\left[(u*v)(t)\right] = \frac{s}{\gamma\eta}\int_0^\infty\int_\tau^\infty \exp\left(-\frac{st}{\gamma\eta}\right)u\left(\alpha\,(t-\tau)^{\frac{1}{\alpha}}\right)v\left((\alpha\tau)^{\frac{1}{\alpha}}\right)d\tau dt. \qquad (4)$$

Substituting $x = t-\tau$ and $dx = dt$ in equation (4), we get

$$\begin{aligned}\mathbb{KH}_\alpha\left[(u_1*u_2)(t)\right] &= \frac{s}{\gamma\eta}\int_0^\infty\int_0^\infty \exp\left(-\frac{s(x+\tau)}{\gamma\eta}\right)u\left((\alpha x)^{\frac{1}{\alpha}}\right)v\left((\alpha\tau)^{\frac{1}{\alpha}}\right)d\tau dx \\ &= \frac{s}{\gamma\eta}\left(\int_0^\infty \exp\left(-\frac{sx}{\gamma\eta}\right)u\left((\alpha x)^{\frac{1}{\alpha}}\right)dx\right) \\ &\quad\times\left(\int_0^\infty \exp\left(-\frac{s\tau}{\gamma\eta}\right)v\left((\alpha\tau)^{\frac{1}{\alpha}}\right)d\tau\right) \\ &= \frac{\gamma\eta}{s}\left(\frac{s}{\gamma\eta}\int_0^\infty \exp\left(-\frac{sx}{\gamma\eta}\right)u\left((\alpha x)^{\frac{1}{\alpha}}\right)dx\right) \\ &\quad\times\left(\frac{s}{\gamma\eta}\int_0^\infty \exp\left(-\frac{s\tau}{\gamma\eta}\right)v\left((\alpha\tau)^{\frac{1}{\alpha}}\right)d\tau\right) \\ &= \frac{\gamma\eta}{s}\mathcal{K}_\alpha(s,\gamma,\eta)\mathcal{H}_\alpha(s,\gamma,\eta).\end{aligned}$$

6 Applications

Application 1 Consider the linear conformable fractional differential equation

$$\mathcal{C}^{(\alpha)}u(t)-u(t)=1, 0<\alpha\leq 1, \tag{5}$$

subject to the initial condition

$$u(0)=0. \tag{6}$$

Applying the conformable fractional Khalouta transform on both sides of equation (5) and using Theorem 5.1, we get

$$\mathbb{KH}_{\alpha}\left[\mathcal{C}^{(\alpha)}u(t)\right]-\mathbb{KH}_{\alpha}\left[u(t)\right]=\mathbb{KH}_{\alpha}\left[1\right]. \tag{7}$$

When using Theorems 3.2 and 4.1, equation (7) becomes

$$\frac{s}{\gamma\eta}\mathbb{KH}_{\alpha}\left[u(t)\right]-\frac{s}{\gamma\eta}u(0)-\mathbb{KH}_{\alpha}\left[u(t)\right]=1. \tag{8}$$

Substituting the initial condition (6) and simplifying equation (8), we have

$$\begin{aligned}\mathbb{KH}_{\alpha}\left[u(t)\right] &= \frac{\gamma\eta}{s-\gamma\eta} \\ &= \frac{s}{s-\gamma\eta}-1.\end{aligned} \tag{9}$$

Taking the inverse conformable fractional Khalouta transform of both sides of equation (9), we get

$$u(t)=\exp\left(\frac{t^{\alpha}}{\alpha}\right)-1. \tag{10}$$

For $\alpha=1$, the result in equation (10) reduces to the exact solution for the standard form of equations (5) and (6) as follows:

$$u(t)=\exp\left(t\right)-1.$$

Application 2 Consider the linear conformable fractional differential equation

$$\mathcal{C}^{(2\alpha)}u(t)-u(t)=\sin\left(\frac{2t^{\alpha}}{\alpha}\right), 0<\alpha\leq 1, \tag{11}$$

subject to the initial conditions

$$u(0)=2, \mathcal{C}^{(\alpha)}u(0)=0. \tag{12}$$

Applying the conformable fractional Khalouta transform on both sides of equation (11) and using Theorem 5.1, we get

$$\mathbb{KH}_{\alpha}\left[\mathcal{C}^{(2\alpha)}u(t)\right]-\mathbb{KH}_{\alpha}\left[u(t)\right]=\mathbb{KH}_{\alpha}\left[\sin\left(\frac{2t^{\alpha}}{\alpha}\right)\right]. \tag{13}$$

When using Theorems 3.3 and 4.1, equation (13) becomes

$$\frac{s^2}{\gamma^2\eta^2}\mathbb{KH}_{\alpha}\left[u(t)\right]-\frac{s^2}{\gamma^2\eta^2}u(0)-\mathbb{KH}_{\alpha}\left[u(t)\right]=\frac{2s\gamma\eta}{s^2+4\gamma^2\eta^2}. \tag{14}$$

Substituting the initial conditions (12) and simplifying equation (14), we have

$$\begin{aligned}\mathbb{KH}_{\alpha}[u(t)] &= \frac{2s\gamma^3\eta^3}{(s^2+4\gamma^2\eta^2)(s^2-\gamma^2\eta^2)}+\frac{2s^2}{s^2-\gamma^2\eta^2}\\ &= -\frac{1}{5}\frac{2s\gamma\eta}{s^2+4\gamma^2\eta^2}+\frac{2}{5}\frac{s\gamma\eta}{s^2-\gamma^2\eta^2}+\frac{2s^2}{s^2-\gamma^2\eta^2}. \end{aligned} \tag{15}$$

Taking the inverse conformable fractional Khalouta transform of both sides of equation (15), we get

$$u(t)=-\frac{1}{5}\sin\left(\frac{2t^{\alpha}}{\alpha}\right)+\frac{2}{5}\sinh\left(\frac{t^{\alpha}}{\alpha}\right)+2\cosh\left(\frac{t^{\alpha}}{\alpha}\right). \tag{16}$$

For $\alpha=1$, the result in equation (16) reduces to the exact solution for the standard form of equations (11) and (12) as follows:

$$u(t)=-\frac{1}{5}\sin(2t)+\frac{2}{5}\sinh(t)+2\cosh(t).$$

Application 3 Consider the linear conformable fractional differential equation

$$\mathcal{C}^{(3\alpha)}u(t)+\mathcal{C}^{(\alpha)}u(t)=\frac{t^{\alpha}}{\alpha}, 0<\alpha\leq 1, \tag{17}$$

subject to the initial conditions

$$u(0)=0, \mathcal{C}^{(\alpha)}u(0)=0, \mathcal{C}^{(2\alpha)}u(0)=0. \tag{18}$$

Applying the conformable fractional Khalouta transform on both sides of equation (17) and using Theorem 5.1, we get

$$\mathbb{KH}_{\alpha}\left[\mathcal{C}^{(3\alpha)}u(t)\right]+\mathbb{KH}_{\alpha}\left[\mathcal{C}^{(\alpha)}u(t)\right]=\mathbb{KH}_{\alpha}\left[\frac{t^{\alpha}}{\alpha}\right]. \tag{19}$$

When using Theorems 3.3 and 4.1, equation (19) becomes

$$\frac{s^3}{\gamma^3\eta^3}\mathbb{KH}_{\alpha}[u(t)]-\frac{s^3}{\gamma^3\eta^3}u(0)+\frac{s}{\gamma\eta}\mathbb{KH}_{\alpha}[u(t)]-\frac{s}{\gamma\eta}u(0)=\frac{\gamma\eta}{s}\Gamma(2). \tag{20}$$

Substituting the initial conditions (18) and simplifying the equation (20), we have

$$\begin{aligned}\mathbb{KH}_{\alpha}[u(t)] &= \frac{\gamma^4\eta^4}{s^2(s^2+\gamma^2\eta^2)}\\ &= \frac{\gamma^2\eta^2}{s^2}+\frac{s^2}{s^2+\gamma^2\eta^2}-1\\ &= \frac{1}{2}\frac{\gamma^2\eta^2}{s^2}\Gamma(3)+\frac{s^2}{s^2+\gamma^2\eta^2}-1. \end{aligned} \tag{21}$$

Taking the inverse conformable fractional Khalouta transform of both sides of equation (21), we get

$$u(t)=\frac{t^{2\alpha}}{2\alpha^2}+\cos\left(\frac{t^{\alpha}}{\alpha}\right)-1. \tag{22}$$

For $\alpha=1$, the result in equation (22) reduces to the exact solution for the standard form of equations (17) and (18) as follows:

$$u(t)=\frac{t^2}{2}+\cos(t)-1.$$

7 Conclusion

In this paper, we carefully proposed a new conformable fractional integral transform known as the conformable fractional Khalouta transform which presents a promising tool for solving fractional differential equations. The conformable fractional Khalouta transform was successfully applied to find solutions of conformable fractional differential equations. It can be concluded that the proposed methodology is very powerful and effective in finding analytical solutions for wide categories of fractional differential equations.

Acknowledgment

The author sincerely thanks the reviewers for their careful reading of the paper and for their important comments and suggestions that led to improving the paper.

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Complex Dynamics of Novel Chaotic System with No Equilibrium Point: Amplitude Control and Offset Boosting Control, Its Adaptive Synchronization

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Received: February 3, 2024; Revised: September 19, 2024

Abstract: In this paper, a novel chaotic system with dissipative nature is introduced. In the proposed system, a conservative system can be modified to a dissipative system by adding a linear term. The chaotic dynamics of the new system such as Lyapunov exponents, Lyapunov dimensions, Poincare plots and attractor plots are verified through numerical simulations. The dynamical analysis is also conducted to verify the existence of chaotic attractors for the particular system parameters. It is found that the amplitude and position of the proposed chaotic attractors can be controlled. The numerical simulations revealed that the new system has many complex dynamics which can be used for various applications. Finally, the chaos synchronization for the proposed system is demonstrated by designing the nonlinear adaptive controllers. The efficiency of the synchronization methodology is verified theoretically by the Lyapunov stability theorem and numerical simulation in MATLAB environment.

Keywords: *chaotic system; no equilibrium points; amplitude control; offset boosting control; chaos synchronization*

Mathematics Subject Classification (2010): 93A30, 93B05, 93D21, 93-XX.

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1 Introduction

Chaotic systems are characterized by unpredictability, sensitivity dependent on initial conditions and system parameters, fractal dimension and Lyapunov exponents. Due to this complex behavior, for the past decades, researchers have reported many chaotic systems with unique features such as stable equilibrium [1], different equilibria [2], non-hyperbolic equilibrium points [3], fractional order [4], hidden attractors [5, 6] and found many applications in science and engineering fields [7–9]. The Lyapunov exponent values and Lyapunov dimension play important roles in the design of chaotic systems. A nonlinear system must have at least one positive Lyapunov exponent to have chaotic dynamics. For a chaotic system with dissipative flow, the sum of Lyapunov exponent values should be non-zero and negative. The Lyapunov dimension value for a dissipative chaotic system must be a real number. In a dissipative system, the system limit sets are confined to a specific limit set of zero volume, and the asymptotic motion of the chaotic system settles onto a strange attractor of the system. The chaotic system with conservative flow is reported in [10] as described in Equation (1):

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x + ayz, \\ \dot{z} &= b - y^4, \end{aligned} \tag{1}$$

where the parameter values are $a = b = 1$. The Lyapunov exponent values of the system (1) are $l_1 = 0.0836$, $l_2 = 0$ and $l_3 = -0.0836$. The Kaplan-Yorke dimension of the system (1) is $D_L = 3$. In this paper, a new chaotic system with dissipative flow is constructed by adding a liner term x and by adjusting the parameter in the third differential equation of (1).

The amplitude control [11, 12] and offset boosting control [13, 14] are the important research problems in the chaotic systems. The amplitude control in the chaotic system can be achieved by multiplying the chaotic signals with a constant parameter. The amplitude control of chaotic signals has many applications such as chaotic modulation and signal processing. The offset boosting control of the chaotic attractors can be achieved by adding a constant parameter with the particular signal. The offset boosting control can be used to convert the bipolar chaotic signal into the unipolar signal or vice versa. The proposed system has the complete amplitude control feature, that is, the amplitude of all the signals can be controlled in the new system. The attractors of the proposed system can be offset boosted along the x and z directions.

Due to the butterfly effect and sensitivity to the initial conditions, the synchronization of even the identical chaotic systems is a challenging research work. The chaos synchronization has engineering applications such as secure transmission system. In the last three decades, many methods have been reported to address the problems in chaos synchronization [15–18]. The adaptive chaos synchronization deals with the design of adaptive controllers to stabilize the chaotic signals globally. In the chaos synchronization methodology, a particular system can be considered as a master system and another or same system can be considered as a slave system. The main idea of the chaos synchronization is to design an adaptive controller so that the slave system follows the output of the master system asymptotically. In this paper, an adaptive controller is designed for the proposed system and verified the stability of the controlled system through the Lyapunov stability theory. The numerical simulation indicates the efficiency of the proposed synchronization methodology.

2 Description of New Chaotic System

The new chaotic system can be described mathematically as the following dynamics:

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x + ayz, \\ \dot{z} &= \alpha - y^6 + x, \end{aligned} \tag{2}$$

where x, y, z are the state signals and α is the system parameter. The parameter values are chosen as $a = 1$ and $\alpha = 4$.

3 No Equilibrium Points

The equilibrium points of the system (2) can be calculated by equating $\dot{x} = 0$, $\dot{y} = 0$ and $\dot{z} = 0$ in Eq.(2),

$$y = 0, \tag{3a}$$

$$-x + ayz = 0, \tag{3b}$$

$$\alpha - y^6 + x = 0. \tag{3c}$$

From (3a), $y = 0$. If we replace $y = 0$ in (3b) and (3c), we will get $x = 0$ and $x = -\alpha$, respectively. It is clear that Eq.(3b) and Eq.(3c) contradict each other and no solution exists for Eq.(3). It can be concluded that the new system (2) has no equilibrium points. The system (2) can be able to generate chaotic attractors even though there are no equilibrium points.

4 Chaotic Dynamics

In this section, the various chaotic dynamics of the system (2) such as the Lyapunov exponents and Lyapunov dimension, dissipativity, sensitivity, Poincare plot, and attractors diagram are verified by numerical simulations.

4.1 Lyapunov exponents and Lyapunov dimension

The Lyapunov exponent values of the system (2) are calculated using classical Wolf's algorithm with the simulation time of 10000 sec and step size 0.1. The Lyapunov values of the system (2) are $LE_1 = 0.05223, LE_2 = 0, LE_3 = -0.07433$ for the parameters $a = 1$, $\alpha = 4$ and the initial condition $(1, -1, 1)$. The Lyapunov dimension D_L of the system (2) can be calculated as

$$D_L = 2 + \frac{l_1 + l_2}{|l_3|} = 2 + \frac{0.05223}{0.074333} = 2.7026.$$

Since the Maximum Lyapunov Exponent (MLE) of the system (2) has a positive value and the Lyapunov dimension D_L has a fractional value, the proposed system (2) satisfies the conditions required for it to be chaotic. Figure 1 represents the time history of Lyapunov exponents of the system (2) over $t \in [0, 10000]$, which indicates that the system has at least one positive Lyapunov exponent value for the entire simulation time.

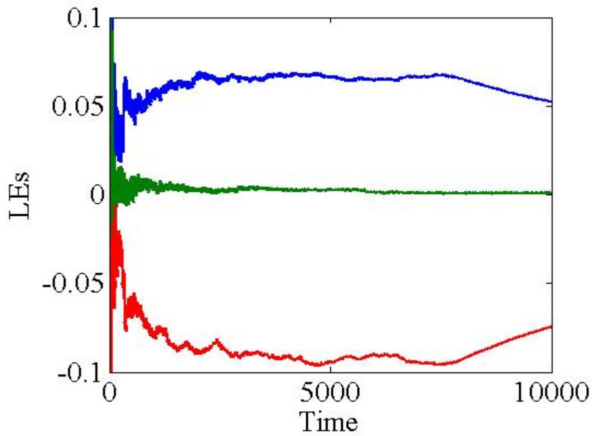


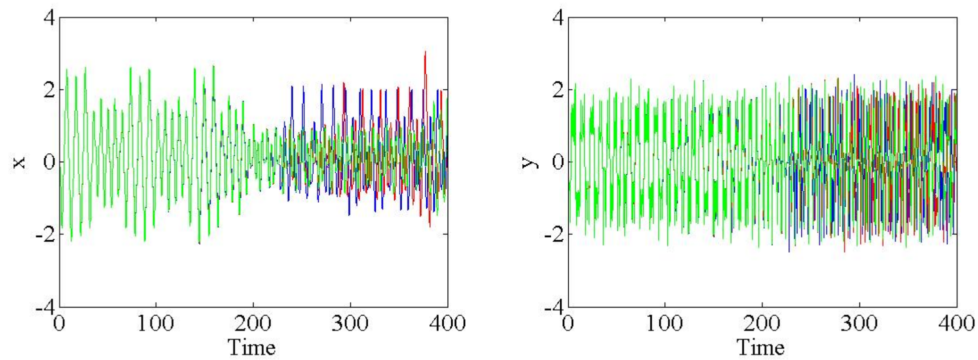
Figure 1: Time variation of the Lyapunov exponents of the system (2).

4.2 Dissipative nature

Since the sum of all the Lyapunov exponent values, i.e., $l_1 + l_2 + l_3 = -0.092494$, has a negative value, the proposed system (2) has dissipative behavior. It can be concluded that a dissipative system (2) can be constructed from the conservative system (1).

4.3 Sensitivity to initial conditions

The sensitivity to small variations in the initial conditions and unpredictability are the essential characteristics of the nonlinear dynamical system to be chaotic. Figure 2 shows the time variation of x and y signals of the system (2) with the initial conditions $(1, -1, 1)$ (Blue), $(1, -1, 1.0001)$ (Red) and $(1.0001, -1, 1)$ (Green). Figure 2 depicts that the state signals of the system (2) have different trajectories after the simulation time $t = 220sec$.

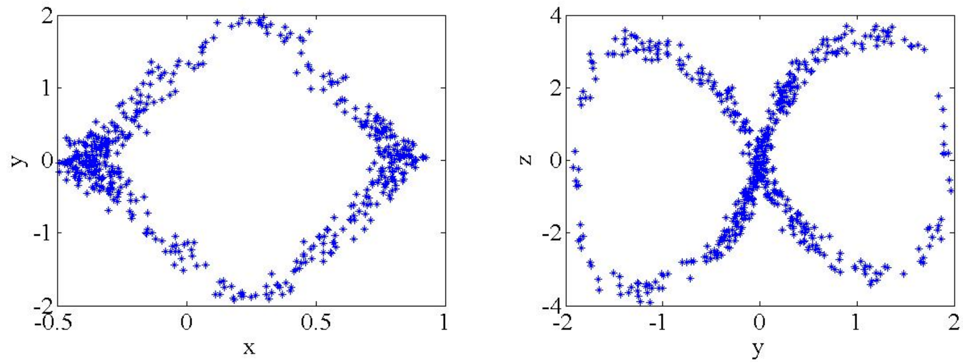


(a) x signal (b) y signal

Figure 2: Time variation of x and y signals with the initial conditions $(1, -1, 1)$ (Blue), $(1, -1, 1.0001)$ (Red) and $(1.0001, -1, 1)$ (Green).

4.4 The Poincaré plot

The Poincaré plot can be used to find out the chaotic behavior in a dynamical system by embedding a data set in a higher-dimensional state space. The Poincaré sections of the system (2) are plotted with the simulation time 50000 and step size 0.1 and presented in Figure 3. It can be observed from Figure 3 that the Poincaré sections have a set of distinct points, which indicates that the system (2) has chaotic behavior. It is interesting



(a) xy plane (b) yz plane

Figure 3: Poincaré plot of the system (2) with the initial conditions $(1, -1, 1)$.

to note that the system (2) can have chaotic behavior even though there is no equilibrium point within itself. For the parameters $a = 1$ and $\alpha = 4$, the system (2) produces chaotic attractors as shown in Figure 4.

5 Dynamical Analysis

The dynamical analysis of the proposed system (2) is conducted with the help of classical non linear tools such as the bifurcation diagram and Lyapunov exponent spectrum. The bifurcation and Lyapunov exponent plots can be obtained by varying the system parameter gradually up to a particular level. The bifurcation diagram can be used to analyze the various states of a nonlinear dynamic system such as chaotic state, periodic state and limit cycle oscillation under the particular range of system parameter. The Lyapunov exponent spectrum having at least one positive Lyapunov exponent value in the region of system parameter indicates the sensitivity to the initial conditions or the existence of the chaotic dynamics in the particular system. Figure 5 depicts the bifurcation and Lyapunov exponent plots of the system (2) for the variation of the parameter $\alpha \in [0, 5]$. Figure 5a indicates that the system (2) has chaotic dynamics from $\alpha = 1.2$ to $\alpha = 5$. Figure 5b also represents that the system has chaotic dynamics in the regions $\alpha \in [1.2, 5]$ at which the system (2) has positive Lyapunov exponent values.

6 Amplitude Control

The amplitude of the chaotic signals can be controlled in the proposed system (2) by rescaling the system variables using the control parameter δ. If the signals are rescaled

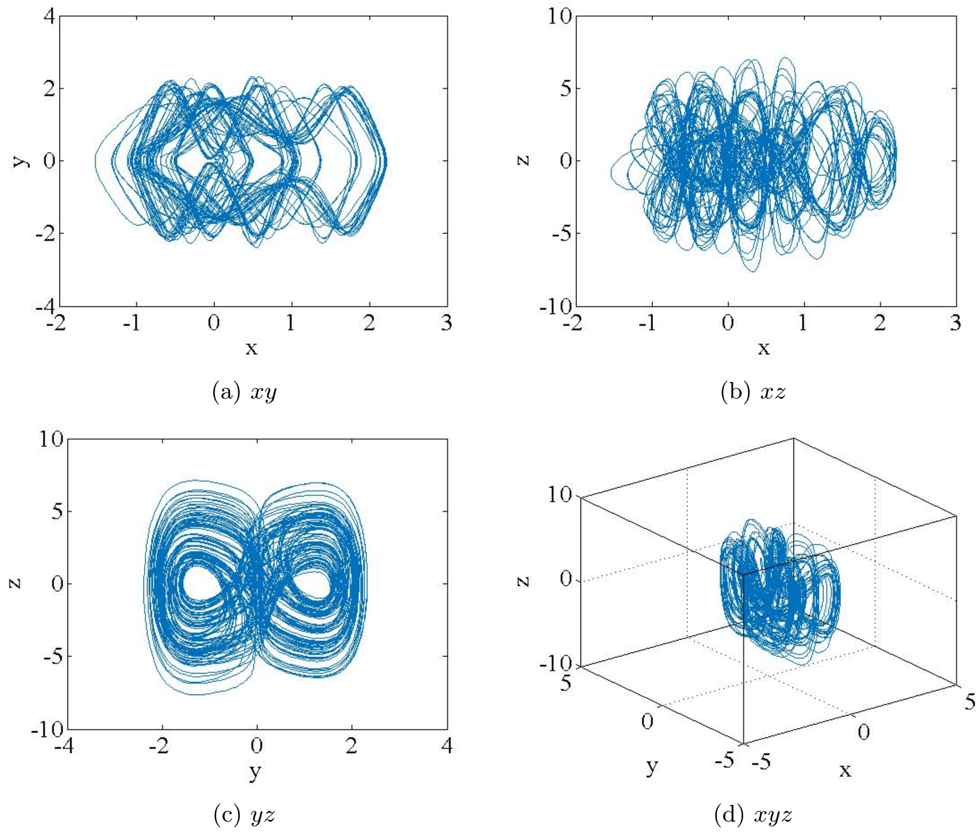


Figure 4: Chaotic attractors of the new system (2) with the parameter $\alpha = 4$ and the initial conditions $(1, 1, 1)$.

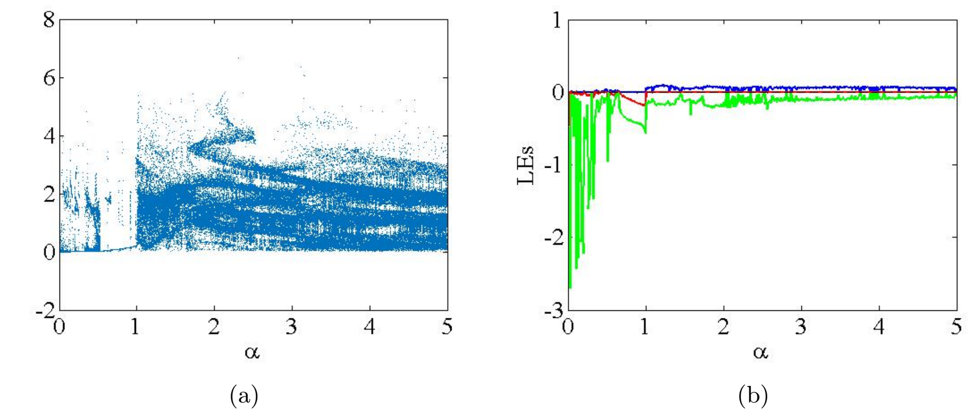
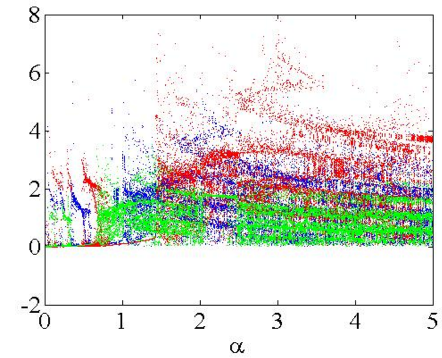


Figure 5: (a) Bifurcation diagram, (b) Lyapunov spectrum of the system (2) with the initial conditions $(1, -1, 1)$.

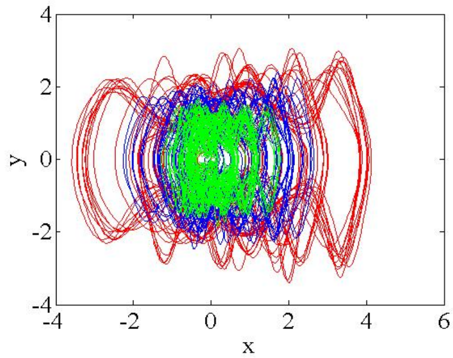
as δx, δy and δz in the system (2), then the system (2) becomes a complete amplitude controllable system (4):

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -ax + yz\delta, \\ \dot{z} &= \alpha - \delta^5 y^6 + x. \end{aligned} \tag{4}$$

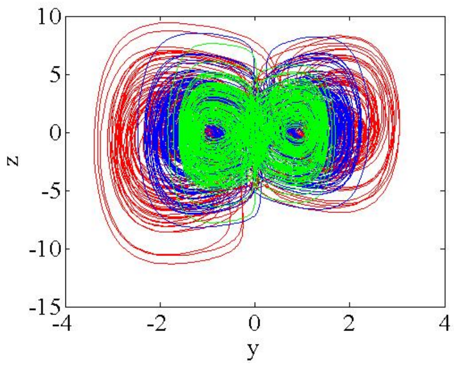
The system (4) also has no equilibrium points because of the contradiction between the second and third equations of (4). The bifurcation diagram of the system (4) with $\delta = 1$ (Blue), $\delta = 0.7$ (Red) and $\delta = 1.5$ (Green) is given in Figure 6a which indicates that the bifurcation level increases for $\delta < 1$ and reduces for $\delta > 1$. Figures 6b - 6d are the chaotic attractors of the amplitude controlled system (4) with $\delta = 1$ (Blue), $\delta = 0.7$ (Red) and $\delta = 1.5$ (Green).



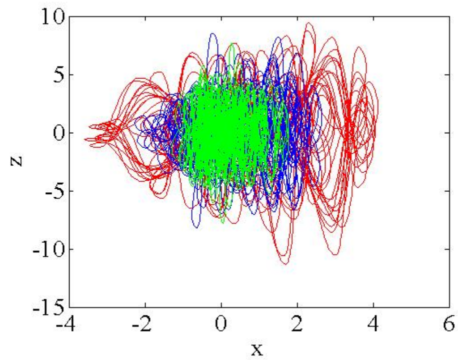
(a)



(b)



(c)



(d)

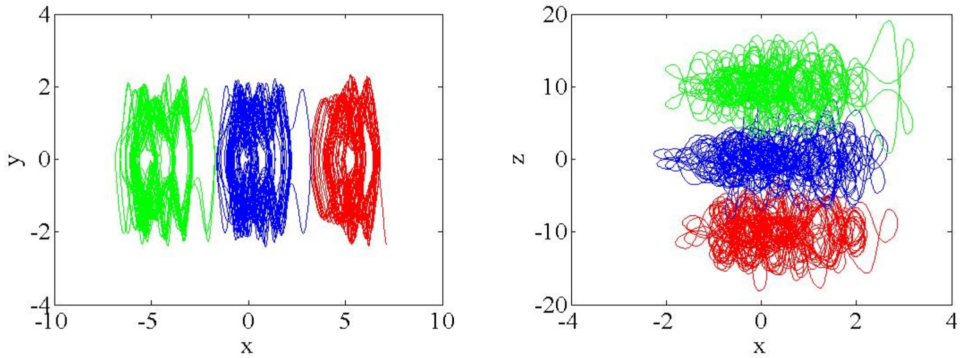
Figure 6: (a) Bifurcation diagram of the system (4), (b-d) Chaotic attractors of the system (4) with $\delta = 1$ (Blue), $\delta = 0.7$ (Red) and $\delta = 1.5$ (Green).

7 Offset Boosting Control

The offset boosting control of the chaotic signal is important for the various applications where the conversion of bipolar chaotic signals into unipolar chaotic signals and vice versa is required. The system (2) can be offset boostable along the x and z directions when we add a booster parameter with the signals. Equations (5) and (6) indicate the dc offset boosted system along the x and z directions, respectively, where β and α are the booster parameters,

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -a(x+\beta) + yz, \\ \dot{z} &= \alpha - y^6 + (x+\beta), \end{aligned} \tag{5}$$

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -ax + y(z+\sigma), \\ \dot{z} &= \alpha - y^6 + x. \end{aligned} \tag{6}$$



(a) (b)

Figure 7: The offset boosted attractors of the system (2) with the initial conditions $(1, -1, 1)$. (a) x direction, (b) z direction.

Figure 7 represents the dc offset boosted attractors of the system (2) along the x and z directions with the control parameter $\beta = 0$ (Blue), $\beta = -5$ (Red) and $\beta = 5$ (Green), $\sigma = 0$ (Blue), $\sigma = 10$ (Red) and $\sigma = -10$ (Green). Note that the signals are shifted to a positive direction for the negative booster and the signals are shifted to a negative direction if the booster parameter has a positive value.

8 Adaptive Synchronization

One of the important practical applications of chaotic systems is secure communication in which the chaotic systems can act as the transmitter (master system) and receiver (slave system). In the past few decades, chaos synchronization has received great attention owing to its applications in designing secure communication systems. In this section, the adaptive synchronization of the proposed system is addressed for practical applications. The adaptive synchronization between the system (2) is achieved using nonlinear feedback control and master-slave synchronization methodology. To achieve chaos synchronization, the system (7) is considered as the master system and the system (8) is considered as the slave system. The main idea of the synchronization is to design the adaptive controllers

u by which the slave system(8) follows the master system (7) in an adaptive manner,

$$\begin{aligned} \dot{x_1} &= y_1, \\ \dot{y_1} &= -x_1 + ay_1z_1, \\ \dot{z_1} &= \alpha - y_1^6 + x_1, \end{aligned} \tag{7}$$

$$\begin{aligned} \dot{x_2} &= y_2 + u_1, \\ \dot{y_2} &= -x_2 + ay_2z_2 + u_2, \\ \dot{z_2} &= \alpha - y_2^6 + x_2 + u_3. \end{aligned} \tag{8}$$

The adaptive synchronization errors can be defined as given in Equation (9):

$$\begin{aligned} e_1 &= x_2 - x_1, \\ e_2 &= y_2 - y_1, \\ e_3 &= z_2 - z_1. \end{aligned} \tag{9}$$

Thus, the error dynamics can be written from the equations (7)-(9) as given in (10):

$$\begin{aligned} \dot{e}_1 &= \dot{x}_2 - \dot{x}_1 = e_2 + u_1, \\ \dot{e}_2 &= \dot{y}_2 - \dot{y}_1 = -e_1 + a(y_2z_2 - y_1z_1) + u_2, \\ \dot{e}_3 &= \dot{z}_2 - \dot{z}_1 = e_1 - y_2^6 + y_1^6 + u_3. \end{aligned} \tag{10}$$

According to the adaptive methodology, the adaptive controllers can be written from (10) as follows:

$$\begin{aligned} u_1 &= -e_2 - k_1e_1, \\ u_2 &= e_1 - \hat{a}(y_2z_2 - y_1z_1) - k_2e_2, \\ u_3 &= -e_1 + y_2^6 - y_1^6 - k_3e_3, \end{aligned} \tag{11}$$

where $\hat{a}$ is the estimate value of the unknown parameter a and k_1, k_2, k_3 are the gain of the controllers. By substituting (11) in (10), the error dynamics can be modified as follows:

$$\begin{aligned} \dot{e}_1 &= -k_1e_1, \\ \dot{e}_2 &= e_a(y_2z_2 - y_1z_1) - k_2e_2, \\ \dot{e}_3 &= -k_3e_3. \end{aligned} \tag{12}$$

Here, the parameter error $e_a = a - \hat{a}$ and thus the parameter error dynamics can be written as

$$\dot{e}_a = -\dot{\hat{a}}. \tag{13}$$

Now, consider a positive Lyapunov definite function as follows:

$$\dot{V} = e_1\dot{e}_1 + e_2\dot{e}_2 + e_3\dot{e}_3 + e_a\dot{e}_a. \tag{14}$$

By substituting (12) and (13) in the equation (14), we get

$$\dot{V} = -k_1e_1^2 - k_2e_2^2 - k_3e_3^2 + e_a[e_2(y_2z_2 - y_1z_1) - \dot{\hat{a}}]. \tag{15}$$

If we choose the parameter error dynamics $\dot{\hat{a}} = -e_2(y_2z_2 - y_1z_1)$, then the Lyapunov definite function becomes as follows:

$$\dot{V} = -k_1e_1^2 - k_2e_2^2 - k_3e_3^2$$

which is a negative definite Lyapunov function. According to the Lyapunov stability theory, the equation (15) indicates that the adaptive synchronization errors e_1, e_2, e_3 and the parameter error e_a decay exponentially to zero with time. For the simulation purpose, the initial conditions are chosen for master (7) and slave system (8) as $\{1, -0.5, -2\}$ and $\{-1, 2.5, 3\}$, respectively.

Figures 8a – 8c represents the synchronized signals of the master and slave chaotic systems. Figure 8d represent the time variation of adaptive synchronization errors between the master and slave systems. Figure 8d indicates that all the signals are synchronized after 12 seconds and the error becomes zero.

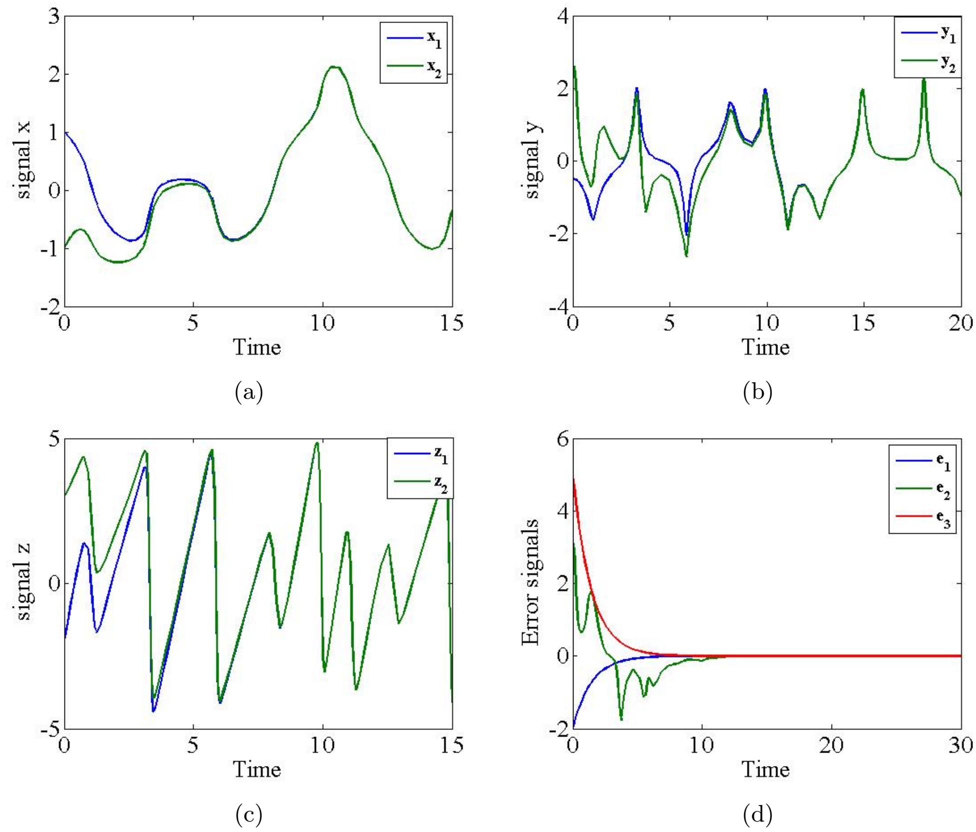


Figure 8: (a-c) The time variation of adaptively synchronized signals x, y, z, respectively, (d) The time variation of the error signals.

9 Conclusion

A novel three-dimensional chaotic system having no equilibrium points has been studied in this work. The results of the numerical simulation indicate that the proposed system satisfies all the conditions required for it to be chaotic. The dynamical analysis indicates that the new system does not lose its chaotic dynamics with a small variation of the parameter value. It is proved that the amplitude of all the signals of the proposed system can be controlled up to a particular level, which can be used to design a chaotic amplifier. The numerical simulation of the offset boosting control indicates that the x and z signals of the new system can be DC offset boosted. The DC offset boosting control behavior of the new system can be used to reduce the number of modulator devices in communication systems. The practical application of the proposed system is addressed by achieving chaos synchronization between identical systems. The adaptive synchronization methodology based on the Lyapunov stability method has been applied to synchronize the proposed system. The MATLAB results proved that the designed adaptive controllers can achieve chaos synchronization within a very short simulation time.

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Nonlinear Dynamics and Systems Theory, **24** (5) (2024) 517–525

A Remotely Operated Vehicle Tracking Model Estimation Using Square Root Ensemble Kalman Filter and Particle Filter

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Received: September 8, 2023;   Revised: September 29, 2024

Abstract: A ROV (Remotely Operated Vehicle) is a product of technological development that functions to perform tasks in the water. Major tasks are coral reef exploration, oil refineries, underwater monitoring, and at-sea accident rescue. The ROV or unmanned submarines have 6 degrees of freedom. In operation, the ROV requires a navigation system, that is, ROV position estimation in its diving and emerging. Some reliable motion estimation methods frequently used are the Ensemble Kalman Filter Square Root (EnKF-SR) and Particle Filter methods. The EnKF method is used to estimate the state of a dynamic system, and it is used in various fields such as meteorology, hydrology, ecology, geophysics, and robotics. Whereas the Particle Filter one is a powerful tool to handle monitoring, estimation, and prediction problems in various contexts involving uncertainty and dynamic changes. And this paper performs the ROV diving and emerging motion estimation using the EnKF-SR and Particle Filter methods. Both methods are proven to be reliable on other platforms. The simulation results in this paper showed that the EnKF method has a higher accuracy than the Particle Filter one by showing an accuracy of 98% by the Particle Filter method and an accuracy of 99% by the EnKF-SR method.

Keywords: *ROV; estimation; ensemble Kalman filter, EnKF-SR, particle filter.*

Mathematics Subject Classification (2010): 93C10, 93D05.

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1 Introduction

Indonesia is an archipelago country, therefore it requires a vehicle for its underwater mapping and maintenance. One of such vehicles is a Remotely Operated Vehicle (ROV), a vehicle that operates under the sea and is controlled by an operator from land. In operation, the ROV requires a navigation system to know the position of the ROV. Its position is then used by the operator to control the ROV motion to the desired place [1].

The ROV can be used for the inspection of underwater structures such as oil and gas pipelines, subsea cables, renewable energy installations, and other underwater facilities [2]. ROVs allow repairs and maintenance to be done without the need for human diving which can be dangerous and costly, and they are frequently used for search and rescue operations, especially in the case of missing submarines or aircraft. The ROV can assist in searches, identify locations, and provide vital information for rescue teams [3, 4].

In addition to ROVs, there are AUVs (Autonomous Underwater Vehicles) having almost the same function. In the AUV, the control is automatic. So, to carry out a mission, the AUV is given a special program to perform certain tasks, then the robot does them according to the program without being controlled by a pilot/operator. The AUV itself requires batteries to do its job, so the time to do its mission is very limited. The advantage of the ROV is that since it is connected to a data cable and a power cable, its use can be very long and maximized [5]. The pilot/operator in moving the ROV usually uses the camera in the ROV and then displays it on the monitor screen to monitor whatever is seen in the water. The ROV various sensors are usually installed to monitor various indicators while its operation in the water, namely water density, temperature, current, etc [6].

This study started with the modeling of the 6-DOF motion equation by combining rotational and translational motions. Then the position estimation method was applied to determine the movements of the ROV when diving underwater. Some of the position estimation methods for ROV or AUV motion estimation ever applied in the previous studies were the Fuzzy Kalman Filter [7], Extended Kalman Filter [8, 9], EnKF-SR [10], Unscented Kalamn Filter [11], H-Infinity [12] and EnKF [13, 14]. Of the six position estimation methods above, the EnKF and Particle Filter methods were applicable to both linear and nonlinear models. The EnKF method is used to estimate the state of the dynamic system. It can be used in various fields such as meteorology, hydrology, ecology, geophysics, and robotics. The Particle Filter is a powerful tool to handle monitoring, estimation, and prediction problems in various contexts involving uncertainty and dynamic changes. And the study of this paper is to estimate the ROV diving motion by using the EnKF and Particle Filter methods.

2 Remotely Operated Vehicle (ROV) with 6-DOF

The profile of the ROV can be seen in Figure 1 and the specifications of the ROV can be seen in Table 1.

This study focused on the diving and emerging motion only so as to observe 6 Degrees of Freedom for which the motion equations are as follows [14]:

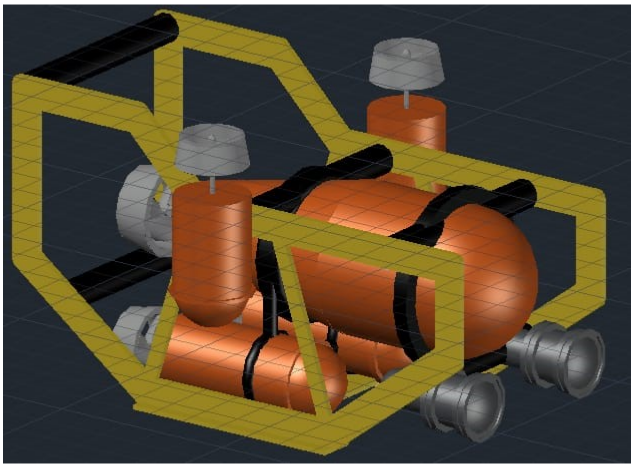

Figure 1: Profile of Rescue ROV [5,6].

Table 1: Specifications of Rescue ROV [5,6].

Weight	15 Kg
Length Length	900 mm
Beam	300 mm
Controller	Wired Control ArduSUB with Joystick
Sensors	Depth Sensor, Sonar
Camera	TTL Camera
Lighting	1500 LM, 145° Beam Dimmable
Battery	11.8 V Li Po 5200 mAh
Material	Carbon Fiber
Main Propulsion	T200 Motor Thruster Include Propeller
Maneuver Propulsion	T200 Motor Thruster Include Propeller
Service Speed	1,6 knots
Operation Depth	5 – 10 m

Surge:

$$\dot{u} + \frac{mz_G\dot{q}}{m - X_{\dot{u}}} - \frac{my_G\dot{r}}{m - X_{\dot{u}}} = \left(\frac{1}{m - X_{\dot{u}}}\right)\big(X_{res} + X_{|u|u}u|u| + X_{\dot{u}}\dot{u} + X_{wq}wq + X_{qq}qq + X_{vr}vr + X_{rr}rr + X_{prop} - m\left[-vr + wq - x_G(q^2 + r^2) + pqy_G + prz_G\right]\big)\,; \quad (1)$$

Sway:

$$\dot{v} + \frac{mz_G\dot{p}}{m - Y_{\dot{v}}} + \frac{(mx_G - Y_{\dot{r}})\dot{r}}{m - Y_{\dot{v}}} = \left(\frac{1}{m - Y_{\dot{v}}}\right)\big(Y_{res} + Y_{|v|v}v|v| + Y_{r|r|}r|r| + Y_{\dot{v}}\dot{v} + Y_{\dot{r}}\dot{r} + Y_{ur}ur + Y_{wp}wp + Y_{pq}pq + Y_{uv}uv + Y_{uu\delta_r}u^2\delta_r - m[-wp + ur - y_G(r^2 + p^2) + qrz_G + pqx_G]\big)\,; \quad (2)$$

Heave:

$$\dot{w} + \frac{(mx_G + Z_{\dot{q}})\dot{q}}{m - Z_{\dot{w}}} + \frac{my_G\dot{p}}{m - Z_{\dot{w}}} = \left(\frac{1}{m - Z_{\dot{w}}}\right)\big(Z_{res} + Z_{|w|w}w|w| + Z_{q|q|}q|q| + Z_{\dot{w}}\dot{w} + Z_{\dot{q}}\dot{q} + Z_{uq}uq + Z_{vp}vp + Z_{rp}rp + Z_{uw}uw + Z_{uu\delta_s}u^2\delta_s - m[-uq + vp - z_G(p^2 + q^2) + rpx_G + rqy_G]\big)\,; \quad (3)$$

Roll:

$$\dot{p} + \frac{my_G\dot{w}}{I_x - K_{\dot{p}}} - \frac{mz_G\dot{v}}{I_x - K_{\dot{p}}} = \left(\frac{1}{I_x - K_{\dot{p}}}\right)\big(K_{res} + K_{p|p|}p|p| + K_{prop} -$$
$$((I_z - I_y)qr + m[y_G(-uq + vp) - z_G(-wp + ur)])\big)\,; \tag{4}$$

Pitch:

$$\dot{q} + \frac{mz_G\dot{u}}{I_y - M_{\dot{q}}} - \frac{(mx_G + M_{\dot{w}})\dot{w}}{I_y - M_{\dot{q}}} = \left(\frac{1}{I_y - M_{\dot{q}}}\right)\big(M_{res} + M_{w|w|}w|w| + M_{q|q|}q|q| +$$
$$M_{\dot{w}}\dot{w} + M_{\dot{q}}\dot{q} + M_{uq}uq + M_{vp}vp + M_{rp}rp + M_{uw}uw + M_{uu\delta_s}u^2\delta_s -$$
$$(I_x - I_z)rp + m[z_G(-vr + wq) - x_G(-uq + vp)]\big)\,; \tag{5}$$

Yaw:

$$\dot{r} + \frac{(mx_G - N_{\dot{v}})\dot{v}}{I_z - N_{\dot{r}}} - \frac{my_G\dot{u}}{I_z - N_{\dot{r}}} = \left(\frac{1}{I_z - N_{\dot{r}}}\right)\big(N_{res} + N_{|v|v}v|v| + N_{r|r|}r|r| + N_{ur}ur +$$
$$N_{wp}wp + N_{pq}pq + N_{uv}uv + N_{uu\delta_r}u^2\delta_r - ((I_y - I_z)pq +$$
$$m[x_G(-wp + ur) - y_G(-vr + wq)])\big)\,. \tag{6}$$

3 Algorithm of Square Root Ensemble Kalman Filter and Particle Filter

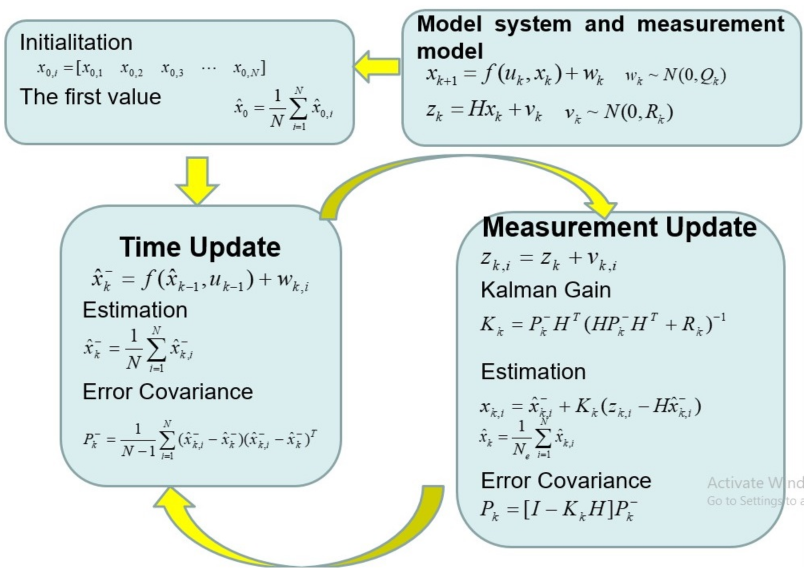

Figure 2: Flowchart of the application of the EnkF-SR algorithm to the dynamic system model [15, 16].

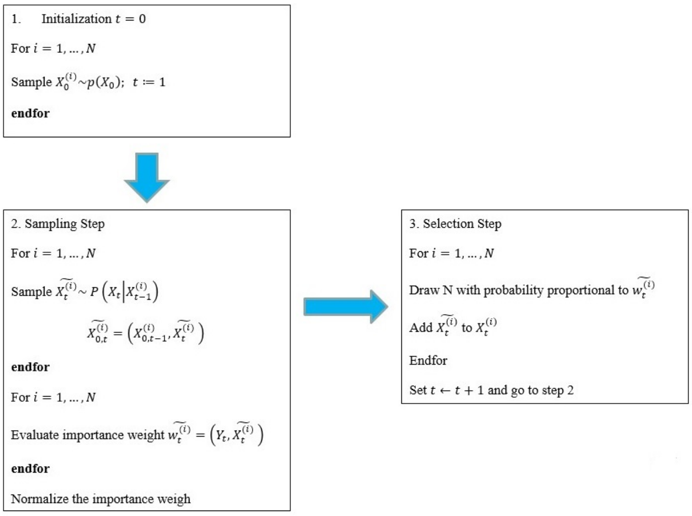

Figure 3: Flowchart of the application of the Particle Filter algorithm to the dynamic system model [17].

4 Simulation Results

The simulation is used and generated 250 and 500 ensembles. The starting point given in each trajectory is x(0)=0, y(0)=0, and z(0)=0. With the trajectories in diving and emerging motions, the position estimation results in the XY, XZ, and XYZ planes by using the EnKF-SR and Particle Filter methods with generation of 250 ensembles were obtained as in Figures 4 and 5. In addition, a table of RMSE values for the EnKF-SR and Particle Filter methods is displayed, in which the EnKF-SR method uses 300 ensembles as shown in Table 2.

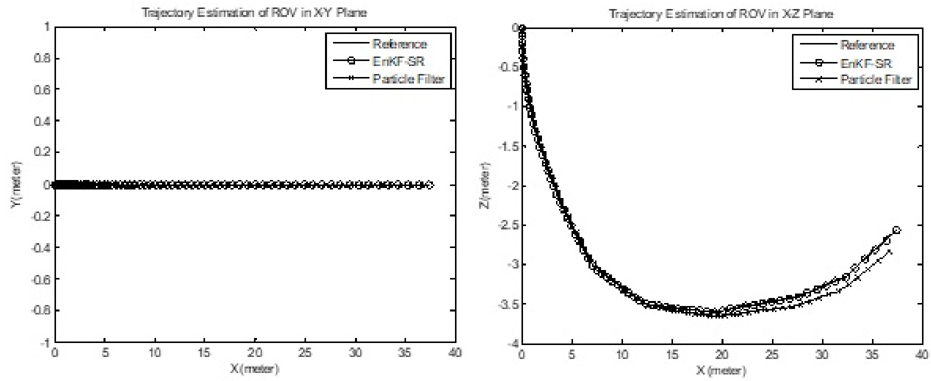

Figure 4: Position Estimation of ROV Diving and Emerging Motions in XY and XZ Planes with 250 ensembles.

Figure 4 shows that the ROV follows the trajectory predetermined in the XY plane and in diving and emerging motions. The error obtained in the simulation by generating 250 ensembles is the X position with the smallest error of 0.08235 m or a position error of 8.2 cm from the target passed, which is 35 m or an error of 0.0023% by the EnKF-SR method, for the Z position, 0.00467 m, there is a position error of 0.47 cm from the target passed, which is 4 m or an error of 0.0011% by the EnKF-SR method. The small position error by the three methods is due to the small RMSE of each DOF. Meanwhile the results by the Particle Filter method produced a position error of 0.1235 m for the X position and 0.00895 m for the Z position.

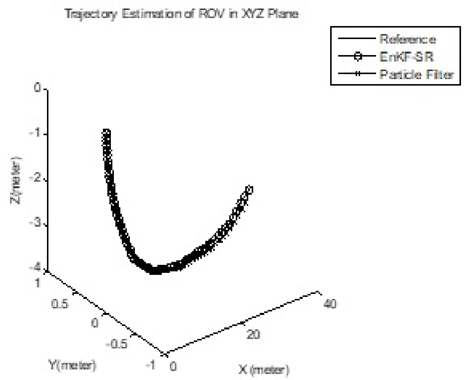

Figure 5: Position Estimation of ROV Diving and Emerging Motions in the XYZ Plane with 250 ensembles.

The trajectory combinations predetermined in the XY and XZ planes are then displayed in the three-dimensional plane as shown in Figure 5. In the XYZ plane, the ROV follows the trajectory where the ROV moves forward and dives then rises upward (emerging) without turning motion. The three methods are very accurate as shown in Table 2 indicating that by the EnKF-SR method, in the X position, the position error is 8.2 cm from the passed target of 35 m or an error of 0.0023% and in the Z position, the position error is 0.47 cm from the passed target of 4 m or an error of 0.0011%, while the Paticle Filter method has an error in the X position of about 0.0035% and that in the Z position of about 0.0022%.

Figure 6 shows that the ROV follows the trajectory predetermined in the XY plane with diving and emerging motions. The error obtained in the simulation by generating 250 ensembles is the X position, the smallest error is 0.06978 m or it has a position error of 6.97 cm from the traversed target of 35 m or an error of 0.0019% by the EnKF-SR method, for the Z position, 0.00359 m, it has a position error of 0.359 cm from the traversed target of 4 m or an error of 0.00089% by the EnKF-SR method. The small position error in the three methods is due to the small RMSE of each DOF. Meanwhile the results by the Particle Filter method produced a position error of 0.1051 m for the X position and 0.00722 m for the Z position.

The XYZ plane is a combination of the trajectories made in the XY and XZ planes and then displayed in the three-dimensional plane as shown in Figure 7. In the XYZ plane, the ROV follows the trajectory where the AUV moves forward, dives, then rises upward (emerging) without turning movement. The three methods are very accurate, shown in Table 2. By the EnKF-SR method, in the X position, the position error is 6.97

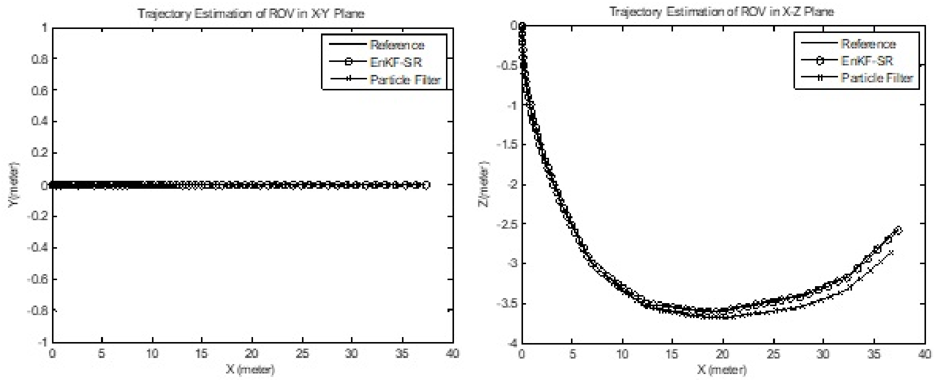

Figure 6: Position Estimation of ROV Diving and Emerging Motion in XY and XZ Planes with 500 ensembles.

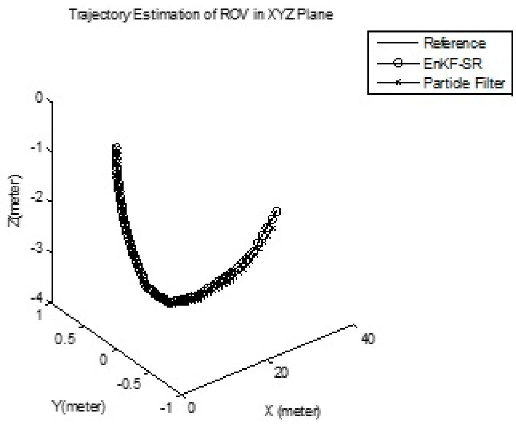

Figure 7: Position Estimation of ROV Diving and Emerging Motions in the XYZ Plane with 500 ensembles.

cm from the target passed, which is 35 m or an error of 0.0019%, and in the Z position, the error is 0.359 cm from the target passed, which is 4 m or it has an error of 0.00089%, while by the Paticle Filter method, in the X position, it has an error of about 0.003% and in the Z position, it has an error of about 0.0018%.

Table 2 shows that by generating 250 and 500 ensembles, the EnKF-SR method has a higher accuracy than the Particle Fililter method for the position in the translational and rotational motion. The EnKF SR method has a more accurate position estimation in the translational and rotational motions than the Particle Filter. This shows that the X position error is affected by the translational and rotational motions in the X axis, that is, surge and roll, while the Z position error is affected by the translational and rotational motions in the Z axis, that is, heave and pitch. Thus, overall the EnKF-SR method has a higher accuracy than the Particle Filter with either 250 ensembles or 500 ensembles. However, both estimation methods can be implemented for the position estimation of the ROV and other autonomous vehicle systems.

	300 ensembles		500 ensembles	
	Particle Filter	EnKF-SR	Particle Filter	EnKF-SR
Position X	0.0035%	0.0023%	0.003%	0.0019%
Position Y	0	0	0	0
Position Z	0.0022%	0.0011%	0.0018%	0.00089%
Simulation Time	9.5114 s	9.8378 s	12.5114 s	12.7624 s

Table 2: Comparison of Errors generated by EnKF-SR and Particle Filter in Diving and Emerging Motions.

5 Conclusion

Based on the results of the discussion and analysis above, by generating 250 and 500 ensembles, the EnKF-SR method is more accurate than the Particle Filter method for the position in translational and rotational motions. The EnKF-SR method has a position error in translational and rotational motions and is more accurate than the Particle Filter. This shows that the X position error is influenced by the translational and rotational motions in the X axis, that is, surge and roll, while the Z position error is influenced by the translational and rotational motions in the Z axis, that is, heave and pitch. In conclusion, overall the EnKF-SR method is more accurate than the Particle Filter one with either 250 ensembles or 350 ensembles. However, both motion estimation methods can be implemented for the position estimation of the ROV and other autonomous vehicle systems.

Acknowledgment

High appreciation to Kemdikbudristek for the very fund support for the completion of this research conducted in the year of 2024 with contract numbers 109/E5/PG.02.00.PL/2024, 083/SP2H/PT/LL7/2024, and 01/KP/LPPM/ITATS/2024.

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Diagnosis of Diabetes Mellitus Symptoms Using Simple Additive Weighting and Weighted Product Methods

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Received: January 31, 2024; Revised: October 10, 2024

Abstract: Diabetes Mellitus is a disease when the body has abnormalities in insulin secretion, insulin performance or both, maintaining excess sugar in the blood. Diabetes Mellitus is caused by an imbalance between the supply and demand of insulin facilitating the entry of glucose into cells. Reduced or absent insulin makes glucose retained in the blood and leads to an increase in blood sugar, while cells become deficient in glucose badly needed for cell survival and function [8]. The frightening consequence of diabetes mellitus is that patients are at a high risk of cardiovascular disease, kidney disease, rupture of blood vessels, heart attack, stroke, leg ulcers, infection, amputation and all risks. Diabetes Mellitus is also a disease that shows an increase in glucose due to insulin deficiency which can cause macrovascular, microvascular and neurological complications. Considering those as described above, this study is intended to provide a decision support system for public to get informed of the risk of diabetes militus so as to take an immediate action. The methods used in this research are the SAW(Simple Additive Weighting) and WP (Weighted Product) methods to diagnose the diabetes militus symptoms.

Keywords: *Diabetes Melitus; Decision Making Support System; Simple Additive Weighting (SAW), Weighted Product (WP).*

Mathematics Subject Classification (2010): 90B50, 68U35.

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1 Introduction

Diabetes is a disease familiar in the world of medicine and society. Diabetes mellitus usually affects various social classes and public circles. Diabetes mellitus is a chronic metabolic disease characterized by increased blood glucose (hyperglycemia) due to an imbalance between the supply and demand of insulin facilitating the entry of glucose into cells so that it can be used for cell metabolism and growth. Reduced or absent insulin makes glucose retained in the blood and causes an increase in blood sugar, while cells become deficient in glucose very much needed for cell survival and function [8].

Diabetes or commonly called 'kencing manis' (in Bahasa Indonesia) is a dangerous disease that can lead to the death of patients. Diabetes mellitus is a chronic disease that may last a lifetime. The frightening consequence of diabetes mellitus is that patients are at a high risk of cardiovascular disease, kidney disease, rupture of blood vessels, heart attack, stroke, leg ulcers, infection, amputation and all risks. Diabetes Mellitus is also a disease that shows an increase in glucose due to insulin deficiency which can cause macrovascular, microvascular and neurological complications.

Most people are often unaware of the bad effects caused by diabetes and do not know that they may be at the risk of suffering from this disease [5]. The reason for this is the lack of information for people regarding diabetes mellitus, their limited funds and time to consult a doctor [4]. Considering the several problems above, enough information is needed to help solve those problems. For this, an effective analysis tool is required. That is a decision making support system, information system used to make decisions effectively and efficiently on structured and unstructured problems.

The decision support system is intended to overcome the problems and to assist people in diagnosing diabetes symptoms. The benefits of the decision support systems include providing solutions that deliver faster and more reliable results, increasing decision makers' confidence in their decisions, and saving time, effort, and money with on-demand decision support system. It is very much needed to solve problems, especially the problems that are very complex and unstructured [12]. In this research, we need a method reliable and effective to solve the existing problems. The methods used are the Simple Additive Weighting (SAW) Method and the Weighted Product (WP) Method to be applied with the Matlab application. The basic concept of the SAW method is to find a weighted sum of performance ratings for each alternative on all attributes. Meanwhile, the WP method uses multiplication to connect attribute ratings, of which the rating of each attribute must be raised to the power of the weight of the attribute in question.

Both of these methods are simple methods to provide a more accurate assessment because they are based on predetermined criteria values and preference weights used to complete the decision-making process and choose the best alternative. Therefore, to assist the process of determining the results of the diagnosis of the diabetes mellitus symptoms in this study, the SAW method and the WP method were used. The use of these two methods is expected to help people find out whether they have diabetes mellitus or not with a fast and precise process. As a result, the community immediately knows and it is not too late to handle it.

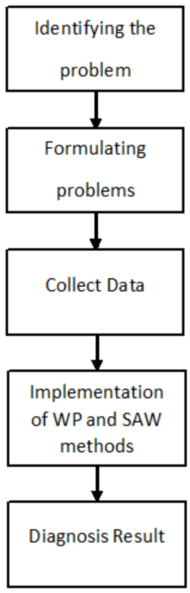

Figure 1: Research Flowchart.

2 Method

2.1 Research flow chart

The flow of this research began with identifying the problem of diabetes mellitus with many criteria required, followed by the formulation of the problem. Through existing problems such as determining the criteria for diagnosing diabetes mellitus symptoms, solutions can then be determined. Further, to assist in finding out solutions, this can be resolved through observation and interviews in the data collection process, then from the data obtained, analyzed and implemented into the SAW method and the WP method. After all these processes have been conducted, conclusions can be drawn in the form of the results of the diagnosis of diabetes mellitus symptoms.

2.2 Data collection method

This research was conducted by applying data collection techniques of questionnaires made with the Google form. If you want to get the data being convincing and real, the authors review it for direct interviews with people with diabetes mellitus. For the literature study at this stage, the researcher collects information and data from several different sources such as journals, e-journals, proceedings, books, e-books and the internet, after which the researcher studies them in order to get valid results. From this data collection technique, the researcher needs several research objects because if only one sample is taken, then it cannot be used as a conclusion in this study, therefore severalty of research objects are needed and later used as comparisons when carrying out later calculations.

3 Theoretical Framework

3.1 Diabetes Mellitus

Diabetes mellitus is a condition in which the body cannot produce insulin effectively, resulting in excessive sugar in the blood. Based on the results of collecting information from several sources and those of interviews with experts in the health field, it can be concluded that diabetes mellitus can be risky due to several factors, including heredity, overweight, unhealthy lifestyles, and age. This disease can also be triggered by the presence of other diseases such as hypertension and cholesterol due to high blood pressure which can make the sugar distribution to cells not run optimally so that it leads to accumulation of sugar and cholesterol in the blood. On the contrary, if the condition of blood pressure is within the normal range, then blood sugar is maintained within the normal range since the insulin is working properly. Considering the factors causing diabetes mellitus, they are used as a criterion determinant in this study, which is essentially expected to assist in the decision-making process.

3.2 Decision Making Support System (SPK)

Decision Support System is an information system that is used to assist in decision making by using data and several decision models effectively and efficiently to solve semi-structured and unstructured problems. The Decision Support System usually does not change the function of decision makers but only provides support or strengthens the results in making decisions. The purpose of the Decision Support System is to provide information, forecasts, and guidance for information users so that they can make decisions by doing the calculations using predetermined methods so that the results obtained are more accurate.

3.3 Simple Additive Weighting (SAW) method

The SAW (Simple Additive Weighting) method is often called the weighted sum method. The basic concept of the SAW method is to find a weighted sum of performance ratings for each alternative of all attributes. The SAW method requires the process of normalizing the decision matrix (x) to a scale that can be compared to all existing alternative ratings. This method requires the decision maker to determine the weight for each attribute. The rating of each attribute must be dimension-free in the sense that it has gone through the previous normalization process. The SAW method recognizes the existence of two attributes, that is, the profit criterion and cost criterion.

The formula for doing the normalization is as follows :

$$r_{ij} \begin{cases} \frac{x_{ij}}{Max\ x_{ij}}\ if\ j : atribute\ of\ benefit, \\ \frac{Min\ x_{ij}}{x_{ij}}\ if\ j : atribute\ of\ cost, \end{cases} \tag{1}$$

where r_{ij} is the normalized performance rating, x_{ij} is the attribute of each criterion, $Max\ x_{ij}$ is the highest value of each criterion and $Min\ x_{ij}$ is the lowest value of each criterion.

Preference value for each alternative (V_i) is given as

$$V_i = \sum_{j=1}^{n} w_j r_{ij}, \tag{2}$$

where V_i is the final value of the alternative, w_j is the predetermined weight, r_{ij} is the normalised matrix. The higher value of V_i indicates that the alternative A_i is preferred.

3.4 Weighted Product (WP) method

The Weighted Product (WP) method is a multi-criteria decision analysis, and it is a multi-criteria decision-making method. The WP method is a set of decision alternatives described in terms of several criteria. The weighted product method uses multiplication to link attribute ratings, of which the rating of each attribute must be raised first to the power of the attribute weight in question. This process is the same as the normalization process. In the WP method, matrix manipulation is not required because this method multiplies the results of the assessment of each attribute. The multiplication results have not been compared to (divided by) the standard value, in this case, the ideal alternative is often used as the standard weight value. The weight for the benefit attribute functions as a positive rank in the multiplication process between attributes, while the cost weight functions as a negative rank. This process is the same as the normalization process. The preference for the Ai alternative is given as follows:

$$S_i = \prod_{j=1}^{n} X_{ij}^{w_i}; i = 1, 2, \ldots, m, \tag{3}$$

where $\Sigma w_j = 1$, w_j is the power of positive value for the benefit attribute and of negative value for the cost attribute.

Then the ranking process uses the vector v, and the vector v can be obtained by applying the following formula:

$$V_i = \frac{\prod_{j=1}^{n} X_{ij}^{w_i}}{\prod_{j=1}^{n} \left(X_j^*\right)^{w_i}}; i = 1, 2, \ldots, m. \tag{4}$$

4 Discussion

4.1 Determining criteria (C_i)

The data obtained from the questionnaire of the Google form are evaluated then used as a reference for the decision making process criteria for the cases of diabetes mellitus symptoms, see Table 1.

No.	Criteria	Description
1	C_1	Hereditary
2	C_2	Age
3	C_3	BMI
4	C_4	Diet
5	C_5	History of other diseases

Table 1: Criteria.

4.2 Determining the criterion value based on the weight value

For each criterion of heredity, age, difference of ideal body weight, diet, history of other diseases, the value of the criterion is determined with the reference to the value of the variable given to each criterion. Then this value is considered as an indicator criterion which later becomes the value determining factor. For reference, see Table 2.

No.	Value	Description
1	1	Low risk
2	2	Average risk
3	3	High risk

Table 2: Weight value reference.

1. Hereditary Criterion
   The criteria for heredity (offspring of diabetics) are categorized into three types: first, neither father nor mother have diabete, second, either father or mother has diabete, and third, both father and mother have diabetes. When converted with reference to the weight value determination, the values are as follows:

No.	Heredity(C_1)	Value
1	Neither	1
2	Either father or mother	2
3	Both father and mother	3

Table 3: Heriditary Criterion.

2. Age Criterion
   The age category is converted into the weight value, and the weight value determining reference is shown in Table 4.

No.	Age(C_2)	Value
1	0-30	1
2	31-45	2
3	> 45	3

Table 4: Age Criterion.

3. BMI Criterion
   The BMI is obtained by using the BMI formula, that is, $BMI = \frac{berat\ badan\ (kg)}{tinggi\ badan\ (m^2)}$. The obtained result is converted into the weight value, and the weight value determining reference is shown in Table 5.
4. Diet Criteria
   The diet criterion is converted into the weight value, and the weight value determining reference is shown in Table 6.

No.	BMI (C_3)	Value
1	18-25	1
2	< 18	2
3	> 25	3

Table 5: BMI Criterion.

No.	Diet (C_4)	Value
1	1-2 times a day	1
2	3 times a day	2
3	> 3 times a day	3

Table 6: Diet Criterion.

5. Other disease history Criterion
   The criterion of the other disease history is converted into the weight value, and the weight value determining reference can be seen in Table 7.

No.	Other Disease History (C_5)	Value
1	Not suffering any other disease	1
2	Suffering an internal disease	2
3	High blood tension or high cholesterol	3

Table 7: Criterion of other disease history.

4.3 Determining the weight of each criterion applied

The next step is to determine the weight for each criterion as shown in Table 8.

No.	Criteria (C_i)	Attribute	Value
1	Heredity (C_1)	Benefit	$45\% = \frac{45}{100} = 0,45$
2	Age (C_2)	Benefit	$25\% = \frac{25}{100} = 0,25$
3	BMI (C_3)	Benefit	$15\% = \frac{15}{100} = 0,15$
4	Diet (C_4)	Benefit	$10\% = \frac{10}{100} = 0,10$
5	History of other disease (C_5)	Benefit	$5\% = \frac{5}{100} = 0,05$

Table 8: Determining the weight of each criterion.

Each criterion in this study has a benefit attribute because all types of criteria prioritize the highest value as a reference for selection.

4.4 Determining the alternatives

Determining the alternatives is done by taking the data based on the criteria predetermined by the researcher. The data were obtained from 15 respondents who filled out the Google form. The following is a table for each alternative determined based on the determined criteria, that is, hereditary, age, BMI, diet, and history of other diseases as shown in Table 9.

No.	Name	Parents With Diabete	Age	BMI	Diet	Disease History
1	SI	1	2	1	2	1
2	AN	2	3	3	2	2
3	SK	1	3	1	2	1
4	ST	1	3	1	2	3
5	PL	2	3	1	2	1
6	AZ	1	3	3	1	3
7	AS	1	3	1	2	1
8	E	2	3	1	2	1
9	MY	2	3	3	1	2
10	MT	1	2	3	2	1
11	KT	2	3	3	1	2
12	SA	1	2	1	2	1
13	KS	1	3	3	3	3
14	S	2	2	3	2	1
15	SD	2	2	3	2	1

Table 9: Determining alternatives.

4.5 Normalizing the matrix by the SAW method

The following is a decision matrix formed in accordance with the value of each alternative obtained by the researchers by calculation as follows:

$$X = \begin{pmatrix} 1 & 2 & 1 & 2 & 1 \\ 2 & 3 & 3 & 2 & 2 \\ 1 & 3 & 1 & 2 & 1 \\ 1 & 3 & 1 & 2 & 3 \\ 2 & 3 & 1 & 2 & 1 \\ 1 & 3 & 3 & 1 & 3 \\ 1 & 3 & 1 & 2 & 1 \\ 2 & 3 & 1 & 2 & 1 \\ 2 & 3 & 3 & 1 & 2 \\ 1 & 2 & 3 & 2 & 1 \\ 2 & 3 & 3 & 1 & 2 \\ 1 & 2 & 1 & 2 & 1 \\ 1 & 3 & 3 & 3 & 3 \\ 2 & 2 & 3 & 2 & 1 \\ 2 & 2 & 3 & 2 & 1 \end{pmatrix}.$$

The next step is to normalize the matrix based on the types of attributes predefined so as to get the normalized matrix results by calculation as follows:

$$r_{11} = \frac{1}{2} = 0,5; r_{21} = \frac{2}{2} = 1; r_{31} = \frac{1}{2} = 0,5.$$

Proceed up to r_{155} to get the results of the normalized matrix as follows:

$$X = \begin{pmatrix} 0.500 & 0.667 & 0.333 & 0.667 & 0.333 \\ 1.000 & 1.000 & 1.000 & 0.667 & 0.667 \\ 0.500 & 1.000 & 0.333 & 0.667 & 0.333 \\ 0.500 & 1.000 & 0.333 & 0.667 & 1.000 \\ 1.000 & 1.000 & 0.333 & 0.667 & 0.333 \\ 0.500 & 1.000 & 1.000 & 0.333 & 1.000 \\ 0.500 & 1.000 & 0.333 & 0.667 & 0.333 \\ 1.000 & 1.000 & 0.333 & 0.667 & 0.333 \\ 1.000 & 1.000 & 1.000 & 0.333 & 0.667 \\ 0.500 & 0.667 & 1.000 & 0.667 & 0.333 \\ 1.000 & 1.000 & 1.000 & 0.333 & 0.667 \\ 0.500 & 0.667 & 0.333 & 0.667 & 0.333 \\ 0.500 & 1.000 & 1.000 & 1.000 & 1.000 \\ 1.000 & 0.667 & 1.000 & 0.667 & 0.333 \\ 1.000 & 0.667 & 1.000 & 0.667 & 0.333 \end{pmatrix}.$$

4.6 Ranking process by SAW method

The next process is to have the sum of the matrix R, and later it is multiplied by the weight of each criterion, then the obtained value is used as a benchmark in determining the diagnosis of diabetes mellitus symptoms. The following is the alternative ranking based on the calculation results as shown in Table 10.

No.	Alternative	Reference	Diagnosis Results
1	SI	0.52	Low risk
2	SA	0.52	Low risk
3	SK	0.60	Fair risk
4	AS	0.60	Fair risk
5	MT	0.62	Fair risk
6	ST	0.64	Fair risk
7	AZ	0.70	Fair risk
8	KS	0.77	High risk
9	E	0.83	High risk
10	PL	0.83	High risk
11	S	0.85	High risk
12	SD	0.85	High risk
13	MY	0.91	High risk
14	KT	0.91	High risk
15	AN	0.95	High risk

Table 10: The Results of the Ranking by the SAW method.

4.7 Determining the preference and alternative ranking by WP method

Because the total weight is equal to 1, the next step is to determine the preference for each alternative.

$$S_1 = (1^{0,45})(2^{0,25})(1^{0,15})(2^{0,1})(1^{0,05}) = 1,274561,$$
$$S_2 = (2^{0,45})(3^{0,25})(3^{0,15})(2^{0,1})(2^{0,05}) = 2,352158,$$
$$S_3 = (1^{0,45})(3^{0,25})(1^{0,15})(2^{0,1})(1^{0,05}) = 1,410533.$$

Proceed up to S_{15}. Then, it is followed by the calculation of the relative preference.

$$V_1 = \frac{1,274561}{26,533754} = 0,048035,$$
$$V_2 = \frac{2,352158}{26,533754} = 0.088648,$$
$$V_3 = \frac{1,410533}{26,533754} = 0.053160.$$

And it is continued up to V_{15}. The following is the alternative ranking determined based on the calculation results as shown in Table 11.

No.	Alternative	Reference	Alternative Preference	Diagnosis Results
1	SI	1.274561	0.048035	Low Risk
2	SA	1.274561	0.048035	Low risk
3	SK	1.410533	0.053160	Low risk
4	AS	1.410533	0.053160	Fair risk
5	MT	1.502895	0.056641	Fair risk
6	ST	1.490182	0.056162	Fair risk
7	AZ	1.639474	0.061788	Fair risk
8	KS	1.829855	0.068963	High risk
9	E	1.926845	0.072619	High risk
10	PL	1.926845	0.072619	High risk
11	S	2.053015	0.077374	High risk
12	SD	2.053015	0.077374	High risk
13	MY	2.194641	0.082711	High risk
14	KT	2.194641	0.082711	High risk
15	AN	2.352158	0.088648	High risk

Table 11: Results of Preference and Ranking by the WP method.

5 Conclusion

Based on the calculation results as seen above, there were 8 people indicated to have high risk of suffering from diabete mellitus, that is, KS, E, PL, S, SD, MY, KT, and AN due to the high values of heredity, body weight, and diet criteria. And, there were 5 people indicated to be at medium risk of suffering from diabetes mellitus, they are SK, AS, MT, ST, AZ. Meanwhile, there were 2 people indicated to be at low risk of suffering from diabetes mellitus, they are SI and SA. Thus, the decision support system for diabetes mellitus symptoms was effectively done by using the Simple Additive Weighting (SAW) method, or by the Weighted Product (WP) method, with the aim of providing information for the public regarding the risk of diabetes mellitus for their immediate action to take.

Acknowledgment

We gratefully acknowledge the support from LPPM - Universitas Nahdlatul Ulama Surabaya (UNUSA).

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