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Well-Posedness of Boundary Control System of Nonlinear Chemical Reaction

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Abstract: We consider mixed boundary control systems inducted by non-isothermal axial dispersion chemical tubular reactors. We characterize the well-posedness, approximate controllability, and transfer function of the mixed boundary control system. There exists an admissible operator control such that the mixed boundary control system is well-posed. By constructing an extended space, the classical solution can be obtained explicitly. Sufficient conditions for approximate controllability of the mixed boundary control system are identified by the eigenvalues and eigenvectors of the Sturm-Loiuville operator using an equivalence in the extended space. A proper transfer function of the associated boundary control system equipped with an output can be constructed. The proper transfer function shows that the associated boundary control system is well-posed.

Keywords: *chemical tubular reactor; boundary control system; well-posed; approximately controllable; Sturm-Liouville operator; transfer function.*

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1 Introduction

A non-isothermal reaction is a reaction in the process taking place at a temperature varying from one point to another. Dynamical analysis of non-isothermal tubular reactors has been studied massively recently, see [1–6]. The dynamics of non-isothermal axial dispersion chemical tubular reactors are described by nonlinear partial differential equations (PDEs) derived from mass and energy balance equations. The nonlinearities are usually located in the kinetic terms due to the Arrhenius law for non-isothermal reactors. In particular, let L be the length of the tubular reactor and if the reaction is

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characterized by first-order kinetics with respect to the reactant concentration $C = 1$, then the reactor temperature $T(\xi, \tau)$ at the position ξ, $0 \leq \xi \leq L$, and the time τ govern the boundary problem of the nonlinear PDEs:

$$\begin{aligned} \frac{\partial T}{\partial \tau} &= D\frac{\partial T}{\partial \xi^2} - \upsilon\frac{\partial T}{\partial \xi} - \frac{\Delta H}{\rho C_p}k_0 e^{-E/RT} - \frac{4h}{\rho C_p d}(T - T_c(\tau)), \\ D\frac{\partial T}{\partial \xi}(0,\tau) &= \upsilon[T(0,\tau) - T_{\text{in}}(\tau)], \\ \frac{\partial T}{\partial \xi}(L,\tau) &= 0, \end{aligned} \tag{1}$$

where D, υ, ΔH, ρ, C_p, k_0, E, R, h, d, T_c, and T_{in} denote the energy dispersion coefficients, the superficial fluid velocity, the heat of reaction, the density, the specific heat, the kinetic constant, the activation energy, the ideal gas constant, the wall heat transfer coefficient, the reactor diameter, the coolant temperature, and the inlet temperature, respectively.

Well-posedness for the nonlinear problem (1) is crucial. There is a complex problem when dealing with the nonlinearity of infinite dimensional system. Linearization about the steady state is an approximate solution to the problem. The linearization transforms the system into a boundary control system, specifically, a mixed boundary control system with inner and boundary controls u and v, respectively, see (5). Therefore, we will focus on the linearized system, addressing its well-posedness, approximate controllability, and the well-posedness of the related input-output system.

In general, the well-posedness for the control systems is determined by the well definability and boundedness of the mappings of input to state, input to output, initial state to input, and initial state to final state [7]. For the mixed boundary control system of the linearized system of system (1), the well-posedness requires the boundedness of the existence of the admissible control operator (input-state map) B, see Definition 2.1 below. Thus, the well-posedness of the linearized system depends on the well-posedness of the state-space formulation (A, B). In the state-space, sufficient and necessary conditions for (approximate) controllability have been investigated, see [8–11]. However, for sufficiently smooth inputs, we can redefine the state space to be an extended state space, for illustration, see (15). By this construction, the sufficiency for the well-posedness and controllability of the associated problems (systems) with respect to some state space have been investigated, see [8, 12–14]. These facts guide investigations to the well-posedness and controllability for the linearized system of system (1).

Henceforth, we consider the associated state-space (A, B, C) of the linearized system of system (1), where C is the input-output map. In this space, the boundedness of C implies the well-posedness for the control system [15]. On the other hand, the boundedness of C can be identified by a system transfer function. Curtain and Weiss [16] proved that C is bounded if and only if the transfer function is uniformly bounded in a right half-plane. Therefore, to prove the well-posedness, it is enough to show that the transfer function is bounded in some right half-plane. Unfortunately, this approach is for only a few systems. In a class of structural control systems that measure acceleration at a point, the boundedness of C is proved by showing that the transfer function is proper [17]. In the paper, the justification of the transfer function was not computed directly but the properness of the transfer function is shown due to the fact that the infinitesimal generator generates an analytic semigroup. Now, one should justify the transfer function of the state-space (A, B, C) for the linearized system of system (1).

2 A Mixed Boundary Control Problem of Chemical Reactor

To facilitate analysis, the dynamic model (1) will be converted into an equivalent dimensionless distributed parameter system. By putting the new state variables

$$
\begin{aligned}
t &= \frac{\tau\upsilon}{L}, & x &= \frac{\xi}{L}, & P_e &= \frac{\upsilon}{D},\\
B &= -\frac{\Delta H k_0 L D e^{-E/RT_0}}{\upsilon T_0}, & \gamma &= \frac{E}{RT_0}, & \beta &= \frac{4hD^2L}{d\upsilon},\\
z &= \frac{T-T_0}{T_0}, & u &= \frac{T_c-T_0}{T_0}, & v &= \frac{T_{\text{in}}-T_0}{T_0},
\end{aligned}
$$

where T_0 is a reference temperature and P_e is a Peclet number, we get the nonlinear dimensionless model

$$
\begin{aligned}
z_t(x,t) &= \tfrac{1}{P_e} z_{xx}(x,t) - z_x(x,t) + Be^{\gamma z(x,t)/(1+z(x,t))} + \beta[u(t) - z(x,t)],\\
z(0,t) &- \tfrac{1}{P_e} z_x(0,t) = v(t), \\
z_x(1,t) &= 0.
\end{aligned}
\tag{2}
$$

Let $z_s(x), u_s, v_s$ be the steady states of the system (2), so these functions satisfy

$$
\begin{aligned}
0 &= \frac{1}{P_e}\frac{d^2 z_s}{dx^2}(x) - \frac{dz_s}{dx}(x) + Be^{\gamma z_s(x)/(1+z_s(x))} + \beta[u_s - z_s(x)],\\
v_s &= z_s(0) - \frac{1}{P_e}\frac{dz_s}{dx}(0),\\
0 &= \frac{dz_s}{dx}(1).
\end{aligned}
\tag{3}
$$

By linearizing the nonlinear system (2) about the steady states and using the same symbols again, we have the linearized system

$$
\begin{aligned}
z_t(x,t) &= \tfrac{1}{P_e} z_{xx}(x,t) - z_x(x,t) + J(x)z(x,t) + \beta u(t),\\
z(0,t) &- \tfrac{1}{P_e} z_x(0,t) = v(t),\\
z_x(1,t) &= 0,
\end{aligned}
\tag{4}
$$

where $J(x) = e^{\gamma z_s(x)/(1+z_s(x))}/(1+z_s(x))^2$. We note that this problem is a distributed parameter system controlled both internally and at the boundary, and referred to as the mixed boundary control problem. We will focus on analyzing the mixed boundary control problem (4) with the interior control $u = u(t)$ and the boundary control $v = v(t)$, $t \geq 0$.

We recall the abstract mixed boundary control problem [8, 18]

$$
\begin{aligned}
\dot{z}(t) &= \mathcal{A}z(t) + B_d u(t), \quad z(0) = z_0,\\
\mathcal{B}z(t) &= v(t),
\end{aligned}
\tag{5}
$$

where $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset Z \to Z$, $B_d \in \mathcal{L}(U, Z)$, $\mathcal{B} : \mathcal{D}(\mathcal{B}) \subset Z \to V$ such that $\mathcal{D}(\mathcal{A}) \subset \mathcal{D}(\mathcal{B})$, and Z, U, V are separable Hilbert spaces. We simplify the mixed boundary control system (5) by $(\mathcal{A}, \mathcal{B})$.

Definition 2.1 The mixed boundary control system $(\mathcal{A}, \mathcal{B})$ (5) is said to be well-posed if:

(a) The operator $A : \mathcal{D}(A) \to Z$, where $\mathcal{D}(A) = \mathcal{D}(\mathcal{A}) \cap \ker(\mathcal{B})$ and

$$Az = \mathcal{A}z, \quad \text{for all} \quad z \in \mathcal{D}(A), \tag{6}$$

is the infinitesimal generator of a C_0-semigroup $T(t)$ on Z;

(b) There is an admissible control operator $B \in \mathcal{L}(V, Z)$ for $T(t)$ such that for each $v \in V$, $Bv \in \mathcal{D}(\mathcal{A})$, $\mathcal{A}B \in \mathcal{L}(V, Z)$ and

$$\mathcal{B}Bv = v. \tag{7}$$

Condition (b) implies that the operator $\mathcal{B}$ is onto on V. Therefore, $\mathcal{B}$ has at least one bounded right inverse $\mathcal{F} \in \mathcal{L}(V, Z)$. In this case, we can put $B = (\mathcal{A} - A)\mathcal{F}$. Further, we can show that

$$\mathcal{A} = A + B\mathcal{B} \quad \text{and} \quad \mathcal{B}(sI - A)^{-1}B = I \tag{8}$$

for all $s \in \rho(A)$.

We begin to analyze the linearized system (4). We set $Z = L_2(0,1)$, $U = V = \mathbb{C}$, and define the operator $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset Z \to Z$ as

$$\mathcal{A} := \frac{1}{P_e}\frac{d^2}{dx^2} - \frac{d}{dx} + J(x) \tag{9}$$

with

$$\mathcal{D}(\mathcal{A}) = \left\{ h \in L_2(0,1) : h \text{ and } \tfrac{dh}{dx} \text{ are a.c.}, \tfrac{d^2h}{dx^2} \in L_2(0,1), \tfrac{dh}{dx}(1) = 0 \right\},$$

where a.c. denotes "absolutely continuous".

We define an operator $\mathcal{B} : \mathcal{D}(\mathcal{B}) \subset Z \to V$ by

$$\mathcal{B}h := h(0) - \frac{1}{P_e}\frac{dh}{dx}(0) \quad \text{with} \quad \mathcal{D}(\mathcal{B}) = \mathcal{D}(\mathcal{A}) \tag{10}$$

and an operator $A : \mathcal{D}(A) \subset Z \to Z$ by

$$A := \frac{1}{P_e}\frac{d^2}{dx^2} - \frac{d}{dx} + J(x) \tag{11}$$

with $\mathcal{D}(A) = \mathcal{D}(\mathcal{A}) \cap \ker \mathcal{B}$.

Theorem 2.1 *The linearized system* (4) *is a well-posed mixed boundary control problem.*

Proof. Let $Z = L_2(0,1)$, $U = V = \mathbb{C}$. We consider the operators $\mathcal{A}, \mathcal{B}$, and A defined in (9), (10), and (11) on their domains, respectively. It is clear that $Az = \mathcal{A}z$ for all $z \in \mathcal{D}(A)$. We see that $A = -A_0$, where A_0 is the Sturm-Liouville operator, where $\rho(x) = P_e e^{-P_e x}$ and $p(x) = e^{-P_e x}$, see [8]. Therefore, A is closed, negative, and self-adjoint with respect to the weighted inner product

$$\langle h_1, h_2 \rangle_\rho := \int_0^1 h_1(x)\overline{h_2(x)}\rho(x)\,dx.$$

Additionally, the eigenvalues of A are real, simple, and form a decreasing sequence. The corresponding eigenfunctions are also orthogonal with respect to the weight function w. Therefore, A is the infinitesimal generator of an exponentially stable semigroup $T(t)$ on Z and $T(t) \geq 0$ for all $t \geq 0$. Let (λ_n, ϕ_n), $n \in \mathbb{N} \cup \{0\}$, be the pairs of the eigenvalue and the corresponding eigenfunction of A, we have

$$T(t)z = \sum_{n=0}^{\infty} e^{\lambda_n t} \langle z, \phi_n \rangle_\rho \phi_n \quad \text{for all} \quad z \in Z. \tag{12}$$

Henceforth, if we define an operator $Bv = b(x)v$ for all $v \in V$, where $b(x) = 1 + ce^{P_e x}$ for some constants c, then B satisfies (7). We confirm that the operators $\mathcal{A}$, A, $\mathcal{B}$, and B satisfy Definition 2.1 on Z, U, and V. We conclude that the mixed boundary control problem (4) is well-posed.

To investigate the solution explicitly, we need to reformulate equation (5) to be an abstract Cauchy problem. In this context, we have a relationship of the solution of the mixed boundary control problem (5) and the solution of the related Cauchy problem. For this purpose, several assumptions are required.

Consider the mixed boundary control system $(\mathcal{A}, \mathcal{B})$ (5) of problem (4) for $\dot{v} \in L_1([0,\tau], V)$ and $u \in L_1([0,\tau], U)$, where $\mathcal{A}, \mathcal{B}$ and A are defined in (9), (10) and (11), respectively. The related abstract Cauchy problem of (5) is

$$\begin{aligned} \dot{w}(t) &= Aw(t) - B\dot{v}(t) + \mathcal{A}Bv(t) + B_d u(t), \\ w(0) &= w_0. \end{aligned} \tag{13}$$

The assumptions guarantee the existence and uniqueness of a classical solution to problem (13) for $w_0 \in \mathcal{D}(A)$.

Theorem 2.2 *If $v \in C^2([0,\tau], V)$, $u \in C^1([0,\tau], U)$, and $w_0 \in \mathcal{D}(A)$, where $w_0 = z_0 - Bv(0)$, then problems* (5) *and* (13) *have the classical solutions related by*

$$w(t) = z(t) - Bv(t). \tag{14}$$

Moreover, problem (5) *has a unique classical solution.*

Proof. Let w be the classical solution of problem (13). This gives $w(t) \in \mathcal{D}(A) \subset \mathcal{D}(\mathcal{A}) \cap \mathcal{D}(\mathcal{B})$ and $Bv(t) \in \mathcal{D}(\mathcal{B})$. Since $w(t) \in \ker(\mathcal{B})$, (7) gives

$$\mathcal{B}z(t) = \mathcal{B}[w(t) + Bv(t)] = \mathcal{B}w(t) + \mathcal{B}Bv(t) = v(t).$$

Further, from equations (13) and (14), we have

$$\dot{z}(t) = \dot{w}(t) + B\dot{v}(t) = \mathcal{A}z(t) + B_d u(t).$$

Thus, the function z in (14) is the classical solution of problem (5) when w is the classical solution of problem (13).

The converse is shown similarly and the uniqueness of z follows from the uniqueness of w.

Alternately, we can reformulate problem (5) to be the abstract Cauchy problem (13) without the derivative of the boundary control term. In this context, we define an extended state space $\mathcal{P} := V \oplus Z$ and reformulate problem (13) on $\mathcal{P}$:

$$\begin{aligned} \dot{p}(t) &= \mathfrak{A}p(t) + \mathfrak{B}\mathfrak{u}(t), \\ p(0) &= p_0, \end{aligned} \tag{15}$$

where $\mathfrak{A} = \begin{bmatrix} 0 & 0 \\ \mathcal{A}B & A \end{bmatrix}$, $\mathfrak{B} = \begin{bmatrix} I & 0 \\ -B & B_d \end{bmatrix}$, $p(t) = \begin{bmatrix} v(t) \\ w(t) \end{bmatrix}$, $\mathfrak{u}(t) = \begin{bmatrix} \dot{v}(t) \\ u(t) \end{bmatrix}$, and $p_0 = \begin{bmatrix} v_0 \\ w_0 \end{bmatrix}$. We verify that the operator $\mathfrak{A}$ generates a C_0-semigroup $K(t)$ on $\mathcal{P}$, given by

$$K(t) = \begin{bmatrix} I & 0 \\ S(t) & T(t) \end{bmatrix}, \tag{16}$$

where $S(t)p_1 = \int_0^t T(t-s)\mathcal{A}Bp_1\, ds$, $p_1 \in V$.

Theorem 2.3 *If $v \in C^2([0,\tau], V)$, $u_d \in C^1([0,\tau], U)$, and $w_0 \in \mathcal{D}(A)$, then $p(t) = \begin{bmatrix} v(t) \\ w(t) \end{bmatrix}$ is the unique classical solution of problem* (15)*, where w is a unique classical solution of problem* (13)*. Moreover, if $z_0 = w_0 + Bv(0)$, then the classical solution of problem* (5) *is defined by*

$$\begin{aligned} z(t) &= \begin{bmatrix} B & I \end{bmatrix} p(t) \\ &= Bv(t) - T(t)Bv(0) + T(t)z_0 - \int_0^t T(t-s)B\dot{v}(s)\, ds + \int_0^t T(t-s)\mathcal{A}Bv(s)\, ds \\ &\quad + \int_0^t T(t-s)B_d u(s)\, ds. \end{aligned} \tag{17}$$

Proof. We see that $\mathfrak{A} := \begin{bmatrix} 0 & 0 \\ \mathcal{A}B & A \end{bmatrix}$ on $\mathcal{D}(\mathfrak{A}) = V \oplus \mathcal{D}(A)$ is the infinitesimal generator of a C_0-semigroup on $\mathcal{P}$ and $\begin{bmatrix} I & 0 \\ B & B_d \end{bmatrix} \in \mathcal{L}(V \oplus U, \mathcal{P})$. This gives that system (15) is well-defined. Moreover, the mild solution of problem (15) is given by

$$\begin{aligned} p(t) = &\begin{bmatrix} I & 0 \\ S(t) & T(t) \end{bmatrix} \begin{bmatrix} v_0 \\ w_0 \end{bmatrix} \\ &+ \int_0^t \begin{bmatrix} I & 0 \\ S(t-s) & T(t-s) \end{bmatrix} \begin{bmatrix} I & 0 \\ -B & B_d \end{bmatrix} \begin{bmatrix} \dot{v}(s) \\ u(s) \end{bmatrix} ds, \end{aligned} \tag{18}$$

where $S(t)p_1 = \displaystyle\int_0^t T(t-s)\mathcal{A}Bp_1\, ds$, $p_1 \in V$. The first component of (18) is

$$p_1(t) = v_0 + \int_0^t \dot{v}(s)\, ds = v(0) + \int_0^t \dot{v}(s)\, ds = v(t).$$

Since $p(0) = p_0 = \begin{bmatrix} v_0 \\ w_0 \end{bmatrix} \in \mathcal{D}(\mathfrak{A})$, the hypothesis and uniqueness theorem gurantee that $p(t)$ is the unique classical solution of problem (15) satisfying $\dot{p}_1(t) = \dot{v}(t)$. Also, from the second component of (18), we have

$$\dot{p_2}(t) = \mathcal{A}Bv(t) + Aw(t) - B\dot{v}(t) + B_d u(t).$$

Since $p_2(0) = w_0$, problem (13) has a unique classical solution $p_2(t) = w(t)$.

Next, if $z_0 = w_0 + Bv(0)$, then Theorem 2.2 gives

$$\begin{bmatrix} B & I \end{bmatrix} p(t) = Bv(t) + p_2(t) = Bv(t) + w(t) = z(t).$$

On the other hand, the mild solution of (13) is given by

$$w(t) = T(t)w_0 - \int_0^t T(t-s)B\dot{v}(s)\,ds + \int_0^t T(t-s)\mathcal{A}Bv(s)\,ds + \int_0^t T(t-s)B_d u(s)\,ds.$$

The last two results give (17).

3 Approximate Controllability for Mixed Boundary Control System

We will investigate an approximate controllability for the mixed boundary control system (4). If v is a differentiable control satisfying $\dot{v} \in L_2([0,\tau],V)$, then the mild solution of problem (4) is well-defined. We define the customized reachability subspace:

$$\mathcal{R}_b = \Big\{ z \in Z : \text{there are a } \tau > 0 \text{ and a differentiable control } v, \text{ with } v(0) = 0, \\ v, \dot{v} \in L_2([0,\tau],V) \text{ and } z(\tau) \text{ is the classical solution (17)} \Big\}.$$

The mixed boundary control system is said to be approximately controllable if $\mathcal{R}_b$ is dense in Z. Let $\mathcal{R}^e$ be the reachability subspace of the extended system $(\mathfrak{A},\mathfrak{B})$ on $\mathcal{P}$, i.e.,

$$\mathcal{R}^e = \Big\{ p \in \mathcal{P} : \text{there exists a } \tau > 0 \text{ and } \mathfrak{u} \in L_2([0,\tau], V \oplus U) \text{ such that} \\ p(\tau) = \int_0^\tau K(\tau - s)\mathfrak{B}\mathfrak{u}(s)ds \Big\}.$$

Theorem 3.1 *If the extended system $(\mathfrak{A},\mathfrak{B})$ is approximately controllable, then the mixed boundary control system (4) is also approximately controllable.*

Proof. Refer to the proof of Theorem 2.3, we have $\mathcal{R}_b = \begin{bmatrix} B & I \end{bmatrix} \mathcal{R}^e$. This implies that if $\mathcal{R}^e$ is dense in $\mathcal{P}$, then $\mathcal{R}_b$ is dense in Z.

We recall that A has real eigenvalues $\{\lambda_n : n \geq 1\}$ and a biorthogonal pair $\{(\phi_n, \phi_n) : n \geq 1\}$ due to A is self-adjoint. We consider that the operator B is finite-rank defined by

$$Bv = \sum_i^m b_i v_i, \quad b_i \in Z, \tag{19}$$

where $v = (v_1, v_2, \ldots, v_m) \in V = \mathbb{C}^m$ and $b_i(x) = 1 + c_i e^{P_e x}$ for some constants c_i. In the following, we have some results of system (15) on $\mathcal{P} = V \oplus Z$.

Lemma 3.1 *The operator $\mathfrak{A}$ in (15) has the biorthogonal pair $\{(\tilde{\phi}_n, \tilde{\psi}_n) : n \geq 1\}$, where*

$$\tilde{\phi}_n = \begin{bmatrix} 0 \\ \phi_n \end{bmatrix} \quad \textit{and} \quad \tilde{\psi}_n = \begin{bmatrix} \frac{1}{\lambda_n}(\mathcal{A}B)^*\phi_n \\ \phi_n \end{bmatrix},$$

whenever $\lambda_n \neq 0$.

Proof. From the hypothesis, we see that $A\phi_n = \lambda_n \phi_n$, where λ_n is real for all $n \in \mathbb{N}$. For $\mu_n \neq 0$, let $\tilde{\phi}_n = \begin{bmatrix} \tilde{\phi}_n^1 & \tilde{\phi}_n^2 \end{bmatrix}^{tr}$ and $\tilde{\psi}_n = \begin{bmatrix} \tilde{\psi}_n^1 & \tilde{\psi}_n^2 \end{bmatrix}^{tr}$, where Q^{tr} denotes the transpose of Q. Taking into account $\mathfrak{A}\tilde{\phi}_n = \mu_n \tilde{\phi}_n$ gives $\tilde{\phi}_n^1 = 0$, $\tilde{\phi}_n^2 = \phi_n$ and $\mu_n = \lambda_n$.

Similarly, $\mathfrak{A}^*\tilde{\psi}_n = \overline{\mu_n}\tilde{\psi}_n = \lambda_n\tilde{\psi}_n$ gives $(\mathcal{A}B)^*\tilde{\psi}_n^2 = \lambda_n\tilde{\psi}_n^1$ and $A^*\tilde{\psi}_n^2 = \lambda_n\tilde{\psi}_n^2$. This forces that $\tilde{\psi}_n^2 = \phi_n$ and $\tilde{\psi}_n^1 = \frac{1}{\lambda_n}(\mathcal{A}B)^*\phi_n$.

This lemma stresses that the operators A and $\mathfrak{A}$ have common nonzero eigenvalues. However, if 0 is not an eigenvalue of A $(0 \in \rho(A))$, where $\rho(A)$ is the resolvent set of A, we have the following.

Lemma 3.2 *If $0 \in \rho(A)$, then $\lambda_0 = 0$ is the eigenvalue with multiplicity m of $\mathfrak{A}$ and the corresponding biorthogonal pair*

$$\tilde{\phi}_0^i = \begin{bmatrix} e_i \\ -A^{-1}(\mathcal{A}B)e_i \end{bmatrix}, \quad \tilde{\psi}_0^i = \begin{bmatrix} e_i \\ 0 \end{bmatrix},$$

where $\{e_i : i = 1, 2, \dots, m\}$ is the usual orthonormal basis of $V = \mathbb{C}^m$.

Proof. The fact that $0 \in \rho(A)$ implies that A is invertible. Let $\tilde{\phi}_0^i = \begin{bmatrix} \tilde{\phi}_0^{i1} & \tilde{\phi}_0^{i2} \end{bmatrix}^{tr}$ and $\tilde{\psi}_0^i = \begin{bmatrix} \tilde{\psi}_0^{i1} & \tilde{\psi}_0^{i2} \end{bmatrix}^{tr}$, $i = 1, 2, \dots, m$, be the corresponding biorthogonal pair of $\mathfrak{A}$ associated with $\lambda_0 = 0$. The equation $\mathfrak{A}\tilde{\phi}_0^i = 0$ gives $\tilde{\phi}_0^{i2} = -A^{-1}(\mathcal{A}B)\tilde{\phi}_0^{i1}$. Chooshing $\tilde{\phi}_0^{i1} = e_i$ gives the assertion. Finally, $\mathfrak{A}^*\tilde{\psi}_0^i = 0$ gives $\tilde{\psi}_0^{i2} = 0$ and $\tilde{\psi}_0^{i1} = e_i$ for $i = 1, 2, \dots, m$.

Next, we assume the interior control B_d in (5) is given by

$$B_d u = \sum_{i=1}^{m} d_i u_i, \quad d_i \in Z, \tag{20}$$

where $u = (u_1, u_2, \dots, u_m) \in V = \mathbb{C}^m$. The following theorem gives the sufficiency for the approximate controllability of system (5).

Theorem 3.2 *Let $0 \in \rho(A)$. The mixed boundary control system* (5) *with the interior control* (20) *is approximately controllable if for each $n \in \mathbb{N}$,*

$$\text{rank}\,(\langle \mathcal{A}b_1 - \lambda_n b_1 + \lambda_n d_1, \phi_n\rangle_\rho, \cdots, \langle \mathcal{A}b_m - \lambda_n b_m + \lambda_n d_m, \phi_n\rangle_\rho) = 1. \tag{21}$$

Proof. According to Theorem 3.1, we prove the approximate controllability of system (15) in $\mathcal{P}$. Following the proof of Theorem 4.2.3 of [8], we need to prove that $\overline{\mathcal{R}^e} = \mathcal{P}$. However, $\overline{\mathcal{R}^e} = \mathcal{P}$ is equivalent to for each $n \geq 1$, there is a $\mathfrak{u}(t) = \begin{bmatrix} \dot{v}(t) \\ u(t) \end{bmatrix}$, where $v(t) = te_0$, $u(t) = e_0$, $e_0 = (1, 1, \dots 1) \in \mathbb{C}^m$, implying $\langle \mathfrak{B}\mathfrak{u}, \tilde{\psi}_n\rangle \neq 0$. From the definitions of $\mathfrak{B}$, $\mathfrak{u}$ in (15) and $\tilde{\psi}_n$ in Lemma 3.1, we have

$$\begin{aligned} \langle \mathfrak{B}\mathfrak{u}, \tilde{\psi}_n\rangle &= \left\langle \dot{v}, \tfrac{1}{\lambda_n}(\mathcal{A}B)^*\phi_n \right\rangle_\rho + \langle -B\dot{v} + B_d u, \phi_n\rangle_\rho \\ &= \langle \dot{v}, (\mathcal{A}B)^*\phi_n\rangle_\rho + \langle -\lambda_n B\dot{v} + \lambda_n B_d u, \phi_n\rangle_\rho \\ &= \langle \mathcal{A}B\dot{v}, \phi_n\rangle_\rho + \langle -\lambda_n B\dot{v} + \lambda_n B_d u, \phi_n\rangle_\rho \\ &= \langle \mathcal{A}B\dot{v} - \lambda_n B\dot{v} + \lambda_n B_d u, \phi_n\rangle_\rho . \end{aligned}$$

Equations (19) and (20) give $\langle \mathcal{A}b_i - \lambda_n b_i + \lambda_n d_i, \phi_n\rangle_\rho \neq 0$, $i = 1, 2, \dots, m$, for each $n \in \mathbb{N}$, and (21) follows.

Lemma 3.3 *Assume that A has the eigenvalue $\lambda_1 = 0$ with the eigenvector ϕ_1.*

(a) If $\langle \mathcal{A}b_i, \phi_1\rangle_\rho = 0$ for $i = 1, 2, \dots, r$, then $\mathfrak{A}$ has the corresponding eigenvectors $\tilde{\phi}_0^i = \begin{bmatrix} e_i \\ y_i \end{bmatrix}$ and biorthogonal pairs $\tilde{\psi}_0^i = \begin{bmatrix} e_i \\ 0 \end{bmatrix}$, where $y_i = -\sum_{n=2}^{\infty} \frac{1}{\lambda_n}\langle \mathcal{A}b_i, \phi_n\rangle_\rho \phi_n$, for $i = 1, 2, \dots, r$.

(b) If $\langle \mathcal{A}b_i, \phi_1\rangle_\rho \neq 0$ for $i = r+1, \dots, m$, then $\mathfrak{A}$ has the generalized eigenvectors $\tilde{\phi}_0^i = \begin{bmatrix} \langle \mathcal{A}b_i, \phi_1\rangle_\rho^{-1} e_i \\ x_i \end{bmatrix}$ of order 2 satisfying $\mathfrak{A}\tilde{\phi}_0^i = \begin{bmatrix} 0 \\ \phi_1 \end{bmatrix}$ with the biorthogonal pair $\tilde{\psi}_0^i = \begin{bmatrix} \langle \mathcal{A}b_i, \phi_1\rangle_\rho e_i \\ 0 \end{bmatrix}$, where $x_i = -\sum_{n=2}^{\infty} \frac{\langle \mathcal{A}b_i, \phi_n\rangle_\rho}{\lambda_n \langle \mathcal{A}b_i, \phi_1\rangle_\rho}\phi_n$ for $i = r+1, \dots, m$.

Proof. (a) Lemma 3.2 implies that $\lambda_1 = 0$ is the eigenvalue of $\mathfrak{A}$ with multiplicity r. Let the corresponding eigenvector of $\mathfrak{A}$ have the form $\tilde{\phi}_0^i = \begin{bmatrix} e_i \\ y_i \end{bmatrix}$. It gives $\mathcal{A}b_i + Ay_i = 0$. Multiplying this equation by $\phi_n\rho$ and simplifying, we have $\langle y_i, \phi_n\rangle_\rho = -\frac{1}{\lambda_n}\langle \mathcal{A}b_i, \phi_n\rangle_\rho$. The form of y_i follows. Then the direct calculation of $\mathfrak{A}^*\tilde{\psi}_0^i = 0$ gives $\tilde{\psi}_0^i = \begin{bmatrix} e_i \\ 0 \end{bmatrix}$.

(b) For i fixed, $r < i \le m$, we can verify that

$$\mathfrak{A}^2\tilde{\phi}_0^i = \begin{bmatrix} 0 \\ \frac{A(\mathcal{A}b_i)}{\langle \mathcal{A}b_i, \phi_1\rangle_\rho} + A^2 x_i \end{bmatrix}.$$

The facts that $\mathcal{A}b_i \in \mathcal{D}(A)$ and A is self-adjoint with respect to the inner product $\langle \cdot, \cdot\rangle_\rho$ give the second component of $\mathfrak{A}^2\tilde{\phi}_0^i$ is 0. This confirms that $\tilde{\phi}_0^i$ is the generalized eigenvector of $\mathfrak{A}$ of order 2. Further, $\frac{A(\mathcal{A}b_i)}{\langle \mathcal{A}b_i, \phi_1\rangle_\rho} + A^2 x_i = 0$ implies that $\langle x_i, \phi_n\rangle_\rho = -\frac{\langle A(\mathcal{A}b_i), \phi_n\rangle_\rho}{\lambda_n \langle \mathcal{A}b_i, \phi_1\rangle_\rho}$. The required form of x_i follows. The biorthogonal $\tilde{\psi}_0^i$ is found easily. Finally, since $\mathcal{A}b_i$ can be expanded in ϕ_n, we have

$$\mathfrak{A}\tilde{\phi}_0^i = \begin{bmatrix} 0 \\ \frac{\mathcal{A}b_i}{\langle \mathcal{A}b_i, \phi_1\rangle_\rho} + Ax_i \end{bmatrix} = \begin{bmatrix} 0 \\ \phi_1 \end{bmatrix}.$$

Remark 3.1 For $r < i \le m$, the set $\{\tilde{\phi}_n, \tilde{\phi}_0^i : n \in \mathbb{N}, i = 1, 2, \dots, m\}$ generates a Riesz basis of $\mathcal{P} = \mathbb{C}^m \oplus Z$ and the operator $\mathfrak{A}$ has a spectral decomposition

$$\mathfrak{A}p = \sum_{i=r+1}^{m} \langle p, \tilde{\psi}_0^i\rangle \begin{bmatrix} 0 \\ \phi_1 \end{bmatrix} + \sum_{n=2}^{\infty} \lambda_n \langle p, \tilde{\psi}_n\rangle \begin{bmatrix} 0 \\ \phi_n \end{bmatrix}, \quad p \in \mathcal{P}.$$

Theorem 3.3 *Let $\lambda_1 = 0$ and $\mathfrak{A} = \tilde{A} + A_h$, where $\tilde{A}$ is a Riesz operator on $\tilde{Y} = \mathrm{span}_{n\ge 2}\{\tilde{\phi}_n\}$ and the operator A_h is finite-rank on $Y_h = \mathrm{span}_{i=1,\dots,m}\{\tilde{\phi}_1, \tilde{\phi}_0^i\}$. If (21) holds for all $n \ge 2$ and $\langle \mathcal{A}b_i, \phi_1\rangle \neq 0$ for all $i \ge r+1$, then the mixed boundary control system (5) with the interior control (20) is approximately controllable.*

Proof. From Exercise 4.17 of [8], the extended system $(\mathfrak{A}, \mathfrak{B})$ is approximately controllable if and only if the systems $(\tilde{A}, \tilde{B})$ and (A_h, B_h) are approximately controllable. We verify that (A_h, B_h) is exactly controllable. By Theorem 3.2, $(\tilde{A}, \tilde{B})$ is approximately controllable when (21) holds for all $n \ge 2$ and $\langle \mathcal{A}b_i, \phi_1\rangle \neq 0$ for all $i \ge r+1$. The assertion follows by Theorem 3.1.

4 Transfer Function for Boundary Control Systems

The mixed boundary control system allows to be converted to the boundary control system. Therefore, without loss of generality, we consider the boundary control system (4) with $u(t) = 0$. Moreover, we assume that the output $y(x, t)$ is a substrate concentration measured at x, $0 \leq x \leq 1$. The system equations are

$$\begin{aligned} z_t(x,t) &= \tfrac{1}{P_e} z_{xx}(x,t) - z_x(x,t) + J(x)z(x,t), \\ z(0,t) - \tfrac{1}{P_e} z_x(0,t) &= v(t), \\ z_x(1,t) &= 0, \\ y(x,t) &= Kz(x,t), \end{aligned} \tag{22}$$

where $K \in \mathcal{L}(Z, Y)$ and Y is the output space. System (22) is rewritten as the abstract boundary control system

$$\begin{aligned} \dot{z}(t) =& \mathcal{A}z(t), \quad z(0) = z_0, \\ \mathcal{B}z(t) =& v(t), \\ y(t) =& Kz(t). \end{aligned} \tag{23}$$

The triple $(\mathcal{A}, \mathcal{B}, K)$ denotes the boundary control system (23) with the output operator K.

In this paper, we focus on the boundedness of the input-output map from $v \in L_2([0, \tau], V)$ to $y \in L_2([0, \tau], Y)$.

Definition 4.1 Let $\hat{y}(s)$ and $\hat{v}(s)$ be the Laplace transform of the output and input of system (23), respectively. A system transfer function is an operator $G(s)$ such that

$$\hat{y}(s) = G(s)\hat{v}(s)$$

for all s, $\operatorname{Re} s > \sigma$ for some real σ.

The definition implies that the input-output map is well-defined and the output is Laplace transformable. Further, the system transfer function can be used to determine the boundedness of the input-output map.

Theorem 4.1 ([16]) *Let $(\mathcal{A}, \mathcal{B}, K)$ be any boundary control system. The input-output map of the system is bounded if and only if there exists a real number σ such that the transfer function $G(s)$ associated with $(\mathcal{A}, \mathcal{B}, K)$ satisfies*

$$\sup_{\operatorname{Re} s > \sigma} \|G(s)\|_{\mathcal{L}(V,Y)} < \infty.$$

The function $G(s)$ is said to be proper if the above inequality holds.

The boundary control system (23) can be written in the state-space form (A, B, C), see [19]. Here, the operators A and B satisfy Definition 2.1. The operator $C \in \mathcal{L}(W, Y)$ is defined by $C = K|_W$, where $W = \ker(\mathcal{B})$. The following refers to Theorem 2.6 of [19].

Theorem 4.2 *The input-output map of boundary control system* (23) *is well-defined for all inputs $v \in \mathcal{H}^2([0, \tau], V)$, $v(0) = 0$. The output can be written as*

$$y(t) = g(t) * v(t),$$

where $g(t)$ is a distribution with the Laplace transform $G(s)$. For each $s \in \rho(A)$, the operator $G(s) \in \mathcal{L}(V, Y)$ is the system transfer function given by

$$G(s) = K(sI - A)^{-1}B.$$

Proof. We consider the state-space formulation (A, B, C) of the boundary control system $(\mathcal{A}, \mathcal{B}, K)$ (23) constructed by the procedure above:

$$\begin{aligned} \dot{z}(t) &= Az(t) + Bv(t), \quad z(0) = z_0, \\ y(t) &= Cz(t), \end{aligned} \tag{24}$$

where A is the infinitesimal generator of a C_0-semigroup $T(t)$ on Z. Let z be the solution of (24). For any $\mu \in \rho(A)$, we can rewrite

$$\begin{aligned} z(t) &= (\mu I - A)^{-1}(\mu I - A)z(t) \\ &= (\mu I - A)^{-1}(\mu z(t) - \dot{z}(t)) + (\mu I - A)^{-1}Bv(t). \end{aligned} \tag{25}$$

For all initial conditions $z(0) = 0$ and smooth controls $v \in \mathcal{H}^2([0, \tau], V)$ with $v(0) = 0$, the first term in (25) is in $W \subset Z$ for each time t because $Bv \in \mathcal{D}(\mathcal{A})$, see Definition 2.1. Since A is the infinitesimal generator on Z with the domain $\mathcal{D}(A)$, $(\mu I - A)^{-1}B \in \mathcal{L}(V, \mathcal{D}(A))$. Further, for any $\mu \in \rho(A)$, Range$(\mu I - A)^{-1}B \subset Z$ and so $(\mu I - A)^{-1}B \in \mathcal{L}(V, Z)$. By applying the operator K to the solution z, we obtain the output y:

$$y(t) = K(\mu I - A)^{-1}(\mu z(t) - \dot{z}(t)) + K(\mu I - A)^{-1}Bv(t). \tag{26}$$

Since $W \subset Z$, $K(\mu I - A)^{-1} \in \mathcal{L}(\mathcal{D}(A), Y)$ and $K(\mu I - A)^{-1}B \in \mathcal{L}(V, Y)$. Since both v and z are Laplace transformable and due to the fact that z is the solution of (24), the Laplace transform of both sides of (26) gives

$$\hat{y}(s) = K(\mu I - A)^{-1}(\mu - s)(sI - A)^{-1}B\hat{v}(s) + K(\mu I - A)^{-1}B\hat{v}(s).$$

This gives the system transfer function

$$G(s) = K(\mu I - A)^{-1}(\mu - s)(sI - A)^{-1}B + K(\mu I - A)^{-1}B.$$

By replacing $\mu = s$, we obtain

$$G(s) = K(sI - A)^{-1}B \tag{27}$$

for any $s \in \rho(A)$.

On the other hand, the input-output map of system (23) is

$$y(t) = K\int_0^t T(t - r)u(r)\, dr.$$

From (27), the distribution of $G(s)$ is $g(t) = KT(t)B$. Therefore, the output can be written as

$$y(t) = \int_0^t g(t - r)u(r)\, dr = g(t) * u(t).$$

Remark 4.1 The resolvent operator $R(s) := (sI - A)^{-1}$, $s \in \rho(A)$, is given explicitly by Theorem 7.1 in [20]. The reference initiated the study of the maximal and minimal Sturm–Liouville operators, all self-adjoint restrictions of the maximal operator $T_{\max}$. Moreover, the spectrum properties of A have been also comprehensively characterized.

The form of the transfer function can be based entirely on the boundary control description (23) not on the construction of a state-space realization. The transfer function is defined in terms of an elliptic problem associated with the boundary control system.

Definition 4.2 The abstract elliptic problem $(\mathcal{A}, \mathcal{B})_e$ corresponding to the boundary control system $(\mathcal{A}, \mathcal{B})$ in (23) is

$$\begin{aligned} \mathcal{A}z =& sz, \quad s \in \mathbb{C}, \\ \mathcal{B}z =& v. \end{aligned} \tag{28}$$

The solution $z \in Z$ is denoted by $z(s)$.

Let α be the growth bound of the semigroup associated with $(\mathcal{A}, \mathcal{B})$. The elliptic problem (28) has a unique solution $z(s)$ for all v and $\operatorname{Re} s > \alpha$. The system transfer function may be described through the solution to the abstract elliptic problem (28).

Theorem 4.3 *If $(\mathcal{A}, \mathcal{B}, K)$ is the boundary control system (23), then there exists an $\alpha \in \mathbb{C}$ such that the system transfer function $G(s)$ is given by*

$$G(s)v = Kz(s) \quad \textit{for all} \quad s \in \mathbb{C}, \ \textit{with} \operatorname{Re} s > \alpha, \tag{29}$$

where $z(s)$ is the solution to the abstract elliptic problem (28) with input v.

Proof. Let α be the growth bound of the C_0-semigroup $T(t)$ generated by A. From Theorem 4.2, for all $s \in \mathbb{C}$ with $\operatorname{Re} s > \alpha$, $s \in \rho(A)$, we have the transfer function $G(s) = K(sI - A)^{-1}B$. Here, we note that $\mathcal{A} = A + B\mathcal{B}$ and $\mathcal{B}(sI - A)^{-1}B = I$, see (8). Therefore, we obtain $\mathcal{A}(sI - A)^{-1}B = s(sI - A)^{-1}B$. This implies that $z(s) = (sI - A)^{-1}Bv$ is the solution of abstract elliptic problem (28). The assertion follows.

Alternatively, since $\mathcal{B}$ is onto, for any given $v \in V$, we can choose $z \in Z$ such that $\mathcal{B}z = v$. We define $G \in \mathcal{L}(V, Y)$ by

$$G(s)\mathcal{B}z := Kz - C(sI - A)^{-1}(sz - \mathcal{A}z). \tag{30}$$

The definitions of A and C guarantee that G is well-defined. If z solves the associated elliptic problem, then for any $v \in V$ and $s \in \mathbb{C}$ with $\operatorname{Re} s > \alpha$, (30) gives

$$G(s)v = Kz(s).$$

Example 4.1 If the desired steady-state temperature profile of system (22) is essentially uniform and the output $y(t)$ is measured at x_1, $0 \le x_1 \le 1$, this shows that the control system is well-posed.

From the assumptions, we may linearize about a uniform temperature $z_s(x) =$ constant, so $J(x)$ becomes a constant. Let $J(x) = c$. We use the notations relating with (9), (10), and (11) for $J(x) = c$, $Z = L_2(0, 1)$ and $V = \mathbb{C}$. We have the associated C_0-semigroup $T(t)$ is given in the form (12), where λ_n is the solution of the transcendental equation

$$\begin{aligned} \tan \beta_n &= \frac{4P_e\beta_n}{4\beta_n^2 - P_e^2}, \\ \phi_n(x) &= B_n e^{\frac{P_e}{2}x} \left[\tfrac{2\beta_n}{P_e} \cos \beta_n x + \sin \beta_n x \right], \\ \beta_n^2 &= \frac{4P_e(c - \lambda_n) - P_e^2}{4}, \end{aligned}$$

where B_n are some constants. We note that the eigenvalues λ_n, $n \in \mathbb{N} \cup \{0\}$, form the decreasing sequence of negative real numbers, see Lemma 5.1 of [1].

The elliptic problem corresponding to system (22) when the output $y(t)$ is measured at x_1, $0 \leq x_1 \leq 1$, is

$$\begin{aligned} \frac{1}{P_e}\frac{d^2 z}{dx^2} - \frac{dz}{dx} + cz = sz, \quad s \in \mathbb{C}, \\ z'(1) = 0, \\ z(0) - \frac{1}{P_e} z'(0) = v, \end{aligned} \tag{31}$$

with the output equation

$$y = Kz(x_1).$$

The solution of the elliptic problem (31) is

$$z(x, s) = e^{\frac{P_e}{2}x}\left[A(v, s)\cos\beta x + B(v, s)\sin\beta x\right],$$

where

$$\begin{aligned} A(v, s) &= \frac{v[4P_e\beta\cos\beta + 2(P_e^2 - 4\beta^2 + 4\beta)\sin\beta]}{4P_e\beta\cos\beta + (P_e^2 - 4\beta^2)\sin\beta}, \\ B(v, s) &= \frac{2P_e v[2\sin\beta - P_e\cos\beta]}{4P_e\beta\cos\beta + (P_e^2 - 4\beta^2)\sin\beta}, \\ \beta^2 &= \frac{4P_e(c - s) - P_e^2}{4}. \end{aligned}$$

In this problem, we have $\|T(t)\| \leq Me^{\lambda_0 t}$, where $\lambda_0 = \sup_{n\geq 0}\lambda_n$. Therefore, the growth bound $\alpha = \lambda_0$. For all $s \in \mathbb{C}$ with $\operatorname{Re} s > \lambda_0$, Theorem 4.3 gives the system transfer function

$$G(s) = Ke^{\frac{P_e}{2}x_1}\left[A(s)\cos\beta x_1 + B(s)\sin\beta x_1\right],$$

where

$$\begin{aligned} A(s) &= \frac{4P_e\beta\cos\beta + 2(P_e^2 - 4\beta^2 + 4\beta)\sin\beta}{4P_e\beta\cos\beta + (P_e^2 - 4\beta^2)\sin\beta}, \\ B(s) &= \frac{2P_e[2\sin\beta - P_e\cos\beta]}{4P_e\beta\cos\beta + (P_e^2 - 4\beta^2)\sin\beta}. \end{aligned}$$

We see that the transfer function is proper, and in virtue of Theorem 4.1, the input-output map is bounded. This implies that the control system is well-posed.

Remark 4.2 In the case $J(x)$ is not a constant, using the fact that the set $\{\phi_n\}$ is a basis for the state space Z, we can assume that $J(x)z(x, t)$ can be expressed by

$$J(x)z(x, t) = \sum_{n=0}^{\infty} f_n(t)\phi_n(x).$$

Therefore, by solving for f_n, we can use the procedure above to find the transfer function.

5 Conclusions

We concern with the mixed boundary control systems inducted by non-isothermal axial dispersion chemical tubular reactors. The well-posedness, approximate controllability, and transfer function of the mixed boundary control system can be determined. The well-posedness is identified by the operator control $B = 1 + ce^{P_e x}$. Moreover, the classical solution can be obtained using the extended space. The sufficiency for approximate controllability is justified by the eigenvalues and eigenvectors of the Sturm-Loiuville operator using the equivalence in the extended space. The proper transfer function of the associated boundary control system equipped with an output can be constructed. The transfer function ensures the well-posedness for the associated boundary control system.

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