Nonlinear Dynamics and Systems Theory, **24** (6) (2024) 549–560

The Advection-Diffusion-Reaction Equation: A Numerical Approach Using a Combination of Approximation Techniques

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Received: April 19, 2024; Revised: November 3, 2024

Abstract: In this paper, we propose a numerical technique for solving the advection-diffusion-reaction equation. The presented approach is based on coupling two numerical methods to address the problem posed with the Robin boundary conditions perturbed with a small parameter ε, in terms of spatial and temporal variables. We start by employing a Galerkin method for the spatial descretization, using a compact basis of the Legendre polynomials to derive a system of ordinary differential equations. This system is then solved using a Crank-Nicolson scheme, with the temporal domain uniformly discretized. The obtained numerical results demonstrate the effectiveness of the proposed numerical method and the convergence of the approximate solution to the analytic solution of the classical problem with the homogeneous Dirichlet boundary conditions when ε approaches zero. This makes it particularly useful for approximating the solutions of such problems of partial differential equations appearing in reaction-diffusion systems, where the explicit solution is unknown under various types of boundary conditions.

Keywords: *advection-diffusion-reaction equation; Galerkin method; Legendre polynomials; Robin boundary conditions; Crank-Nicolson scheme.*

Mathematics Subject Classification (2010): 65N35, 65N30, 65N22, 33C45, 70K99, 93C20.

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1 Introduction

Partial differential equations (PDEs) represent a mathematical tool that connect different functions and their derivatives. These functions reflect, generally, some physical quantities like heat and waves [1]. Until now, their study is still very active, gathering the resolution of systems, daily problems and the development of mathematics. The problem for the majority of PDEs appears in the calculation of the analytic solution, which is generally impossible. For this reason, mathematicians headed to other tools such as numerical resolution to approximate the solution of such problems in an effective way in terms of time and results.

In this paper, we focus on the numerical study of the advection-diffusion-reaction equation. The latter brings together three important processes: advection, diffusion and reaction. It is formulated as follows:

$$\frac{\partial C}{\partial t} = D_0 \frac{\partial^2 C}{\partial x^2} - V_0 \frac{\partial C}{\partial x} - K_0 C + f(x,t),$$

where D_0 is the diffusion coefficient, V_0 is the convective velocity, K_0 is the reaction constant and $f(x,t)$ is a scalar function often called the source term. It models, according to the problem, a heat source, chemical reaction, injection/production wells, etc.

This equation occurs in several scientific disciplines such as biology, astrophysics, and industrial and environmental issues. It models many phenomena, for example energy transfer, mass transfer and also the transfer of the heat through a permeable medium and the transport of a chemical or biological pollutant through an underground aquifer system [2–4]. Generally, it describes the phenomenon of the distribution of some quantities in space and time. The advection-diffusion-reaction equation can be viewed as a special case of a reaction-diffusion system (systems that describe the dynamics of chemical concentrations reacting and diffusing in space), where the process of advection is also taken into account. In other words, this equation is an extension of reaction-diffusion equations to include the effects of advection [5].

Moving to numerical solution, different methods have been introduced and show hight accuracy to approximate the desired solution, we can cite: finite differences, finite volumes and finite elements. The principle of these methods is the same for all the numerical methods "*searching for discrete numerical values that approach the exact solution*" [6].

Another branch of numerical methods recently appeared are the spectral methods developed by D. Gottieb and S. Orszag in 1970, based on the use of a finite expansion of certain eigenfunctions obtained from the Sturm-Liouville problem [7–10]. This development gives a high level of precision which is superior to the other mentioned methods, thus, it requires a small number of grid points to get the desired precision [11]. Another advantage of these methods is that they are less intensive in terms of time and memory compared to finite elements, but they become less precise if we consider problems with complex geometry. These approaches are applicable for the resolution of different problems such as the resolution of ODEs, linear and nonlinear PDEs and eigenvalue problems [10, 12–14].

In this work, we develop an efficient numerical method basing on a coupling of spectral methods and finite differences schemes. The model problem is posed with perturbed boundary conditions of Robin type, so that we can apply a Legendre-Galerkin approach according to the spatial variable and a Cranck-Nicolson scheme according to the temporal one. The way in which the conditions are perturbed makes it possible to compare the obtained approximation and the exact solution of the same problem with the Dirichlet boundary conditions.

The structure of this paper is as follows. In Section 2, we remind some preliminaries and essential tools required for the elaboration of the presented study. Next, the model problem is presented in Section 3 with the adaptative variational formulation, the existence and uniqueness of the solution are also proved. In Section 4, we outline the principle of the presented approach and we study its convergence and give an estimation of the error of approximation. Then, in the same section, the implementation of the proposed technique and the coupling with the finite differences scheme are exposed to obtain the final system to solve. Section 5 addresses the proof of the efficiency of the algorithm via different numerical examples by showing the convergence of the approximation to the analytic solution of the classic problem with the homogeneous Dirichlet boundary conditions, when ε reaches zero.

2 Preliminaries

Let $I = (-1, 1)$. We define

$$L^2(I) = \{v\,;\ \ v \text{ is mesurable on } I \text{ and } \|v\| < +\infty\}.$$

The scalar product is $\langle u, v\rangle = (u, v)_{L^2} = \int_I u(x)v(x)\,\mathrm{d}x$, and the norm is defined by $\|v\|_{L^2} = (v, v)^{\frac{1}{2}}_{L^2}$.

For every positive m, we define the Sobolev space by

$$H^m(I) = \left\{v\,;\ \ \frac{\partial^k v}{\partial x^k} \in L^2(I), \quad 0 \leq k \leq m\right\},$$

and the standard semi-norm and norm are $|v|_{L^2} = \left\|\frac{\partial v}{\partial x}\right\|_{L^2}$; $\|v\|_{H^m} = \sum_{k=0}^{m} \left\|\frac{\partial^k v}{\partial x^k}\right\|_{L^2}$.
Let $(H^1(-1, 1))^*$ be the dual space of $H^1(-1, 1)$ with a norm defined by

$$\|g\|_{H^{1*}} = \sup_{\substack{v \in H^1(I)\\ v \neq 0}} \frac{\langle g, v\rangle}{\|v\|_{H^1}}.$$

Thus, and since $\|v\|_{L^2} \leq C\|v\|_{H^1}$ for all $v \in H^1(I)$, we can write

$$\|g\|_{H^{1*}} \leq C\|g\|_{L^2}. \tag{1}$$

We denote by $\mathscr{P}_N$ the space of polynomials of degree that is less than or equal to N. Let $L_n(x)$; $x \in I$ be the standard Legendre polynomial of degree n. The family of Legendre polynomials $L_k(x)_{k\in\mathbb{N}}$ constitutes a Hilbert basis of $L^2(I)$ and they are solutions of the following differential Legendre equation:

$$\left(1 - x^2\right) L_n^{''}(x) - 2xL_n^{'}(x) + n\,(n + 1)\, L_n(x) = 0, \quad n \geq 0.$$

The polynomial $L_n(x)$ is of degree n for all $n \in \mathbb{N}$, and the coefficient of its highest degree term is $\dfrac{(2n)!}{2^n(n!)^2}$. They satisfy

$$\forall n \neq m \in \mathbb{N}, \qquad \int_{-1}^{1} L_n(x)L_m(x)\,\mathrm{d}x = 0 \quad \text{ and } \quad \int_{-1}^{1} L_n^2(x)\,\mathrm{d}x = \frac{2}{2n+1}.$$

The Legendre polynomials satisfy the following recurrence relations [9]:

$$L_n(1) = 1 \quad \text{and} \quad L_n(-x) = (-1)^n L_n(x) \Longrightarrow L_n(-1) = (-1)^n,$$
$$(n+1)\, L_{n+1}(x) - (2n+1)\, x L_n(x) + n L_{n-1}(x) = 0, \quad n \geq 1,$$
$$(2n+1) L_n(x) = L'_{n+1}(x) - L'_{n-1}(x), \quad n \geq 1.$$

3 The Model Problem

We consider the advection-diffusion-reaction problem with mixed Robin-type boundary conditions disturbed with a small parameter ε,

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) + \alpha \dfrac{\partial u}{\partial x}(t,x) - \beta \dfrac{\partial^2 u}{\partial x^2}(t,x) + \lambda u(t,x) = f(t,x);\ -1 < x < 1,\ t > 0, \\ u(t,-1) - \varepsilon \dfrac{\partial u}{\partial x}(t,-1) = 0; \\ u(t,1) + \varepsilon \dfrac{\partial u}{\partial x}(t,1) = 0, \end{cases} \tag{2}$$

where $\varepsilon \in]0,1]$, and $u(0,x) = u_0 = g(x)$ is a given initial condition.
In this study, we focus on the case where the reaction, advection and diffusion coefficients are scalars. Let $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}^*_+$ and $\lambda \in \mathbb{R}$.

Multiplying the equation of problem (2) by v which depends only on x, and integrating by parts on I, we obtain

$$\int_{-1}^{1} \frac{\partial u}{\partial t} v(x)\, \mathrm{d}x + \beta \int_{-1}^{1} \frac{\partial u}{\partial x} \frac{dv}{dx}\, \mathrm{d}x + \alpha \int_{-1}^{1} \frac{\partial u}{\partial x}(t,x) v(x)\, \mathrm{d}x + \lambda \int_{-1}^{1} u(t,x) v(x)\, \mathrm{d}x$$
$$=$$
$$\int_{-1}^{1} f(t,x) v(x)\, \mathrm{d}x + \beta \left[\frac{\partial u}{\partial x} v(x) \right]_{-1}^{1}.$$

The boundary conditions give

$$\frac{\partial u}{\partial x}(-1) = \frac{u(-1)}{\varepsilon}, \quad \frac{\partial u}{\partial x}(1) = -\frac{u(1)}{\varepsilon}.$$

Hence the weak formulation of the problem (2) is

$$\begin{cases} \text{Find } u(t) \in H^1(I) \text{ such that} \\ \dfrac{\mathrm{d}}{\mathrm{d}t} \langle u(t), v \rangle + a(u(t), v) = \langle f(t), v \rangle, \end{cases} \tag{3}$$

with the initial condition $u(0) = g(x)$ and where

$$a(u(t),v) = \beta \int_{-1}^{1} \frac{\partial u}{\partial x}(t,x) \frac{dv}{dx}(x)\, \mathrm{d}x + \alpha \int_{-1}^{1} \frac{\partial u}{\partial x}(t,x) v(x)\, \mathrm{d}x + \lambda \int_{-1}^{1} u(t,x) v(x)\, \mathrm{d}x$$
$$+ \frac{\beta}{\varepsilon}(u(1)v(1) + u(-1)v(-1)). \tag{4}$$

Theorem 3.1 *Let $T > 0$ be a final time, $g \in L^2(I)$ be an initial data and $a(.,.)$ be the bilinear form given in (4). The following problem has a unique solution $u \in L^2(]0,T[;H^1(I)) \cap C([0,T];L^2(I))$:*

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\langle u(t), v\rangle + a(u(t), v) = \langle f(t), v\rangle, \quad \forall v \in H^1(I), \quad 0 < t < T, \\ u(t=0) = g(x). \end{cases} \tag{5}$$

In addition, we have

$$\frac{1}{2}\int_{-1}^{1} u^2(x,t)\,\mathrm{d}x + \beta\int_0^t\int_{-1}^{1}\left|\frac{\partial u}{\partial x}(s,x)\right|^2 \mathrm{d}x\,\mathrm{d}s + \lambda\int_0^t\int_{-1}^{1} u^2(s,x)\,\mathrm{d}x\,\mathrm{d}s$$
$$+\frac{\alpha}{2}\int_0^t (u^2(s,1) - u^2(s,-1))\,\mathrm{d}s + \frac{\beta}{\varepsilon}\int_0^t (u^2(s,1) + u^2(s,-1))\,\mathrm{d}s$$
$$=$$
$$\frac{1}{2}\int_{-1}^{1} u^2(0,x)\,x + \int_0^t\int_{-1}^{1} f(s,x)u(s,x)\,\mathrm{d}x\,\mathrm{d}s.$$

This leads to the following energy estimate:

$$\|u\|^2_{C([0,T];L^2(I))} + m\|u\|^2_{L^2(]0,T[;H^1(I))} \le C\left(\|u_0\|^2_{L^2} + \|f\|^2_{L^2([0,T];L^2(I))}\right). \tag{6}$$

Proof. It is clear that $a(.,.)$ is a symmetric bilinear form. So, we prove the continuity and coercivity to ensure the existence and uniqueness.

The continuity is ensured by using the Cauchy-Schwarz inequality and the fact that

$$\mid u(\pm 1) \mid \le \sup_{x\in[-1,1]} \mid u(x) \mid \ = \|u\|_{L^\infty}.$$

So, $\exists \delta > 0,\ \delta = \theta + |\alpha| + \frac{2\beta}{\varepsilon}$ such that

$$\forall u(t), v \in H^1(I) \qquad \mid a(u(t), v) \mid \le \delta \|u(t)\|_{H^1}\|v\|_{H^1}$$

for $\theta = \max_{\lambda\in\mathbb{R},\ \beta\in\mathbb{R}_+^*}\{\beta, |\lambda|\}$.

The coercivity is also ensured. In fact, we have

$$a(u(t), u(t)) \ge \beta \left\|\frac{\partial u}{\partial x}(t)\right\|^2_{L^2} + \frac{\alpha}{2}\left(u^2(t,1) - u^2(t,-1)\right) + \lambda\|u(t)\|^2_{L^2}.$$

So

- If $\alpha \ge 0$ and $\lambda \ge 0$, we have $M = \underset{\lambda\in\mathbb{R},\ \beta\in\mathbb{R}_+^*}{min}\{\beta, \lambda\} > 0$ such that

$$\forall u(t) \in H^1(I), \quad a\left(u(t), u(t)\right) \ge M\|u(t)\|^2_{H^1}.$$

- If $\alpha \ge 0$ and $\lambda \le 0$, we have $M = \beta > 0$ and $\eta = \beta - \lambda$ such that

$$a(u(t), u(t)) + \eta\|u(t)\|^2_{L^2} \ge M\|u(t)\|^2_{H^1}, \quad \forall u(t) \in H^1(I).$$

- If $\alpha < 0$ and $\lambda \geq 0$, we have $\eta = -\dfrac{\alpha}{2}$ and $M = \underset{\lambda\in\mathbb{R},\ \beta\in\mathbb{R}_+^*}{min}\{\beta, \lambda\} > 0$ such that

$$a(u(t), u(t)) + \eta\|u(t)\|_{L^2}^2 \geq M\|u(t)\|_{H^1}^2, \quad \forall u(t) \in H^1(I).$$

- If $\alpha < 0$ and $\lambda \leq 0$, we have $\eta = \beta - \dfrac{\alpha}{2} - \lambda$ and $M = \beta$ such that

$$a(u(t), u(t)) + \eta\|u(t)\|_{L^2}^2 \geq M\|u(t)\|_{H^1}^2, \quad \forall u(t) \in H^1(I).$$

Hence the existence and uniqueness of the solution of the problem (5) is proved.
For the second equality, by integrating the formula (3) on [0,t] and for all $t \in [0, T]$, we obtain the energy equality (6).

Using the previous energy equality and posing $\sigma = \underset{\beta\in\mathbb{R}_+^*,\lambda\in\mathbb{R}}{min}\{\beta, \lambda\}$, we have

$$\begin{aligned}\frac{1}{2}\|u(t)\|_{L^2}^2 + \sigma\int_0^t \|u(s)\|_{H^1}^2\,\mathrm{d}s + \frac{\alpha}{2}\int_0^t (u^2(s,1) - u^2(s,-1))\,\mathrm{d}s \\ \leq \frac{1}{2}\|u_0\|_{L^2}^2 + \int_0^t \|f(t)\|_{L^2}\|u(s)\|_{L^2}\,\mathrm{d}s + \frac{2\beta}{\varepsilon}\int_0^t \|u\|_{H^1}^2\,\mathrm{d}s.\end{aligned}$$

And from Young's algebraic inequality, there exists $k > 0$ such that

$$\begin{aligned}\|u(t)\|_{L^2}^2 + 2\left(\sigma - k - \frac{2\beta}{\varepsilon}\right)\int_0^t \|u(s)\|_{H^1}^2\,\mathrm{d}s + \alpha\int_0^t (u^2(s,1) - u^2(s,-1))\,\mathrm{d}s \\ \leq \|u_0\|_{L^2}^2 + \frac{1}{2k}\int_0^t \|f(s)\|_{L^2}^2\,\mathrm{d}s.\end{aligned}$$

By a simple calculation, we show

$$\|u(t)\|_{L^2}^2 + 2\left(\sigma - k - \frac{2\beta}{\varepsilon} - \frac{\alpha}{2}\right)\int_0^t \|u(s)\|_{H^1}^2\,\mathrm{d}s \leq \|u_0\|_{L^2}^2 + \frac{1}{2k}\int_0^t \|f(s)\|_{L^2}\,\mathrm{d}s.$$

And the desired energy estimate (6) is obtained for $m = 2\left(\sigma - k - \dfrac{2\beta}{\varepsilon} + \dfrac{\alpha}{2}\right)$ and $C = \max\limits_{k\geq 0}\left\{1, \dfrac{1}{2k}\right\}$.

4 Legendre-Galerkin Approximation for the Advection-Diffusion-Reaction Equation

Let N be a positive integer, we consider

$$\mathscr{L}_N = [\{L_0, L_1, \ldots, L_N\}].$$

We define the finite dimensional space V_N included in the space $H^1(I)$ by

$$V_N = \{v \in \mathscr{L}_N \ \ such\ that \quad v(-1) - \varepsilon v'(-1) = 0 \quad and \quad v(1) + \varepsilon\, v'(1) = 0\}.$$

Then the Legendre spectral scheme for (3) is

$$\left\{\begin{array}{c}\text{Find } \ u_N(t) \in V_N \ \text{ such that} \\ \dfrac{d}{dt}\langle u_N(t), v_N\rangle + a(u_N(t), v_N) = \langle f(t), v_N\rangle; \quad \forall\ v_N \in V_N.\end{array}\right. \tag{7}$$

At this stage, we choose as the basis functions for V_N a family of polynomials constructed from the orthogonal Legendre polynomials given as

$$\varphi_k(x) = L_k(x) + a_k L_{k+1}(x) + b_k L_{k+2}(x); \qquad k = 0, 1, 2, \dots, \tag{8}$$

where a_k and b_k are the coefficients that can be determined once φ_k verifies the boundary conditions of the problem (2). Then, for $k \geq 0$ and $\varepsilon > 0$, we obtain

$$a_k = 0; \qquad b_k = -\frac{1 + \frac{\varepsilon}{2}k(k+1)}{1 + \frac{\varepsilon}{2}(k+2)(k+3)}.$$

Since the elements of the polynomial family $\{\varphi_k\}_k$ are linearly independent, we have $V_N = [\{\varphi_0, \varphi_1, \dots, \varphi_{N-2}\}]$, and the desired approximation can be written as

$$u_N(t,x) = \sum_{k=0}^{N-2} u_k(t)\varphi_k(x). \tag{9}$$

4.1 Convergence and error estimation

Theorem 4.1 *Let u_N be the solution of the problem (7). There exists a constant C which depends on M and does not depend on N so that for any $t > 0$, we have*

$$\|u_N(t)\|_{L^2}^2 + M\int_0^t \|u_N(s)\|_{L^2}^2 \,\mathrm{d}s \leq \|u_N(0)\|_{L^2}^2 + C\int_0^t \|f(s)\|_{L^2}^2 \,\mathrm{d}s. \tag{10}$$

Proof. By taking $v = u_N(t)$, in the problem (7), we obtain for every $t > 0$,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_N(t)\|_{L^2}^2 + a(u_N(t), u_N(t)) = \langle f(t), u_N(t)\rangle_{L^2}.$$

From the coercivity of the bilinear form $a(.,.)$ and by making use of Young's algebraic inequality, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u_N(t)\|_{L^2}^2 + M\|u_N(t)\|_{L^2}^2 \leq \frac{1}{M}\|f(t)\|_{L^2}^2. \tag{11}$$

Finally, we integrate the expression (11) for $t \in [0,T]$ to have (10) with $u_N(0) = g(x)$ and the constant $C = \frac{1}{M}$ does not depend on N.

Now, to show the convergence of the proposed spectral method, we introduce the following theorem. First, we define the function e by $\mathrm{e}(t) = R_N u(t) - u_N(t)$, where

$$R_N : H^1(I) \longrightarrow V_N \; ; \qquad \|u - R_N u\|_{H^1} \xrightarrow[N \to +\infty]{} 0, \;\; \forall u \in H^1(I).$$

Theorem 4.2 *Let u be the solution of the problem (5) and u_N be the solution of the problem (7). Then we have the following error estimation:*

$$\|\mathrm{e}(t)\|^2 + M\int_0^t \|\mathrm{e}(s)\|_{H^1(-1,1)}^2 \,\mathrm{d}s \leq \|\mathrm{e}(0)\|^2 + C\int_0^t \left\|\frac{\partial u}{\partial t} - R_N\frac{\partial u}{\partial t}\right\|_{H^{1*}}^2 \mathrm{d}s + \delta\int_0^t \|(u - R_N u)\|_{H^1}^2 \,\mathrm{d}s,$$

where C and δ are two constants not depending on N.

Proof. We can write (cf. Chapter 6 in [9])

$$\frac{1}{2}\frac{d}{dt}\|\mathrm{e}\|^2 + M\|\mathrm{e}\|^2_{H^1} \leq \left|\langle\frac{\partial u}{\partial t} - R_N\frac{\partial u}{\partial t}, \mathrm{e}\rangle + a(u - R_N u, \mathrm{e})\right|. \quad (12)$$

From the continuity of $a(.,.)$ and the formula (1), we have

$$\left|\langle\frac{\partial u}{\partial t} - R_N\frac{\partial u}{\partial t}, \mathrm{e}\rangle + a(u - R_N u, \mathrm{e})\right| \leq \|\mathrm{e}\|_{H^1}\left(C\left\|\frac{\partial u}{\partial t} - R_N\frac{\partial u}{\partial t}\right\|_{H^{1*}} + \delta\|u - R_N u\|_{H^1}\right).$$

By replacing the latter inequality in (12) and integrating for $t > 0$, we obtain the desired estimate.

4.2 Implementation

In order to solve the problem (7), we start by substituting the approximation given in (9) and defined using the spectral basis (8). By taking the test functions v_N as the basis function, the spectral scheme becomes, for all $j = \overline{0, N-2}$,

$$\frac{\mathrm{d}}{\mathrm{dt}}\sum_{k=0}^{N-2} u_k(t)\langle\varphi_k, \varphi_j\rangle + \sum_{k=0}^{N-2} u_k(t)a(\varphi_k, \varphi_j) = \langle f(t), \varphi_j\rangle.$$

So

$$\langle\varphi_k, \varphi_j\rangle = \int_{-1}^{1}\varphi_k(x)\varphi_j(x)\,\mathrm{d}x, \quad \langle f(t), \varphi_j\rangle = \int_{-1}^{1} f(t,x)\varphi_j(x)\,\mathrm{d}x.$$

$$\begin{aligned} a(\varphi_k, \varphi_j) =& \beta\int_{-1}^{1}\varphi'_k(x)\varphi'_j(x)\,\mathrm{d}x + \lambda\int_{-1}^{1}\varphi_k(x)\varphi_j(x)\,\mathrm{d}x + \alpha\int_{-1}^{1}\varphi'_k(x)\varphi_j(x)\,\mathrm{d}x \\ &+ \frac{\beta}{\varepsilon}(\varphi_k(1)\varphi_j(1) + \varphi_k(-1)\varphi_j(-1)). \end{aligned}$$

Then we obtain the matrix form

$$\frac{\mathrm{d}}{\mathrm{dt}}AU(t) + BU(t) = C(t), \quad (13)$$

where $U(t) = (u_0(t), \cdots, u_{N-2}(t))^{\mathrm{T}}$ is the vector of the unknown coefficients and A and B are the $(N-1)\times(N-1)$ matrices defined by

$$A_{kj} = \langle\varphi_k, \varphi_j\rangle, \quad B_{kj} = a(\varphi_k, \varphi_j) \quad \text{and} \quad C(t) = (\langle f(t), \varphi_1\rangle, \cdots, \langle f(t), \varphi_{N-2}\rangle)^{\mathrm{T}}.$$

To solve the obtained system of ordinary differential equations (13), we propose a scheme of Crank-Nicolson. For this, we discretize the domain $[-1, 1]$ using a constant step Δx and the time domaine $[0, T]$ is discretized by a step Δt. We denote by U_i^n the value of the solution U at node x_i and at time t_n and we write the scheme as follows:

$$(A + \frac{\Delta t}{2}B)U_i^{n+1} = (A - \frac{\Delta t}{2}B)U_i^n + \frac{\Delta t}{2}(C(t_n) + C(t_{n+1})); \qquad U_i^0 = (g(x))_i, \quad (14)$$

where $(g(x))_i$ is the value of $g(x)$ in each node x_i of the discretization of $[-1, 1]$.

5 Numerical Results

In order to test the performance of the described method, we propose three examples on $[-1, 1]$, where we solve numerically the equation (2) and show the convergence of the approximate solution, when ε reaches zero, to the analytic solution of the problem with boundary conditions of Dirichlet type.

The figures are obtained for $Nx = 16$ nodes in the domain $[-1, 1]$, and 100 nodes in the time domain $[0, T]$. The error of the approximation, in the case where the Robin boundary conditions are considered, is calculated for ε fixed, according to the following formula:

$$error = \|u_N - u_{N+2}\|_\infty; \qquad N = 2, 4, \ldots, 2\ell, \ldots \tag{15}$$

Example 5.1 We consider the problem posed in (2) with $\alpha = \lambda = 0$, $\beta = 1$, $f(t, x) = 0$, $T = 3$ and the initial condition is given by $u(0, x) = \cos(\frac{\pi}{2}x)$. For the homogeneous Dirichlet boundary conditions, the analytic solution is given by

$$u(t, x) = \exp(-\frac{\pi^2}{4}t)\cos(\frac{\pi}{2}x). \tag{16}$$

In Figure 1, we observe the convergence of the obtained approximate solution, when taking the decreasing values of ε, to the analytic solution (16). The values of ε are taken between 0.08 and 0.0025 when $t = 1.5$ and between 0.25 and 0.04 when $x = 0.2$, to show that its behaviour remains the same all over the domain $[-1, 1]$ at different instances of time.

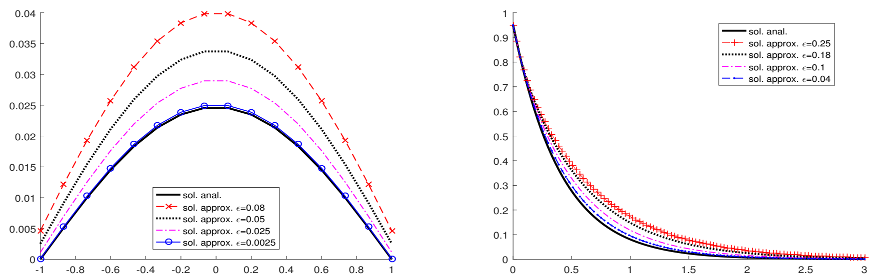


Figure 1: The behavior of the solution of (14) for $t = 1.5$ (left), and for $x = 0.2$ (right), when ε reaches 0 and $N = 6$.

Example 5.2 We consider, in (2), $\alpha = 0$, $\lambda = 0.001$, $\beta = 1$, $T = 3$. The initial condition is given by $u(0, x) = \sin(\pi x)$. For the case of the homogeneous Dirichlet boundary conditions, the analytic solution is given by

$$u(t, x) = \sin(\pi x)\exp(-\lambda t). \tag{17}$$

In Table 1, the error of approximation (15) is calculated for $N = 10$ at different points from $[-1, 1]$ using different values of ε. The obtained results show that the numerical solution of (14) converges to the exact solution of the problem with the homogeneous Dirichlet boundary conditions when ε approaches zero. The variations of the error of approximation (15) are given in Figure 2 as a function of N for fixed $\varepsilon = 0.5$. We mention here that the error of approximation is calculated for $N = 42$ and is 8.129410e − 10.

x	$\varepsilon = 0.1$	$\varepsilon = 0.01$	$\varepsilon = 0.001$	$\varepsilon = 0.0001$
−1.000000	$2.847374 \cdot 10^{-1}$	$3.100927 \cdot 10^{-2}$	$3.128715 \cdot 10^{-3}$	$3.131517 \cdot 10^{-4}$
−0.600000	$1.708432 \cdot 10^{-1}$	$1.860373 \cdot 10^{-2}$	$1.874042 \cdot 10^{-3}$	$1.845087 \cdot 10^{-4}$
−0.200000	$5.693988 \cdot 10^{-2}$	$6.194439 \cdot 10^{-3}$	$6.184256 \cdot 10^{-4}$	$5.532692 \cdot 10^{-5}$
0.066665	$1.897638 \cdot 10^{-2}$	$2.061521 \cdot 10^{-3}$	$2.030002 \cdot 10^{-4}$	$1.532227 \cdot 10^{-5}$
0.466667	$1.328897 \cdot 10^{-1}$	$1.448028 \cdot 10^{-2}$	$1.467799 \cdot 10^{-3}$	$1.536447 \cdot 10^{-4}$
0.866667	$2.467852 \cdot 10^{-1}$	$2.688325 \cdot 10^{-2}$	$2.717881 \cdot 10^{-3}$	$2.774053 \cdot 10^{-4}$

Table 1: The error of approximation as a function of x for $N = 10$.

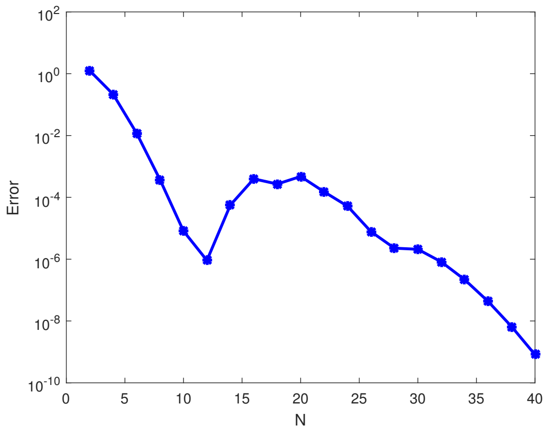
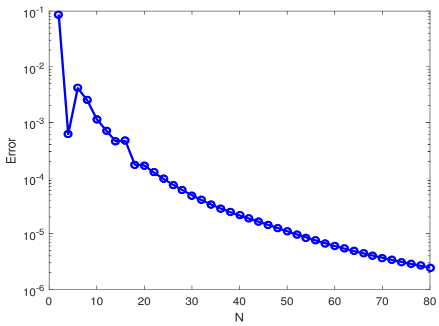


Figure 2: Logarithmic approximation error as a function of N with $\varepsilon = 0.5$.

Example 5.3 We consider, in (2), $\alpha = 0.3$, $\beta = 0.1$, $\lambda = -0.15$, $f(t, x) = 0$ and $T = 1$.The initial condition is given by $u(0, x) = cos(\pi \frac{x}{2})$.

Figure 3 presents, on the left side, the variations of the error as a function of N taking $\varepsilon = 0.1$. We see the decrease in the error curve with the growth of N. Note that the error for $N = 80$ is 2.460133e − 06. The same behavior of the error is depicted in Figure 4 for $\varepsilon = 0.001$. Moreover, on the right side, for the same value of ε and for $N = 18$, we represent the approximate solution of the problem. Finally, in Table 2, we show different values of the approximation error calculated according to the formula (15) and for $\varepsilon = 0.001$.



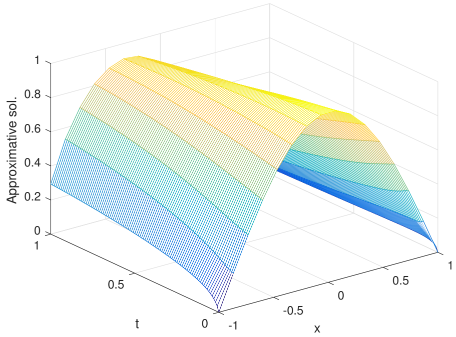


Figure 3: Logarithmic approximation error as a function of N for $\varepsilon = 0.1$ (on the left), and approximate solution of (2) for $\varepsilon = 0.1$ and $N = 18$ (on the right).

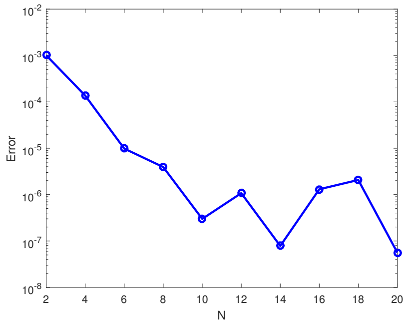


Figure 4: Logarithmic approximation error as a function of N for $\varepsilon = 0.001$.

N	error
2	$1.009365 \cdot 10^{-3}$
4	$1.364608 \cdot 10^{-4}$
6	$9.846206 \cdot 10^{-6}$
8	$3.931209 \cdot 10^{-6}$
10	$2.996409 \cdot 10^{-7}$
12	$1.075871 \cdot 10^{-6}$
14	$7.827772 \cdot 10^{-8}$
16	$1.304010 \cdot 10^{-6}$
18	$2.071257 \cdot 10^{-6}$
20	$5.646181 \cdot 10^{-8}$

Table 2: Error values as a function of N for $\varepsilon = 0.001$.

6 Conclusion

In this work, a combined algorithm of numerical methods is proposed for treating the advection-diffusion-reaction equation posed with the Robin boundary conditions perturbed with a small parameter ε. The technique is based on a Legendre-Galerkin method devoted to the spatial descretization and a Crank-Nicolson scheme to treat the obtained temporal system. The results obtained show high accuracy and good behavior, especially when comparing the approximate solution to the analytical solution of the problem posed with homogeneous boundary conditions allowing us to obtain the solution of the problem treated for different types of boundary conditions. The presented study offers a new accurate technique to approximate the solution of a partial differential equation which can be applied to approximate the solution of reaction-diffusion systems, where the explicit solution is unknown.

Acknowledgment

The authors acknowldge support from Directoriate General for Scientifc Research and Technological Development (DGRSDT), Algeria.

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