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On Stability and Convergence of a Fractional Convection Reaction-Diffusion Model

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Abstract: In this paper, we study the one-dimensional space fractional convection-diffusion problem by using a finite difference method. First, we give the mathematical model of our first initial boundary value problem. In the second step, we develop the discretization of the mathematical model and the development of the scheme for the fractional order type linear diffusion equation. For this scheme, the stability as well as convergence are studied via the Fourier method. At the end, the solutions of some numerical examples are discussed and represented graphically using Matlab. Finally, error analysis shows that the algorithm is convergent.

Keywords: *finite difference schemes; fractional derivative; Caputo fractional derivative; stability; convergence.*

Mathematics Subject Classification (2010): N65M06, 65M12, 35R11, 65L12.

1 Introduction

In this study, we consider the one-dimensional space fractional convection-diffusion problem of Caputo type of order $0 < \alpha < 1$, which is used in the modeling of chemical convection-diffusion. Several techniques for numerical resolution of this type of equation have been studied by several authors [1] - [5]. In most of these techniques, either the solutions of the integrer order differential equation versions of the given problem or the fractional differential equations with initial conditions and boundary conditions are used.

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The rest of this paper is organized as follows. The following section describes the mathematical model. Sections 2 and 3 introduce model equations and discretization of the mathematical model and development of the scheme. Stability of the approximate scheme is illustrated and described in Section 4. Section 5 describes the convergence of the approximate scheme. Finally, in Section 6, two applications of this technique are given to solve a one-dimensional space fractional convection-diffusion model, numerically.

2 Mathematical Model

We establish a novel mathematical model consisting of a one-dimensional space fractional convection-diffusion problem defined in $\Omega = [0; L]$ and $0 < \alpha \leq 1$ by

$$\frac{\partial u(x,t)}{\partial t} = -c(x)\frac{\partial u(x,t)}{\partial x} + d(x)\frac{\partial^{\alpha} u(x,t)}{\partial x^{\alpha}} + p(x,t), \qquad (x,t) \in \Omega\times]0,\mathrm{T}[, \tag{1}$$

with the boundary conditions

$$u(0,t) = u(L,t) = 0, \qquad x \in \partial\Omega, \tag{2}$$

and the initial condition

$$u(x,0) = f_0(x), \qquad x \in \Omega. \tag{3}$$

3 Discretization of the Mathematical Model and Development of the Scheme

The present study deals with the discretization of the mathematical model which describes the one-dimensional space fractional convection-diffusion problem. First, we discretise the domain [L, R]. We define

$$x_i = x_0 + ih, \ \text{ and } \ t_j = t_0 + jk, \quad \forall i = 0,1,\ldots,M \ \text{ and } \forall j = 0,1,\ldots,N, \tag{4}$$

k represents the time step size and h represents the space step length.

Let us assume that

$$u(x_i,t_j) = u_i^j, \quad p(x_i,t_j) = p_i^j, \quad c(x_i) = c_i, \quad d(x_i) = d_i, \quad f_0(x_i) = f_{0,i}. \tag{5}$$

u_i^j is the numerical approximation of $u(x_i,t_j)$.

The Caputo fractional order derivative is formulated by the structure

$$\frac{\partial^{\alpha} u(x,t)}{\partial x^{\alpha}} = \begin{cases} \frac{1}{\Gamma(1-\alpha)}\int_0^x \frac{u_\xi(\xi,t)}{(x-\xi)^{\alpha}}d\xi & \text{if} \quad 0<\alpha\leq 1, \\ u_x(x,t) & \text{if} \quad \alpha = 1. \end{cases} \tag{6}$$

Initially, as the boundary value problem needs to be discretized to be able to solve (1), it is first necessary to discretize the order space-fractional derivative.

The operator $\left(\frac{\partial u}{\partial \xi}\right)_{i+1}^{s}$ is approximated by the following formula

$$\left(\frac{\partial u}{\partial \xi}\right)_{i+1}^{s} = \frac{u_{i+1}^{s} - u_{i}^{s}}{h} + R\left(h\right). \tag{7}$$

According to (6) and (7), we get

$$\frac{\partial^{\alpha} u\left(x_{i+1}, t_{j}\right)}{\partial x^{\alpha}} = \frac{h^{-\alpha}}{\Gamma(2-\alpha)} \left[u_{i+1}^{j} - u_{i}^{j} + \sum_{n=1}^{i} \left(u_{i-n+1}^{j} - u_{i-n}^{j}\right) B_{i+1}^{j}\left(n\right)\right], \tag{8}$$

where

$$B_{i+1}^{j}\left(n\right) = \left(n+1\right)^{1-\alpha)} - n^{1-\alpha}, \quad j = 0, 1, \ldots, N-1. \tag{9}$$

Then, we use the forward difference approximation of time derivative is follows:

$$\frac{\partial u(x_{i+1}, t_{j})}{\partial t} = \frac{u_{i+1}^{j+1} - u_{i+1}^{j}}{k} + R\left(k\right). \tag{10}$$

Using approximations (8) and (10), and the linear convection-diffusion equations (1)–(3), we obtain

$$-\left(\frac{1+C_i}{A_i}\right) u_{i+1}^{j+1} + \left(1 + \frac{1}{A_i}\right) u_{i+1}^{j} = -\frac{C_i}{A_i} u_{i}^{j+1} + u_{i}^{j} - \sum_{n=1}^{i} \left(u_{i-n+1}^{j} - u_{i-n}^{j}\right) B_{i+1}^{j}\left(n\right) + \frac{k}{A_i} p_{i}^{j},$$
$$i = \overline{1, \mathrm{M}}, j = \overline{1, \mathrm{N}}, \tag{11}$$

with the boundary conditions

$$u_{0}^{j} = u_{M}^{j}, \quad j = \overline{0, \mathrm{N}-1}, \tag{12}$$

and the initial condition

$$u\left(x_{i}\right) = f_{0,i}, \quad i = 0, 1, \ldots, \mathrm{M}, \text{ where } \mathrm{A_i} = \frac{\mathrm{d_{i+1} k h^{-\alpha}}}{\Gamma(2-\alpha)} \text{ and } \mathrm{C_i} = \frac{\mathrm{c_{i+1} k}}{\mathrm{h}}. \tag{13}$$

4 Stability of the Approximate Scheme

In this section, we use the method of Fourier analysis to discuss the stability of the approximate scheme (11)–(13). Assume that the solution of the equations (11)–(13) has the form

$$u_{i}^{j} = \zeta_{i} e^{\nu \tau h j}, \quad i = 0, 1, \ldots, \mathrm{M}, \text{ where } \tau = \frac{2\pi \mathrm{m}}{\mathrm{L}} \text{ and } \nu^{2} = -1. \tag{14}$$

After that, we get

$$\zeta_{i+1} = \frac{\zeta_{i}\left(1 - \frac{C_i}{A_i} e^{\nu \tau h}\right) - \sum_{n=1}^{i}\left(\zeta_{i-n+1} - \zeta_{i-n}\right) B_{i+1}^{j}\left(n\right)}{\left[-\left(\frac{1+C_i}{A_i}\right) e^{\nu \tau h} + \left(1 + \frac{1}{A_i}\right)\right]}, \quad i = \overline{0, \mathrm{M}-1}. \tag{15}$$

Theorem 4.1 *The scheme* (11)–(13) *is unconditionally stable for* $0 < \alpha \leq 1$ *if*

$$\max_{0\leq i\leq \mathrm{M}-1}\left|\frac{3-\frac{C_i}{A_i}e^{\nu\tau h}}{-\left(\frac{1+C_i}{A_i}\right)e^{\nu\tau h}+\left(1+\frac{1}{A_i}\right)}\right|<1. \tag{16}$$

Proof. We use the proof by recurrence for $i = 1$, in view of (11)–(13),

$$\begin{aligned}|\zeta_1| &= \left|\frac{\xi_0\left(1-\frac{C_0}{A_0}e^{\nu\tau h}\right)}{\left[-\left(\frac{1+C_0}{A_0}\right)e^{\nu\tau h}+\left(1+\frac{1}{A_0}\right)\right]}\right|,\\ &\leq T\,|\zeta_0| \leq |\zeta_0| \end{aligned}\tag{17}$$

where $T = \left|\frac{1-\frac{C_0}{A_0}e^{\nu\tau h}}{-\left(\frac{1+C_0}{A_0}\right)e^{\nu\tau h}+\left(1+\frac{1}{A_0}\right)}\right| < 1.$

We assume that the statement is true:

$$|\zeta_i| \leq |\zeta_0|, \qquad i=\overline{1,\mathrm{M}} \tag{18}$$

and we prove that the statement is true:

$$|\zeta_{i+1}| \leq |\zeta_0|, \qquad i=\overline{0,\mathrm{M}-1}. \tag{19}$$

Then, we obtain

$$\begin{aligned}|\zeta_{i+1}| &= \left|\frac{\zeta_i\left(1-\frac{C_i}{A_i}e^{\nu\tau h}\right)-\sum_{n=1}^{i}\left(\zeta_{i-n+1}-\zeta_{i-n}\right)B_{i+1}^{j}\left(n\right)}{\left[-\left(\frac{1+C_i}{A_i}\right)e^{\nu\tau h}+\left(1+\frac{1}{A_i}\right)\right]}\right|,\\
&\leq \left|\frac{\zeta_i\left(1-\frac{C_i}{A_i}e^{\nu\tau h}\right)+\left|\sum_{s=0}^{i-1}\zeta_{s+1}-\zeta_s\right|}{\left[-\left(\frac{1+C_i}{A_i}\right)e^{\nu\tau h}+\left(1+\frac{1}{A_i}\right)\right]}\right|,\\
&\leq \left|\frac{3-\frac{C_i}{A_i}e^{\nu\tau h}}{-\left(\frac{1+C_i}{A_i}\right)e^{\nu\tau h}+\left(1+\frac{1}{A_i}\right)}\right|\left|\zeta_0\right|,\\
&\leq \max_{0\leq i\leq \mathrm{M}-1}\left|\frac{3-\frac{C_i}{A_i}e^{\nu\tau h}}{-\left(\frac{1+C_i}{A_i}\right)e^{\nu\tau h}+\left(1+\frac{1}{A_i}\right)}\right|\left|\zeta_0\right|,\\
&\leq \left(\max_{0\leq i\leq \mathrm{M}-1}\left|\frac{3-\frac{C_i}{A_i}e^{\nu\tau h}}{-\left(\frac{1+C_i}{A_i}\right)e^{\nu\tau h}+\left(1+\frac{1}{A_i}\right)}\right|\right)\left|\zeta_0\right|,\\
&\leq \left|\zeta_0\right|.\end{aligned}\tag{20}$$

Finally, the approximate scheme (11)–(13) is unconditionally stable.

5 Convergence of the Approximate Scheme

We start by selecting the following Fourier analysis to discuss the convergence of numerical schemes (11). Now, assume that

$$R_i^j = E_i e^{\mu\eta h j} \ \text{ and } \quad E_i^j = u(x_i,t_j)-u_i^j, \quad i=\overline{0,\mathrm{M}}, \ \ j=\overline{0,\mathrm{N}}, \tag{21}$$

where $\eta = \frac{2\pi m}{L}$ and $\mu^2 = -1$.

After that, we get

$$R_{i+1} = \frac{R_i\left(1 - \frac{C_i}{A_i}e^{\nu\tau h}\right) - \sum_{n=1}^{i}\left(R_{i-n+1} - R_{i-n}\right)B_{i+1}^{j}(n) + \zeta_i}{\left[-\left(\frac{1+C_i}{A_i}\right)e^{\nu\tau h} + \left(1 + \frac{1}{A_i}\right)\right]}, i = \overline{0, \mathrm{M}-1}. \tag{22}$$

Theorem 5.1 *The scheme* (11)–(13) *is convergent for* $0 < \alpha < 1$ *if*

$$\max_{0\le i\le \mathrm{M}-1}\left\{\frac{1}{\left|-\left(\frac{1+C_i}{A_i}\right)e^{\nu\tau h} + \left(1 + \frac{1}{A_i}\right)\right|}, \left|\frac{3 - \frac{C_i}{A_i}e^{\nu\tau h}}{-\left(\frac{1+C_i}{A_i}\right)e^{\nu\tau h} + \left(1 + \frac{1}{A_i}\right)}\right|\right\} < 1. \tag{23}$$

Proof. We use the proof by recurrence for $i = 1$, in view of (11)–(13),

$$\begin{aligned} |R_1| &= \left|\frac{R_0\left(1 - \frac{C_0}{A_0}e^{\nu\tau h}\right) + \zeta_0}{\left[-\left(\frac{1+C_0}{A_0}\right)e^{\nu\tau h} + \left(1 + \frac{1}{A_0}\right)\right]}\right| \\ &\le T\left(|R_0| + |\zeta_0|\right) \le |R_0| + |\zeta_0|, \end{aligned} \tag{24}$$

where $T^* = \max\left\{\frac{1}{\left|-\left(\frac{1+C_0}{A_0}\right)e^{\nu\tau h} + \left(1+\frac{1}{A_0}\right)\right|}, \left|\frac{1-\frac{C_0}{A_0}e^{\nu\tau h}}{-\left(\frac{1+C_0}{A_0}\right)e^{\nu\tau h} + \left(1+\frac{1}{A_0}\right)}\right|\right\} < 1$.

We assume that the statement is true:

$$|R_i| + |\zeta_i| \le |R_0| + |\zeta_0|, \quad \forall i = \overline{1, \mathrm{M}}, \tag{25}$$

and we prove that the statement is true:

$$|R_{i+1}| + |\zeta_{i+1}| \le |R_0| + |\zeta_0|, \quad \forall i = \overline{0, \mathrm{M}-1}. \tag{26}$$

By the convergence of the series on the right-hand side,

$$\exists T^* > 0 \quad |R_0| + |\zeta_0| \le T^*(k + h), \quad i = \overline{0, \mathrm{M}-1}. \tag{27}$$

Then

$$\begin{aligned} |R_{i+1}| &= \left|\frac{R_i\left(1 - \frac{C_i}{A_i}e^{\nu\tau h}\right) - \sum_{n=1}^{i}\left(R_{i-n+1} - R_{i-n}\right)B_{i+1}^{j}(n) + \zeta_i}{\left[-\left(\frac{1+C_i}{A_i}\right)e^{\nu\tau h} + \left(1 + \frac{1}{A_i}\right)\right]}\right|, \\ &\le \left|\frac{R_i\left(1 - \frac{C_i}{A_i}e^{\nu\tau h}\right) + \left|\sum_{s=0}^{i-1} R_{s+1} - R_s\right| + \zeta_i}{\left[-\left(\frac{1+C_i}{A_i}\right)e^{\nu\tau h} + \left(1 + \frac{1}{A_i}\right)\right]}\right|, \\ &\le \left|\frac{3 - \frac{C_i}{A_i}e^{\nu\tau h}}{-\left(\frac{1+C_i}{A_i}\right)e^{\nu\tau h} + \left(1 + \frac{1}{A_i}\right)}\right| |R_i| + \frac{1}{\left|-\left(\frac{1+C_i}{A_i}\right)e^{\nu\tau h} + \left(1 + \frac{1}{A_i}\right)\right|}|\zeta_i|, \\ &\le \left|\frac{3 - \frac{C_i}{A_i}e^{\nu\tau h}}{-\left(\frac{1+C_i}{A_i}\right)e^{\nu\tau h} + \left(1 + \frac{1}{A_i}\right)}\right| |R_0| + \frac{1}{\left|-\left(\frac{1+C_i}{A_i}\right)e^{\nu\tau h} + \left(1 + \frac{1}{A_i}\right)\right|}|\zeta_0|, \\ &\le C \max_{0\le i\le \mathrm{M}-1}\left\{\frac{1}{\left|-\left(\frac{1+C_i}{A_i}\right)e^{\nu\tau h} + \left(1 + \frac{1}{A_i}\right)\right|}, \left|\frac{3 - \frac{C_i}{A_i}e^{\nu\tau h}}{-\left(\frac{1+C_i}{A_i}\right)e^{\nu\tau h} + \left(1 + \frac{1}{A_i}\right)}\right|\right\}, \\ &\le C' \le (k + h), \end{aligned} \tag{28}$$

where the constant C' is given by $C' = |R_0| + |\zeta_0|$. Finally, the scheme (11)–(13) is convergent.

6 Numerical Simulation

In this section, we provide illustrative simulations that demonstrate the theoretical aspects related to stability and convergence of the fractional convection-diffusion.

Example 1. Consider the space-fractional diffusion type of problem :

$$\frac{\partial u(x,t)}{\partial t} = \Gamma(1.2)\, x^{\alpha} \frac{\partial^{\alpha} u(x,t)}{\partial x^{\alpha}} + \left(6x^3 - 3x^2\right) e^{-t}, \quad (x,t) \in \Omega\times]0, \mathrm{T}[, \tag{29}$$

with the boundary conditions $u(0,\mathrm{t}) = u(1,\mathrm{t}) = 0,\ \ x \in \partial\Omega$, and the initial condition $u(x,0) = x^2 - x^3,\ \ x \in \Omega$. The exact solution $u(x,t) = \left(x^2 - x^3\right) e^{-t},\ \ (x,t) \in \Omega\times]0, \mathrm{T}[$. The problem (29) is unconditionally stable and convergent if

$$\|d\|_{\infty} \leq \frac{h^{\alpha}\Gamma(2-\alpha)}{k}. \tag{30}$$

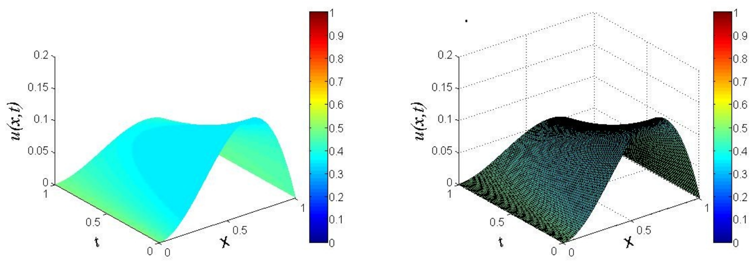

Figure 1: The right figure represents the numerical solution of $u(x,t)$ for $\alpha = 0.93, \mathrm{N} = 100$, while the left figure represents the exact solution.

Example 2. In the second example, we consider the space-fractional diffusion type of problem :

$$\frac{\partial u(x,t)}{\partial t} = x^{\frac{1}{5}} \frac{\partial u(x,t)}{\partial x} + x^{\frac{1}{100}} \frac{\partial^{\alpha} u(x,t)}{\partial x^{\alpha}} +$$
$$e^{-2t}\left(2\left(x - x^{\alpha}\right) - \Gamma(\alpha) + \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)} x^{\alpha-1} - 1\right), \tag{31}$$

with $(x,t) \in \Omega\times]0, \mathrm{T}[$ and the boundary conditions $u(0,t) = u(1,t) = 0$, where $x \in \partial\Omega$ and the initial condition $u(x,0) = x^{\alpha} - x,\ \ x \in \Omega$. The exact solution $u(x,t) = e^{-2t}\left(x^{\alpha} - x\right),\ \ (x,t) \in \Omega\times]0, \mathrm{T}[$. In this example, we present different numerical experiments to support the theoretical and numerical analyses of the previous sections. The problem (31) is unconditionally stable and convergent.

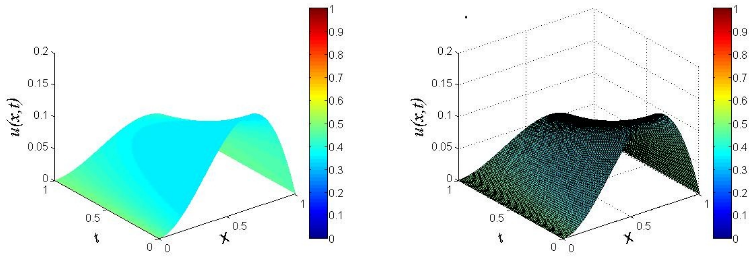

Figure 2: The right figure represents the numerical solution of $u\,(x,t)$ for $\alpha = 0.8, \mathrm{N} = 100$, while the left figure represents the exact solution.

7 Conclusion

In this paper, the one-dimensional space fractional convection-diffusion problem with initial and boundary conditions in a bounded domain is studied by using a finite difference method. The fractional derivative is approximated by the finite difference approximations for space derivatives and Caputo's concept for time-fractional derivatives. Two numerical examples with the known exact solutions are considered to validate theoretical results and demonstrate the accuracy of the method proposed in this paper.

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