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A New Generalization of Fuglede's Theorem and Operator Equations

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Abstract: In this paper, the operator equations $AX - XB = C$ and $AXB - X = C$, where A, B, C and X are bounded linear operators on the Hilbert space $\mathcal{H}$, are investigated and criteria of solvability are established. First, in a Hilbertian framework, by extending the famous Fuglede's theorem to a certain class of operators that are not necessarily normal, we show that some classical criteria, as Roth's removal rule for the first equation, remain valid even under assumptions on A and B weaker than usual. Second, in a Banachian framework, we establish our criteria of solvability by using the inner inverses of the operators $\delta_{A,B}$ and $\Delta_{A,B}$ defined on $L(\mathcal{H})$ by $\delta_{A,B}(X) = AX - XB$ and $\Delta_{A,B}(X) = AXB - X$.

Keywords: *Fuglede-Putnam theorem; elementary operators; operator equations; inner inverses.*

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1 Introduction and Basic Definition

Let $\mathcal{H}$ be an infinite complex Hilbert space and $L(\mathcal{H})$ be the Banach space of all bounded linear operators from $\mathcal{H}$ into $\mathcal{H}$. For $T \in L(\mathcal{H})$, let $\ker(T)$, $\mathcal{R}(T)$, $\sigma(T)$ and $\sigma_p(T)$ stand for the null space, range, spectrum and point spectrum of T, respectively. We recall some definitions of the local spectral theory.

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Definition 1.1 An operator $T \in L(\mathcal{H})$ is said to have the single-valued extension property at $\lambda_0 \in \mathbb{C}$ (abbreviated SVEP at λ_0) if for every open disc $\mathbb{D}$ centered at λ_0, the only analytic function $f : \mathbb{D} \to X$, which satisfies the equation $(T - \lambda I)f(\lambda) = 0$ for all $\lambda \in \mathbb{D}$, is the function $f \equiv 0$. An operator $T \in L(\mathcal{H})$ is said to have the SVEP if T has the SVEP at every $\lambda \in \mathbb{C}$.

Definition 1.2 An operator $T \in L(\mathcal{H})$ is said to have Bishop's property (β) if for any open subset V of $\mathbb{C}$ and any sequence $\{f_n\}$ of $\mathcal{H}$-valued analytic functions on V, the convergence of $(T - \lambda)f_n(\lambda)$ to zero uniformly on each compact subset of V leads to the convergence of $f_n(\lambda)$ to zero again uniformly on each compact subset of V.

Definition 1.3 An operator $T \in L(\mathcal{H})$ is said to be decomposable if for every open cover $\{U, V\}$ of $\mathbb{C}$, there are T-invariant subspaces $\mathcal{X}$ and $\mathcal{Y}$ such that $\mathcal{H} = \mathcal{X} + \mathcal{Y}$, $\sigma(T|_{\mathcal{X}}) \subset \overline{U}$, and $\sigma(T|_{\mathcal{Y}}) \subseteq \overline{V}$.

The following implications are always satisfied:

$$T \text{ is decomposable} \Rightarrow T \text{ has Bishop's property } (\beta) \Rightarrow T \text{ has the SVEP.}$$

Recall that the ascent $p(T)$ and descent $q(T)$ of T are defined by

$$p(T) = \inf\{n \in \mathbb{N} : \ker T^n = \ker T^{n+1}\},$$

$$q(T) = \inf\{n \in \mathbb{N} : \mathcal{R}(T^n) = \mathcal{R}(T^{n+1})\}$$

with $\inf \emptyset = \infty$. It is well known that if $p(T)$ and $q(T)$ are both finite, then $p(T) = q(T)$. We denote by $\Pi(T) = \{\lambda \in \mathbb{C} : p(T - \lambda I) = q(T - \lambda I) < \infty\}$ the set of poles of the resolvent. In the sequel, we shall denote by $accS$ and $isoS$ the set of accumulation points and the set of isolated points of $S \subset \mathbb{C}$, respectively.

Definition 1.4 We say that $T \in L(\mathcal{H})$ is polaroid if for any isolated point λ in $\sigma(T)$, λ is a pole of the resolvent of T (i.e., $iso\sigma(T) \subseteq \Pi(T)$).

Fuglede's theorem states that if an operator commutes with a normal operator, it also commutes with its adjoint, i.e., if X and A are in $L(\mathcal{H})$ with A normal, then

$$AX = XA \Longrightarrow A^*X = XA^*,$$

this was first proven in 1950 by B. Fuglede [19] and then by C.R.Putnam [23] in a more general version. Thanks to its numerous applications, this theorem has a very effective role in the theory of bounded operators. There are different proofs of this theorem, besides, the first two are due to Fuglede and Putnam, see [20]. Perhaps the most elegant proof is due to Rosenblum [24]. Then, with a wonderful matrix operator trick, S.Berberian [10] showed the equivalence between Fuglede's theorem and that of Putnam. Afterwards, it was called the Fuglede-Putnam theorem. This theorem is therefore stated as follows: if X, A and B are bounded Hilbert space operators such that A and B are normal, then

$$AX = XB \Longrightarrow A^*X = XB^*.$$

This theorem has been extended by relaxing the normality hypotheses on A and B to various classes of non-normal operators. It has also been formulated using the elementary operator $\delta_{A,B}$ as follows: if A and B are normal operators, then $\ker \delta_{A,B} \subset \ker \delta_{A^*,B^*}$, where $\delta_{A,B}$ is the generalized derivation defined on $L(\mathcal{H})$ by $\delta_{A,B}(X) = AX - XB$. The Fuglede-Putnam theorem has a (natural) analogue: if A and B are normal, then $\ker \Delta_{A,B} \subset \ker \Delta_{A^*,B^*}$, where $\Delta_{A,B}$ is the elementary operator defined on $L(\mathcal{H})$ by $\Delta_{A,B}(X) = AXB - X$.

In the following, we will denote by $d_{A,B}$ each of elementary operators $\Delta_{A,B}$ or the generalized derivation $\delta_{A,B}$.

In the second section of this paper, we derive a nice generalization of Fuglede's theorem for decomposable operators $A \in L(\mathcal{H})$ which are polaroid with A and A^* being reduced by each of eigenspaces, using examples of non-normal operators, we justify that the set of such operators strictly contains the normal operators. The third section is devoted to the application of these results to give necessary and sufficient conditions for the existence of solutions to the operator equations $AX - XB = C$ and $AXB - X = C$ in this general framework, which presents a generalization of the results obtained by S. Schweinsberg in [27]. In the last section, independently of the previous ones, using the inner inverse of the elementary operator $d_{A,B}$, we give necessary and sufficient conditions for the existence of solutions to the operator equations $d_{A,B}(X) = C$ and also, the form of these solutions.

2 An Extension of Fuglede's Theorem

In [21], the authors proved the following theorem.

Theorem 2.1 *[21, Theorem 2.2, Theorem 2.3]*
Suppose that $A, B \in L(\mathcal{H})$ satisfies the following conditions:

i) A and B^* are reduced by each of their eigenspaces,

ii) A and B^* are polaroid,

iii) A and B^* have property (β).

Then

$$\ker(d_{A,B} - \lambda I) \subseteq \ker(d_{A^*,B^*} - \overline{\lambda} I)$$

holds for every complex number λ, which means that the Fuglede-Putnam theorem holds.

This theorem is established for many classes of operators, we mention, for example, the operators $A \in L(\mathcal{H})$, which satisfy the equation

$$(A^*)^2A^2 - 2A^*A + I = 0,$$

such A are natural generalizations of isometric operators ($A^*A = I$) and are called 2-isometric operators. It is known that an isometric operator is a 2-isometric operator. 2-isometric operators have been studied by many authors and they have many interesting properties, see [2, 3, 13, 16, 22] for example. In [28, Lemma 2.6], the authors proved that 2-isometric operators have Bishop's property (β) and in [28, Corollary 2.5] they proved that 2-isometric operators are reduced by each of their eigenspaces. In [15, Proposition 2.1], B. P. Duggal proved that 2-isometric operators are polaroid. Then we have the following examples.

Example 2.1 Suppose that A and B^* are 2-isometric operators. Then

$$\ker(d_{A,B} - \lambda I) \subseteq \ker(d_{A^*,B^*} - \overline{\lambda} I), \quad \forall \lambda \in \mathbb{C}.$$

Example 2.2 Suppose that A and B^* are unilateral weighted shift operators on l_2 defined by $Ae_n = \alpha_n e_{n+1}$ and $B^*e_n = \beta_n e_{n+1}$ for all $n \geq 0$ and such that $\alpha_n^2\alpha_{n+1}^2 - 2\alpha_n^2 + 1 = 0$ and $\beta_n^2\beta_{n+1}^2 - 2\beta_n^2 + 1 = 0$ for all $n \geq 0$, where $\{e_n\}_n^\infty$ is a canonical orthogonal basis for l_2 and $(\alpha_n)_{n\in\mathbb{N}}$ and $(\beta_n)_{n\in\mathbb{N}}$ are bounded sequences of non-negative numbers. Then A and B^* are 2-isometric, it follows that

$$\ker(d_{A,B} - \lambda I) \subseteq \ker(d_{A^*,B^*} - \overline{\lambda} I), \quad \forall \lambda \in \mathbb{C}.$$

The following theorem forms an interesting generalization of Fuglede's theorem to a set larger than that of the normal operators.

Theorem 2.2 *Suppose that $A \in L(\mathcal{H})$ satisfies the following conditions:*

i) *A is decomposable,*

ii) *A is polaroid,*

iii) *A and A^* are reduced by each of their eigenspaces ($\ker(A - \lambda I) = \ker(A^* - \overline{\lambda I})$, $\forall \lambda \in \sigma_p(A)$).*

Then $\ker(\delta_{A,A} - \lambda I) \subseteq \ker(\delta_{A^*,A^*} - \overline{\lambda} I), \quad \forall \lambda \in \mathbb{C}$.

Proof. If A is decomposable, it follows that A and A^* have property (β), on the other hand, it is well known that A is polaroid if and only if A^* is polaroid. Then we obtain the result.

We note that normal operators A on a Hilbert space are decomposable, polaroid, A and A^* are reduced by each of their eigenspaces. We note also that the class of operators A which are decomposable, polaroid, A and A^* are reduced by each of their eigenspaces, contains strictly normal operators. Since E. Albrecht in [6, Proposition 5.1] constructed a non normal, subnormal operator S which is decomposable and since subnormal operators (their adjoint too) are hyponormal, it follows that S is polaroid, S and S^* are reduced by each of their eigenspaces. Another interesting class of bounded operators from which the conditions of the previous theorem are satisfied, is the class of compact p-symmetric operators. Now we recall the definition of p-symmetric operators.

Definition 2.1 [11, Definition 1.2] Let $A \in L(\mathcal{H})$, where $\mathcal{H}$ is a separable complex Hilbert space. A is called p-symmetric if $AT = TA$ implies $A^*T = TA^*$ for all trace class operators T.

Proposition 2.1 *Let $A \in L(\mathcal{H})$, where $\mathcal{H}$ is a separable complex Hilbert space. If A is compact and p-symmetric, then A is decomposable polaroid, A and A^* are reduced by each of their eigenspaces. Therefore* $\ker(d_{A,A} - \lambda I) \subseteq \ker(d_{A^*,A^*} - \overline{\lambda} I), \quad \forall \lambda \in \mathbb{C}$.

Proof. It is well known that compact operators are decomposable, and from [9, Corollary V.10.3], compact operators are polaroid. We have A is compact, it follows that if $\lambda \in \sigma_p(A)$, then $\overline{\lambda} \in \sigma_p(A^*)$, since A is p-symmetric, then from [11], we deduce that A is reduced by each of its eigenspaces. Since A^* is also compact and p-symmetric, we deduce that A^* is reduced by each of its eigenspaces.

Now we give another example which satisfies the conditions of Theorem 2.2. In [7], S.A. Alzraiqi and A.B. Patel introduced the class of n-normal operators, we recall that an operator $A \in L(\mathcal{H})$ is said to be an n-normal operator if $A^nA^* = A^*A^n$.

Example 2.3 Let $A \in L(\mathcal{H})$ such that A is 2-normal, $\sigma(A) \cap (-\sigma(A)) \subset \{0\}$ and $kerA = kerA^*$, then from [14, Theorem 4.4], A is decomposable and from [14, Theorem 2.3], A is polariod and it is reduced by each of its eigenspaces. Since A^* is also 2-normal, then A^* is polariod and it is reduced by each of its eigenspaces and it follows that $\ker(d_{A,A} - \lambda I) \subseteq \ker(d_{A^*,A^*} - \overline{\lambda} I), \quad \forall \lambda \in \mathbb{C}$.

Example 2.4 Let $A = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}$ be an operator acting in a two-dimensional complex Hilbert space. Then A is 2-normal, $\sigma(A) \cap (-\sigma(A)) \subset \{0\}$ and $kerA = kerA^*$, then A is decomposable, polaroid and A and A^* are reduced by each of their eigenspaces. Then

$$\ker(d_{A,A} - \lambda I) \subseteq \ker(d_{A^*,A^*} - \overline{\lambda} I), \quad \forall \lambda \in \mathbb{C}.$$

3 Solvability Criteria for the Equation $d_{A,B}(X) = C$ in a Hilbertian Framework

Mathematicians often try to find suitable solutions to problems in a wide range of fields by using various methods, and to study the properties of solutions such as existence, uniqueness, stability, and so on. See, for example, [4] and [5]. The previous results are very useful for solving the equation $d_{A,B}(X) = C$ in a more general setting. Let us first recall that in [26], W. E. Roth proved for finite matrices over a field that $AX - XB = C$ is solvable for X if and only if the matrices $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ and $\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ are similar. A considerably briefer proof has been given by Flanders and Wimmer [18]. In [25], Rosenblum showed that the result remains true when A and B are bounded self-adjoint operators on a complex separable Hilbert space. In [27], A. Schweinsberg extended the result to include finite rank operators and normal operators on a Hilbert space. In this part, we generalize it to the operators $A, B \in L(\mathcal{H})$ satisfying the conditions given below.

Theorem 3.1 *Suppose that $A, B \in L(\mathcal{H})$ satisfy the following conditions:*

i) A^* have property (β), and B is decomposable.

ii) A^* and B are polaroid,

iii) A^*, B and B^* are reduced by each of their eigenspaces.

Then the operator equation $AX - XB = C$ has a solution if and only if $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ and $\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ are similar.

Proof. If the equation $AX - XB = C$ has a solution X, then

$$\begin{pmatrix} I & -X \\ 0 & I \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}\begin{pmatrix} I & X \\ 0 & I \end{pmatrix} = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}.$$

Hence $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ and $\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ are similar.

Suppose that $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ and $\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ are similar, then there exists an invertible operator $\begin{pmatrix} Q & R \\ S & T \end{pmatrix}$ such that $\begin{pmatrix} Q & R \\ S & T \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}\begin{pmatrix} Q & R \\ S & T \end{pmatrix}$, this implies that

$$QA - AQ = CS, RB - AR = CT$$
$$SA = BS, TB = BT. \tag{3.1}$$

We apply Theorem 2.2 above, we get

$$\ker(d_{B,B} - \lambda I) \subseteq \ker(d_{B^*,B^*} - \overline{\lambda} I), \quad \forall \lambda \in \mathbb{C},$$

also from [21, Theorem2.2], we obtain

$$\ker(d_{B,A} - \lambda I) \subseteq \ker(d_{B^*,A^*} - \overline{\lambda} I), \quad \forall \lambda \in \mathbb{C}.$$

Thus, the equality (3.1) gives

$$SA^* = B^*S, \quad TB^* = B^*T, \tag{3.2}$$

and by taking the adjoint in (3.2), we have $AS^* = S^*B, \quad BT^* = T^*B$, which ensures that B commutes with SS^* and TT^*. We have also

$$C(SS^* + TT^*) = (QS^* + RT^*)B - A(QS^* + RT^*).$$

We apply the result from [27, Lemma 1], we deduce that there exists $X = -(QS^* + RT^*)(SS^* + TT^*)^{-1}$, which is the solution to the operator equation $AX - XB = C$.

Corollary 3.1 *Suppose that $A, B \in L(\mathcal{H})$ such that $\mathcal{H}$ is a separable Hilbert space. If*

i) A^* is 2-isometric,

ii) B is compact and p-symetric,

then the operator equation $AX - XB = C$ has a solution if and only if $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ and $\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ are similar.

If we set $B = A$ in Theorem 3.1, we get the following corollary.

Corollary 3.2 *Let $A \in L(\mathcal{H})$ satisfy*

i) A is decomposable and polaroid,

ii) A and A^* are reduced by each of their eigenspaces.

Then the operator equation $AX - XA = C$ has a solution if and only if $\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$ and $\begin{pmatrix} A & C \\ 0 & A \end{pmatrix}$ are similar.

Corollary 3.3 *Let $A \in L(\mathcal{H})$ such that $\mathcal{H}$ is a separable Hilbert space. If A is compact and p-symmetric, then the operator equation $AX - XA = C$ has a solution if and only if $\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$ and $\begin{pmatrix} A & C \\ 0 & A \end{pmatrix}$ are similar.*

As a consequence of Corollary 3.2, we obtain a well known theorem of A. Schweinsberg [27, Theorem 1]:

Corollary 3.4 *[27, Theorem 1] Let $A \in L(\mathcal{H})$ be a normal operator. Then the operator equation $AX - XA = C$ has a solution if and only if $\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$ and $\begin{pmatrix} A & C \\ 0 & A \end{pmatrix}$ are similar.*

Example 3.1 Let A^* be a 2-isometric operator and B be a 2-normal operator on a Hilbert space $\mathcal{H}$, $\sigma(A) \cap (-\sigma(A)) \subset \{0\}$ and $kerA = kerA^*$, then the equation $AX - XB = C$ has a solution if and only if $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ and $\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ are similar.

Example 3.2 Let $A \in L(\mathcal{H})$ such that A is 2-normal, $\sigma(A) \cap (-\sigma(A)) \subset \{0\}$ and $kerA = kerA^*$. Then the operator equation $AX - XA = C$ has a solution if and only if $\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$ and $\begin{pmatrix} A & C \\ 0 & A \end{pmatrix}$ are similar.

We also get similar results for the equation $AXB - X = C$.

Theorem 3.2 *Let $A, B \in L(\mathcal{H})$ such that*

i) A is decomposable and polaroid.

ii) A and A^* are reduced by each of their eigenspaces.

iii) B has property (β), is polaroid and reduced by each of its eigenspaces.

Then the equation $AXB - X = C$ has a solution in $L(\mathcal{H})$ if and only if there exist two invertible operators U and V such that $U\begin{pmatrix} A & C \\ 0 & I \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}V$ and $U\begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix}V$.

Proof. If X is a solution of $AXB - X = C$, then $AXB = C + X$.

Let $U = \begin{pmatrix} I & X \\ O & I \end{pmatrix}$ and $V = \begin{pmatrix} I & XB \\ O & I \end{pmatrix}$, it is clear that U and V are invertible, in addition, we have

$$U\begin{pmatrix} A & C \\ 0 & I \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}V \text{ and } U\begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix}V.$$

Conversely, assume that there exist two invertible operators $U = \begin{pmatrix} Q & R \\ S & T \end{pmatrix}$ and $V = \begin{pmatrix} Q_1 & R_1 \\ S_1 & T_1 \end{pmatrix}$ such that

$$U\begin{pmatrix} A & C \\ 0 & I \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}V \text{ and } U\begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix}V,$$

so

$$\begin{cases} QA = AQ_1, & (1) \\ SA = S_1, & (2) \\ QC + R = AR_1, & \\ SC + T = T_1. & (3) \end{cases} \quad \text{and} \quad \begin{cases} Q = Q_1, & (4) \\ RB = R_1, & \\ S = BS_1, & (5) \\ TB = BT_1. & \end{cases}$$

From (1) and (4), we have $AQ = QA$, then, according to Theorem 2.2 and by taking the adjoint, we have $AQ^* = Q^*A$, consequently, we get $AQ^*Q = Q^*QA$. From (2) and (5), we have $BSA = S$; knowing that if A is decomposable, then A^* has property (β), moreover, if A is polaroid, then A^* is too, which allows us to obtain from [21, Theorem 2.3] that $B^*SA^* = S$, and by taking the adjoint, we get $AS^*B = S^*$, which implies that $S^*SA = (AS^*B)SA = (AS^*B)S_1 = AS^*S$. Therefore, A commutes with the sum Q^*Q+S^*S and so with the inverse $(Q^*Q + S^*S)^{-1}$, which exists according to [27, Lemma 1].

In addition, from (3), we have $S^*SC = S^*T_1 - S^*T = A(S^*T)B - S^*T$. Therefore

$$\begin{aligned}(Q^*Q + S^*S)\,C &= Q^*(AR_1 - R) + A(S^*T)B - S^*T,\\ &= Q^*ARB - Q^*R + A(S^*T)B - S^*T,\\ &= A(Q^*R + S^*T)B - (Q^*R + S^*T),\end{aligned}$$

and so

$$\begin{aligned}C &= (Q^*Q + S^*S)^{-1}\,A(Q^*R + S^*T)B - (Q^*Q + S^*S)^{-1}\,(Q^*R + S^*T),\\ &= A\,(Q^*Q + S^*S)^{-1}\,(Q^*R + S^*T)B - (Q^*Q + S^*S)^{-1}\,(Q^*R + S^*T),\end{aligned}$$

which means that $X = (Q^*Q + S^*S)^{-1}\,(Q^*R + S^*T)$ is a solution of the equation $\Delta_{A,B}(X) = C$, and the proof is complete.

Corollary 3.5 *Suppose that $A, B \in L(\mathcal{H})$ such that $\mathcal{H}$ is a separable Hilbert space. If*

i) A is compact and p-symmetric,

ii) B is 2-isometric,

then the operator equation $AXB - X = C$ has a solution if and only if there exist two invertible operators U and V such that
$U\begin{pmatrix} A & C \\ 0 & I \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix} V$ *and* $U\begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix} V.$

If we set $B = A$ in Theorem 3.2, we get the following corollary.

Corollary 3.6 *Let $A \in L(\mathcal{H})$ satisfy*

i) A is decomposable and polaroid,

ii) A and A^* are reduced by each of their eigenspaces.

Then the operator equation $AXA^* - X = C$ has a solution if and only if there exist two invertible operators U and V such that
$U\begin{pmatrix} A & C \\ 0 & I \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix} V$ *and* $U\begin{pmatrix} I & 0 \\ 0 & A^* \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & A^* \end{pmatrix} V.$

Corollary 3.7 *Suppose that $A \in L(\mathcal{H})$ such that $\mathcal{H}$ is a separable Hilbert space. If A is compact and p-symmetric, then the operator equation $AXA^* - X = C$ has a solution if and only if there exist two invertible operators U and V such that* $U\begin{pmatrix} A & C \\ 0 & I \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix} V$ *and* $U\begin{pmatrix} I & 0 \\ 0 & A^* \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & A^* \end{pmatrix} V.$

Example 3.3 Let $A \in L(\mathcal{H})$ such that A is 2-normal, $\sigma(A) \cap (-\sigma(A)) \subset \{0\}$ and $kerA = kerA^*$. Then the operator equation $AXA^* - X = C$ has a solution if and only if there exist two invertible operators U and V such that $U \begin{pmatrix} A & C \\ 0 & I \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix} V$ and $U \begin{pmatrix} I & 0 \\ 0 & A^* \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & A^* \end{pmatrix} V$.

4 Solvability Criteria for the Equation $d_{A,B}(X) = C$ in a Banachian Framework

Let E be a Banach space, and $d_{A,B} \in L(L(E))$, in this section, we give necessary and sufficient conditions for regularity of the elementary operator $d_{A,B}$, then we deduce necessary and sufficient conditions for the existence of solutions to the operator equations $d_{A,B}(X) = C$, using the inner inverses of the elementary operator $d_{A,B}$. First, we recall the following definitions.

Definition 4.1 Let $A \in L(E)$. An operator $B \in L(E)$ is said to be an inner inverse of A if it satisfies the equation

$$ABA = A.$$

We denote the inner inverse by A^-. An operator with an inner inverse will be called regular.

Remark 4.1 We note that

1. $A \in L(E)$ has an inner inverse if and only if $\ker(A)$ and $\mathcal{R}(A)$ are closed and complemented subspaces of E.

2. If A has an inverse A^{-1} in $L(E)$, then A^{-1} is the only inner inverse of A.

Theorem 4.1 *Suppose that $A, B \in L(E)$ are polaroid, $p(\delta_{A,B}) \leq 1$ and $\delta^*_{A,B}$ has the SVEP at 0, then the following conditions are pairwise equivalent:*

1. *$\delta_{A,B}$ has a closed range,*

2. *$L(E) = \ker(\delta_{A,B}) \oplus \mathcal{R}(\delta_{A,B})$,*

3. *$0 \in iso\sigma(\delta_{A,B})$,*

4. *$\delta_{A,B}$ is regular.*

Proof. The equivalences $1 \Leftrightarrow 2 \Leftrightarrow 3$ have been proven by the authors in [17, Theorem 3.2]. The condition (4) is equivalent to (1). Indeed, if $\delta^-_{A,B}$ is an inner inverse of $\delta_{A,B}$, then $\delta_{A,B}\delta^-_{A,B}\delta_{A,B}\delta^-_{A,B} = \delta_{A,B}\delta^-_{A,B}$, i.e., $\delta_{A,B}\delta^-_{A,B}$ is a projection on the closed subspace $\mathcal{R}(\delta_{A,B}\delta^-_{A,B})$. Moreover, $\mathcal{R}(\delta_{A,B}) = \mathcal{R}(\delta_{A,B}\delta^-_{A,B}\delta_{A,B}) \subseteq \mathcal{R}(\delta_{A,B}\delta^-_{A,B}) \subseteq \mathcal{R}(\delta_{A,B})$, so $\mathcal{R}(\delta_{A,B}\delta^-_{A,B}) = \mathcal{R}(\delta_{A,B})$, and it is therefore closed. Conversely, if $\mathcal{R}(\delta_{A,B})$ is closed, then $P_{\mathcal{R}(\delta_{A,B})}$ is a bounded linear operator and, by the Douglas theorem, the equation $\delta_{A,B}X = P_{\mathcal{R}(\delta_{A,B})}$ admits a solution; that is, there exists B in $L(E)$ such that $\delta_{A,B}B = P_{\mathcal{R}(\delta_{A,B})}$. Then $\delta_{A,B}B\delta_{A,B} = \delta_{A,B}$ and therefore $\delta_{A,B}$ has an inner inverse.

Corollary 4.1 *Suppose that $A, B \in L(E)$ are polaroid, $p(\delta_{A,B}) \leq 1$ and $\delta^*_{A,B}$(the dual of $\delta_{A,B}$) has the SVEP at 0 and $C \in L(E)$. If $0 \in iso\sigma(\delta_{A,B})$, then the operator equation $\delta_{A,B}(X) = C$ has a solution if and only if*

$$\delta_{A,B}\delta^-_{A,B}C = C.$$

In this case, the general solution is

$$X = \delta^-_{A,B}C + (I_{L(E)} - \delta^-_{A,B}\delta_{A,B})U,$$

where $U \in L(E)$ is an arbitrary operator.

Proof. We apply Theorem 4.1, we deduce that $\delta_{A,B}$ is regular, and from [12], we get the result.

Corollary 4.2 *Suppose that $A, B \in L(\mathcal{H})$ are normal operators and $C \in L(\mathcal{H})$. If $0 \in iso\sigma(\delta_{A,B})$, then the operator equation $\delta_{A,B}(X) = C$ has a solution if and only if*

$$\delta_{A,B}\delta^-_{A,B}C = C.$$

In this case, the general solution is

$$X = \delta^-_{A,B}C + (I_{L(\mathcal{H})} - \delta^-_{A,B}\delta_{A,B})U,$$

where $U \in L(\mathcal{H})$ is an arbitrary operator.

Proof. If A and B are normal operators, it follows that A and B are polaroid, $p(\delta_{A,B} - \lambda) \leq 1$ holds for every complex number λ, and $\delta^*_{A,B}$ has the SVEP at 0. Hence the result follows from Theorem 4.1 and Corollary 4.1.

Theorem 4.2 *Suppose that $A, B \in L(E)$ are contractions, then the following conditions are pairwise equivalent:*

1. *$\Delta_{A,B}$ has a closed range,*
2. *$L(E) = \ker(\Delta_{A,B}) \oplus \mathcal{R}(\Delta_{A,B})$,*
3. *$0 \in iso\sigma(\Delta_{A,B})$,*
4. *$\Delta_{A,B}$ is regular.*

Proof. The equivalences $1 \Leftrightarrow 2 \Leftrightarrow 3$ have been proven by the authors in [17, Theorem 3.2], and in the same way as in Theorem 4.1, we show that (4) is equivalent to (1).

Corollary 4.3 *Suppose that $A, B \in L(E)$ are contractions and $C \in L(E)$. If $0 \in iso\sigma(\Delta_{A,B})$, then the operator equation $\Delta_{A,B}(X) = C$ has a solution if and only if $\Delta_{A,B}\Delta^-_{A,B}C = C$. In this case, the general solution is $X = \Delta^-_{A,B}C + (I_{L(E)} - \Delta^-_{A,B}\Delta_{A,B})U$, where $U \in L(E)$ is an arbitrary operator.*

5 Conclusion

Many researchers have focused on studying equations of the form $AX - XB = C$ and $AXB - X = C$ due to their significance in solving various problems in many fields such as physics, biology, economics, etc. They have achieved considerable results in this regard.

This work is part of the same context where we presented, in the first part, important results, represented by the provision of the necessary and sufficient conditions for these equations to have solutions in the Hilbertian framework, through an important extension of the theorem of Fugleg, while giving the general form of expression of these solutions.

These results represent a natural and important extension of many previously known results to much broader classes of operators than usual. Examples of applications have been included, as well as some corollaries of these results.

In the second part, within the Banachian framework and using generalized inverse operators, we provided the necessary and sufficient conditions for these equations to have solutions, as well as the expression in general form of these solutions, and also their important implications.

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