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A New Generalization of Fuglede's Theorem and Operator Equations

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Abstract: In this paper, the operator equations $AX - XB = C$ and $AXB - X = C$, where A, B, C and X are bounded linear operators on the Hilbert space $\mathcal{H}$, are investigated and criteria of solvability are established. First, in a Hilbertian framework, by extending the famous Fuglede's theorem to a certain class of operators that are not necessarily normal, we show that some classical criteria, as Roth's removal rule for the first equation, remain valid even under assumptions on A and B weaker than usual. Second, in a Banachian framework, we establish our criteria of solvability by using the inner inverses of the operators $\delta_{A,B}$ and $\Delta_{A,B}$ defined on $L(\mathcal{H})$ by $\delta_{A,B}(X) = AX - XB$ and $\Delta_{A,B}(X) = AXB - X$.

Keywords: *Fuglede-Putnam theorem; elementary operators; operator equations; inner inverses.*

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1 Introduction and Basic Definition

Let $\mathcal{H}$ be an infinite complex Hilbert space and $L(\mathcal{H})$ be the Banach space of all bounded linear operators from $\mathcal{H}$ into $\mathcal{H}$. For $T \in L(\mathcal{H})$, let $\ker(T)$, $\mathcal{R}(T)$, $\sigma(T)$ and $\sigma_p(T)$ stand for the null space, range, spectrum and point spectrum of T, respectively. We recall some definitions of the local spectral theory.

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