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Boundedness in Nonlinear Oscillatory Systems over a Given Time Interval

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Abstract: This paper considers three types of perturbed motion equations with a stable linear (nonlinear) approximation. New sufficient conditions are established for the boundedness of motion on a finite interval with respect to a given Lyapunov function. The conditions are obtained on the basis of the direct Lyapunov method and the method of integral inequalities.

Keywords: *equations of perturbed motion; stable approximation; boundedness with respect to given function.*

Mathematics Subject Classification (2010): 34D40, 34D20, 70K40.

1 Introduction

Non-autonomous systems of equations, applicable in nonlinear mechanics [1], are studied by various methods (see [2–7] and the bibliography therein). The Lyapunov function method [8], combined with the method of integral inequalities (see [1, 9]), allows establishing new conditions for the boundedness of motion over a specified time interval. This paper is structured as follows.

Section 2 discusses a system of two scalar equations with nonlinear stable approximation. Definitions of motion boundedness with respect to a positive definite function are provided.

In Section 3, an estimation of the Lyapunov function is established.

Section 4 presents conditions for the boundedness of motion with respect to a positive definite function.

Section 5 addresses the problem of boundedness of solutions to equation systems with autonomous stable approximation.

In Section 6, conditions for boundedness are established in the case of stability of non-autonomous linear approximation.

Section 7 provides conditions for the boundedness of solutions over a specified time interval for perturbed motion equations in the normal Cauchy form.

The concluding section offers comments on the obtained results.

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2 Formulation of the Problem

Consider a system of perturbed motion equations in the form

$$\frac{dx}{dt} = f(t,x,y) + \sum_{i=1}^{m} \mu^i X_i(t,x,y) + \sum_{i=0}^{\infty} \mu^i \varphi_i(t),$$

$$\frac{dy}{dt} = g(t,x,y) + \sum_{i=1}^{m} \mu^i Y_i(t,x,y) + \sum_{i=0}^{\infty} \mu^i \psi_i(t), \tag{1}$$

where $t \in R_\tau$, $x, y \in D \subset \mathbb{R}$, $0 < \mu^i < \mu_0$ is a small parameter, $f : R_\tau \times D \times D \to R$, $g : R_\tau \times D \times D \to R$. The coefficients of the polynomials X_i and Y_i, and the functions φ_i and ψ_i are bounded functions of time $t \in R_\tau$, where τ is a finite number or the symbol $+\infty$.

Together with the systems of equations ((1) and others), we will consider a positive definite continuously differentiable Lyapunov function $V(t,x,y)$ and its total derivative along the solutions of system (1) and other systems of equations investigated in this paper.

Taking into account certain results from [10,11], we provide the following definitions.

Definition 2.1 A solution $(x(t), y(t))^T$ of system (1) is called bounded for given $t_0 \geq 0$ and $\beta > 0$ with respect to the positive definite function $V(t,x,y)$ on the interval R_τ if from the condition $V(t_0, x_0, y_0) = \beta^* \leq \beta$, it follows that $V(t, x(t), y(t)) \leq \beta$ for all $t \in R_\tau$.

Definition 2.2 A solution $(x(t), y(t))^T$ of system (1) is called bounded on a given interval for a given $t_0 \in R_\tau$ if there exists a positive number $\tau > 0$ and a positive definite function $V(t,x,y)$ such that with respect to it, the solution $(x(t), y(t))^T$ of system (1) is bounded on the finite interval R_τ.

Let us obtain conditions for the boundedness of solutions of system (1) in the sense of Definitions 2.1 and 2.2.

3 Estimation of the Lyapunov Function on Solutions of System (1)

For the Lyapunov function $2V_1(x,y) = x^2 + y^2$, let us compute the total derivative with respect to time:

$$\frac{d}{dt}V_1(x,y) = x\Big(f(t,x,y) + \sum_{i=1}^{m} \mu^i X_i(t,x,y) + \sum_{i=0}^{\infty} \mu^i \varphi_i(t)\Big) +$$

$$+ y\Big(g(t,x,y) + \sum_{i=1}^{m} \mu^i Y_i(t,x,y) + \sum_{i=0}^{\infty} \mu^i \Psi_i(t)\Big). \tag{2}$$

Suppose there exist a non-negative function $a_1(t,\mu)$ and a continuous function $a_2(t,\mu)$, as well as values $\mu_1 \in (0, \mu_0]$, $\mu_2 \in (0, \mu_0]$ such that

$H_1.$: $xf(t,x,y) + yg(t,x,y) \leq 0$ for all $t \in \mathbb{R}_\tau$, $(x,y) \in D \times D$;

$H_2.$: $x \sum\limits_{i=1}^{m} \mu^i X_i(t,x,y) + y \sum\limits_{i=1}^{m} \mu^i Y_i(t,x,y) \leq a_1(t,\mu)(x^2 + y^2)$ for $\mu < \mu_1$;

$H_3.$: $x \sum_{i=0}^{\infty} \mu^i \varphi_i(t) + y \sum_{i=0}^{\infty} \mu^i \Psi_i(t) \le a_2(t,\mu)$, for $(x,y) \in D \times D$ for $\mu < \mu_2$.

Let us show that the following statement holds.

Lemma 3.1 *If conditions* H_1–H_3 *are satisfied for system* (1) *and the function* $V_1(x,y)$*, then*

$$V_1(x(t),y(t)) \le V_1(x_0,y_0)\exp\Big(\int_0^t a_1(s,\mu)ds\Big) + \int_{t_0}^t \exp\Big[\int_\tau^t a_1(s,\mu)ds\Big] a_2(\tau,\mu)d\tau \quad (3)$$

for all $t \in R_\tau$ *and* $0 < \mu < \min(\mu_1,\mu_2)$*, where* $x_0 = x(t_0)$*,* $y_0 = y(t_0)$*.*

Proof. From conditions H_1–H_3 and relation (2), it follows that

$$\frac{d}{dt}V_1(x(t),y(t)) \le a_1(t,\mu)V_1(x(t),y(t)) + a_2(t,\mu)$$

for all $t \in R_\tau$ and $0 < \mu < \min(\mu_1,\mu_2)$.

Let us compute the derivative with respect to time of the product of two functions:

$$\frac{d}{dt}\Big\{V_1(x(t),y(t))\exp\Big[-\int_{t_0}^t a_1(s,\mu)ds\Big]\Big\} =$$

$$= \Big\{\frac{d}{dt}V_1(x(t),y(t)) - a_1(t,\mu)V_1(x(t),y(t))\Big\}\exp\Big[-\int_{t_0}^t a_1(s,\mu)ds\Big]. \quad (4)$$

From equation (4), upon integration from t_0 to t, we obtain

$$V_1(x(t),y(t))\exp\Big[-\int_{t_0}^t a_1(s,\mu)ds\Big] - V_1(x_0,y_0) =$$

$$= \int_{t_0}^t \Big[\frac{d}{dt}V_1(x(\tau),y(\tau)) - a_1(\tau,\mu)V(x(\tau),y(\tau))\Big]\times$$

$$\times\Big[-\int_{t_0}^t a_1(s,\mu)ds\Big]d\tau \le \int_{t_0}^t a_2(\tau,\mu)\Big[\int_\tau^{t_0} a_1(s,\mu)ds\Big]d\tau. \quad (5)$$

From inequality (5), we deduce the estimate (3). □

4 Conditions for the Boundedness of Solutions to System (1)

The estimate (3) allows us to establish the following conditions for the boundedness of solutions to system (1).

Theorem 4.1 *To ensure that the solution* $(x(t), y(t))^T$ *of system* (1) *is bounded on a given interval with respect to the function* $V_1(x, y)$*, it suffices that conditions* H_1*–*H_3 *hold, and if* $V_1(x_0, y_0) = \beta^* < \beta$*, the following estimate holds:*

$$\exp\Big(\int_0^t a_1(s,\mu)ds\Big) + \frac{1}{\beta^*}\int_0^t a_2(s,\mu)\exp\Big(\int_s^t a_1(\tau,\mu)d\tau\Big)ds \le \frac{\beta}{\beta^*} \tag{6}$$

for all $t \in R_\tau$ *and* $0 < \mu < \min(\mu_1, \mu_2)$.

Proof. From estimate (3) under condition (6), we obtain that $V_1(x(t), y(t)) \le \beta$ for all $t \in R_\tau$. This, according to Definition 2.1, proves the statement of Theorem 4.1. □

5 System of Equations with Autonomous Stable Approximation

We consider a system of perturbed motion equations in the form (see [12])

$$\frac{dy_s}{dt} = -\lambda_s z_s + \sum_{i=1}^{\infty}\mu^i Y_{s_i}(t,x,z) + \sum_{i=0}^{\infty}\mu^i\varphi_{s_i}(t),$$

$$\frac{dz_s}{dt} = \lambda_s y_s + \sum_{i=1}^{\infty}\mu^i Z_{s_i}(t,x,z) + \sum_{i=0}^{\infty}\mu^i\psi_{s_i}(t),\ s = 1,2,\dots,n. \tag{7}$$

In system (7), the coefficients of the polynomials Y_{s_i} and Z_{s_i}, as well as the functions $\varphi_{s_i}(t)$, $\psi_{s_i}(t)$, are bounded functions of time $t \in R_\tau$, where τ is a finite number.

It is assumed that there are no external or internal resonances in system (7).

For system (7), we choose the Lyapunov function as $2V_2(y, z) = \sum\limits_{s=1}^{n}(y_s^2 + z_s^2)$ and compute its total derivative with respect to $t \in R_\tau$. Specifically,

$$\frac{d}{dt}V_2(y,z) = \sum_{s=1}^{n}\Big\{y_s\Big(-\lambda_s z_s + \sum_{i=1}^{\infty}\mu^i Y_{s_i}(t,y,z) + $$

$$+\sum_{i=0}^{\infty}\mu^i\varphi_{s_i}(t)\Big) + z_s\Big(\lambda_s y_s + \sum_{i=1}^{\infty}\mu^i Z_{s_i}(t,y,z) + \sum_{i=0}^{\infty}\mu^i\psi_{s_i}(t)\Big)\Big\}. \tag{8}$$

Let there exist a non-negative function $\bar a_1(t, \mu)$ and a continuous function $\bar a_2(t, \mu)$, as well as values μ_1, $\mu_2 \in (0, 1]$ such that

$H_4.$: $\sum\limits_{s=1}^{n}\Big(y_s\sum\limits_{i=1}^{\infty}\mu^i Y_{s_i}(t,y,z) + z_s\sum\limits_{i=0}^{\infty}\mu^i Z_{s_i}(t,y,z)\Big) \le \bar a_1(t,\mu)\sum\limits_{s=1}^{n}(y_s^2 + z_s^2)$
for $\mu < \mu_1$ and $t \in R_\tau$;

$H_5.$: $\sum\limits_{s=1}^{n}\Big(y_s\sum\limits_{i=1}^{\infty}\mu^i\varphi_{s_i}(t) + z_s\sum\limits_{i=0}^{\infty}\mu^i\psi_{s_i}(t)\Big) \le \bar a_2(t,\mu)$
for $\mu < \mu_2$, $t \in R_\tau$ and $|y_s| < k < +\infty$, $|z_s| < k < \infty$, $s = 1, 2, \dots, n$.

Theorem 5.1 *To ensure that the solution* $(y(t), z(t))^T$ *of system* (7) *is bounded on a given interval with respect to the function* $V_2(y, z)$*, it is sufficient that conditions* H_4, H_5 *hold, and if* $V_2(y_0, z_0) = \beta^* < \beta$*, the following estimate holds:*

$$\exp\Big(\int_0^t \bar a_1(s,\mu)ds\Big) + \frac{1}{\beta^*}\int_0^t \bar a_2(s,\mu)\exp\Big(\int_s^t \bar a_1(\tau,\mu)d\tau\Big)ds \le \frac{\beta}{\beta^*}\quad \textit{for all}\ \ t \in R_\tau, \tag{9}$$

and for $\mu \in (0, \mu^*)$, *where* $\mu^* = \min(\mu_1, \mu_2)$.

Proof. Under conditions H_4, H_5, it follows from equation (8) that

$$\frac{d}{dt}V_2(y(t), z(t)) \leq \bar{a}_1(t,\mu)V_2(y(t), z(t)) + \bar{a}_2(t,\mu)$$

for all $t \in R_\tau$ and $0 < \mu < \mu^*$. Hence, we find that

$$V_2(y(t), z(t)) \leq V_2(y_0, z_0) + \int\limits_0^t (\bar{a}_1(s,\mu)V_2(y(s), z(s)) + \bar{a}_2(s,\mu))ds. \tag{10}$$

Applying Lemma 3.1 to inequality (10), we obtain for the function $V_2(y, z)$, an estimate similar to estimate (3). This estimate, together with condition (9), leads to the statement of Theorem 5.1. □

6 Boundedness of Solutions of a Quasilinear System with Stable Nonautonomous Approximation

Let us consider a quasilinear nonautonomous system of equations

$$\frac{dx_i}{dt} = \sum_{s=1}^{n} p_{si}(t)x_s + X_i(t, x_1, \ldots, x_n, \mu) + \psi_i(t), \ i = 1, 2, \ldots, n, \tag{11}$$

where the functions $X_i(t, x_1, \ldots, x_n, \mu)$ have expansions in powers of the parameter μ, $\psi_i(t)$ are bounded functions on any specified time interval $t \in R_\tau$. Let us assume that for system (11), a positive definite function $V_3(t, x)$ differentiable with respect to t has been constructed.

The total derivative of the function $V_3(t, x)$ due to system (11) can be represented as

$$\frac{dV_3}{dt}(t,x) = \frac{\partial V_3}{\partial t}(t,x) + \sum_{i=1}^{n}\sum_{s=1}^{n} \frac{\partial V_3}{\partial x_s}(t,x)p_{si}(t)x_s +$$
$$+ \sum_{s=1}^{n} \frac{\partial V_3}{\partial x_s}(t,x)X_s(t,x,\mu) + \sum_{s=1}^{n} \frac{\partial V_3}{\partial x_s}(t,x)\psi_s(t). \tag{12}$$

Let us assume that for system (11), there exist a positive function $\tilde{a}_1(t, \mu)$ and a bounded function $\tilde{a}_2(t)$ such that the following conditions hold:

H_6. : $\frac{\partial V_3}{\partial t}(t,x) + \sum\limits_{i=1}^{n}\sum\limits_{s=1}^{n} \frac{\partial V_3}{\partial x_s}(t,x)p_{si}(t)x_s \leq 0$ for all $(t, x) \in R_\tau \times D$, where $D \subset \mathbb{R}^n$ is an open set;

H_7. : $\sum\limits_{s=1}^{n} \frac{\partial V_3}{\partial x_s}(t,x)X_s(t,x,\mu) \leq \tilde{a}_1(t,\mu)V_3(t,x)$ for $0 < \mu < \mu_1$ and $(t, x) \in R_\tau \times D$;

H_8. : $\sum\limits_{s=1}^{n} \frac{\partial V_3}{\partial x_s}(t,x)\psi_s(t) \leq \tilde{a}_2(t)$ for all $t \in R_\tau$ and $|x_s| < h$, where $h = const > 0$.

The condition H_6, together with the positive definiteness of the function $V_3(t, x)$, ensures the stability of the zero solution of the linear approximation system within the system of equations (11). Taking into account conditions H_6–H_8, we obtain an estimation from equation (12):

$$\frac{dV_3}{dt}(t,x) \le \tilde{a}_1(t,\mu)V_3(t,x) + \tilde{a}_2(t)$$

for all $t \in R_\tau$ and $0 < \mu < \mu_1$. Hence, we find the estimate of the change of the function $V_3(t, x(t))$ as

$$V_3(t,x(t)) \le V(t_0,x_0) + \int\limits_{t_0}^{t} (\tilde{a}_1(s,\mu)V_3(s,x(s)) + \tilde{a}_2(s))ds \tag{13}$$

for all $t \in R_\tau$ and $0 < \mu < \mu_0$.

Applying Lemma 3.1 to the inequality (13), we can easily obtain the estimate of the function $V_3(t, x(t))$ in the form of (3). The following statement holds.

Theorem 6.1 *To ensure that the solution $x(t)$ of system (11) with a stable linear approximation is bounded on a given interval with respect to the function $V_3(t,x)$, it is sufficient that conditions H_6–H_8 hold, and for a given $\beta > 0$, the inequalities $V_3(t_0,x_0) = \beta^* < \beta$ are satisfied, as well as the inequality*

$$exp\Big(\int\limits_{t_0}^{t} \tilde{a}_1(s,\mu)ds\Big) + \frac{1}{\beta^*}\int\limits_{t_0}^{t} \tilde{a}_2(s)\exp\Big(\int\limits_{s}^{t} \tilde{a}_1(\tau,\mu)d\tau\Big)ds < \frac{\beta}{\beta^*}$$

at all $t \in R_\tau$ and $0 < \mu < \mu_1$.

The proof of Theorem 6.1 is similar to the proof of Theorem 5.1.

7 Conditions for the Boundedness of Solutions of a System in Normal Form

Let us consider the differential equations of perturbed motion

$$\frac{dx_s}{dt} = X_s(t, x_1, \dots, x_n), \quad s = 1, 2, \dots, n, \tag{14}$$

$$x_s(t_0) = x_{s_0}, \tag{15}$$

where $X_s(t, 0, \dots, 0) \neq 0$ for all $t \in R_\tau$. We associate with the system (14) a differentiable function $V(t, x_1, \dots, x_n) > 0$, for which we write a Lyapunov relation

$$V(t,x(t)) = V(t_0,x_0) + \int\limits_{t_0}^{t} \dot{V}(s,x(s))ds, \tag{16}$$

where $\dot{V}(t, x(t))$ is the total derivative of the function $V(t, x)$ due to the system of equations (14) and $x(t) = (x_1(t), \dots, x_n(t))^T$.

Let $V(t, x(t)) = v(t)$, and suppose that the following condition is satisfied:

$$H_9.: \; v(t_0) + \int\limits_{t_0}^{t} \dot{V}(s,x(s))ds \le w(t) + \int\limits_{t_0}^{t} p(s)v(s)ds, \tag{17}$$

where $w(t)$ and $p(t)$ are non-negative bounded functions on the given interval R_τ. The following statement holds.

Lemma 7.1 *If the perturbed motion equations* (14) *admit a differentiable function* $V(t,x)$*, and condition* H_9 *is satisfied, then the function* $V(t,x(t)) = v(t)$ *satisfies the inequality*

$$v(t) \le w(t) + \int_{t_0}^{t} \exp\Big(\int_{\tau}^{t} p(s)ds\Big) p(\tau)w(\tau)d\tau \tag{18}$$

for all $t \in R_\tau$.

Proof. From equation (16) under condition (17), we obtain the inequality

$$v(t) \le w(t) + \int_{t_0}^{t} p(s)v(s)ds$$

for all $t \in R_\tau$. Let us denote $z(t) = \int_{t_0}^{t} p(s)v(s)ds$ and note that $z(t_0) = 0$. Obviously,

$$\frac{dz}{dt} = p(t)v(t) \le p(t)[w(t) + z(t)] = p(t)w(t) + p(t)z(t).$$

From here, it follows that

$$z(t) \le \int_{t_0}^{t} \exp\Big[\int_{\tau}^{t} p(s)ds\Big] p(\tau)w(\tau)d\tau. \tag{19}$$

Since $v(t) \le w(t) + z(t)$, taking (19) into account yields the statement of Lemma 7.1. □

Theorem 7.1 *For the solution* $x(t)$ *of the normal system of equations* (14) *to be bounded on a given interval with respect to the function* $V(x,x)$*, it is sufficient that Lemma 7.1 holds and for a given* $\beta > 0$*, if* $V(t_0,x_0) = \beta^* < \beta$*, then the inequality applies*

$$w(t) + \int_{t_0}^{t} \exp\Big[\int_{\tau}^{t} p(s)ds\Big] p(\tau)w(\tau)d\tau \le \beta \tag{20}$$

for all $t \in R_\tau$.

Proof. If the conditions of Lemma 7.1 are satisfied, then the estimate for the function $V(t,x(t))$ given by (18) holds. From condition (20) and the fact that $V(t_0,x_0) = \beta^*$, it follows that $V(t,x(t)) \le \beta$ for all $t \in R_\tau$. This proves the statement of Theorem 7.1. □

Corollary 7.1 *If in condition* H_9*, we set* $w(t) = \beta^*$*, then the estimate* (18) *takes the form*

$$v(t) \le \beta^* \exp\Big[\int_{t_0}^{t} p(s)ds\Big] \tag{21}$$

for all $t \in R_\tau$.

Based on the estimate (21), the conditions for the boundedness of solutions of system (14) with respect to the function $V(t,x)$ take the form

$$\exp\Big[\int\limits_{t_0}^{t} p(s)ds\Big] \le \frac{\beta}{\beta^*} \tag{22}$$

for all $t \in R_\tau$. If condition (22) is satisfied, then we have the estimate $V(t,x(t)) \le \beta$ for all $t \in R_\tau$.

8 Example

Consider a non-autonomous oscillatory system of the second order [13]

$$\ddot{x} + p(t)\dot{x} + [a^2 + q(t)]x = f(t,x,y), \quad a = \mathrm{const} \ne 0, \tag{23}$$

where $p(t) \ge 0$ for all $t \in R_\tau$ and $\int\limits_0^\infty q(s)ds < +\infty$. The functions $p(t)$, $q(t)$, $f(t,0,0)$ are continuous on $t \in R_\tau$ and $f(t,0,0) \ne 0$ for all $t \in R_\tau$.

Let us rewrite the equation (23) in the form of a system

$$\begin{cases} dx/dt = y, & x(t_0) = x_0, \\ dy/dt = -p(t)y - [a^2 + q(t)]x + f(t,x,y), & y(t_0) = y_0, \end{cases} \tag{24}$$

and for the total derivative of the function $V(x,y) = a^2x^2 + y^2$ on the solutions of system (24), we obtain the estimate

$$\begin{gathered} \frac{d}{dt}V(x(t),y(t)) = -2p(t)y^2(t) - 2q(t)x(t)y(t) + 2yf(t,x,y) \le \\ \le 2|q(t)||x(t)y(t)| - 2p(t)y^2(t) + 2y(t)f(t,x,y) \le \\ \le \frac{|q(t)|}{|a|}\big(a^2x^2(t) + y^2(t)\big) + \big|2y(t)f(t,x,y) - 2p(t)y^2(t)\big| = \overline{a}_1(t)V(x(t),y(t)) + \overline{a}_2(t), \end{gathered} \tag{25}$$

where $a_1(t) = \frac{|q(t)|}{|a|}$, $a_2(t) = \big|2y(t)f(t,x,y) - 2p(t)y^2(t)\big|$.

From inequality (25), it follows that

$$\frac{d}{dt}V(x(t),y(t)) \le \overline{a}_1(t)V(x(t),y(t)) + \overline{a}_2(t)$$

for all $t \in R_\tau$. Hence, we find the estimate of the change of the function $V(x(t),y(t))$ as

$$V(x(t),y(t)) \le V(x_0,y_0) + \int\limits_{t_0}^{t} (\overline{a}_1(s)V(x(s),y(s)) + \overline{a}_2(s))ds \tag{26}$$

for all $t \in R_\tau$.

Applying Lemma 3.1 to the inequality (26), we can easily obtain the estimate of the function $V(x(t),y(t))$ in the form

$$V(x(t),y(t)) \le V(x_0,y_0)\exp\Big(\int\limits_0^t \overline{a}_1(s)ds\Big) + \int\limits_{t_0}^{t} \exp\Big[\int\limits_{\tau}^{t} \overline{a}_1(s)ds\Big]\overline{a}_2(\tau)d\tau \tag{27}$$

for all $t \in R_\tau$, where $x_0 = x(t_0),\ y_0 = y(t_0)$.

The following statement holds.

Applying Theorem 6.1 to inequality (27), we find that the solutions of system (24) are bounded in the sense of Definitions 2.1 and 2.2 if, for given estimates $0 < \beta < \beta^*$ and for $V(x_0, y_0) < \beta$, the following inequality holds:

$$\exp\Big(\int_0^t \overline{a}_1(s)ds\Big) + \int_{t_0}^t \exp\Big[\int_\tau^t \overline{a}_1(s)ds\Big]\overline{a}_2(\tau)d\tau < \frac{\beta^*}{\beta}$$

for all $t \in R_\tau$.

Note that if $a_2(t) = 0$ for all $t \in R_\tau$, then the boundedness of solutions of system (24) occurs under the conditions $V(x_0, y_0) < \beta$ and

$$\int_{t_0}^t \overline{a}_1(s)ds < \ln\left(\frac{\beta^*}{\beta}\right)$$

for all $t \in R_\tau$, where $0 < \beta < \beta^*$ are predefined values.

9 Conclusion

For systems of perturbed motion equations with stable nonlinear or linear approximations, conditions for the boundedness of solutions over a given time interval with respect to a positive definite function have been obtained. This new property of motion applies to nonlinear non-autonomous systems and has broad applications in nonlinear mechanics and system theory.

References

[1] A. A. Martynyuk. *Stability Analysis: Nonlinear Mechanics Equations.* Gordon and Breach Publishers, Amsterdam, 1995.

[2] R. Bellman. *Stability Theory of Differential Equations.* Dover Publ. Inc., New York, 1953.

[3] W. Hahn. *Stability of Motion.* Translated from the German manuscript by Arne P. Baartz. Die Grundlehren der mathematischen Wissenschaften. Band 138. Springer-Verlag New York, Inc., New York, 1967.

[4] F. C. Hoppensteadt. *Analysis and Simulation of Chaotic Systems.* Springer-Verlag, Berlin, 1993.

[5] J. H. Hubbard and B. H. West. *Differential Equations.* Springer-Verlag, Berlin, 1991.

[6] A. A. Martynyuk. Analysis of equi-boundedness and stability of essentially nonlinear systems. *Int. Appl. Mech.* **59** (1) (2023) 69–78.

[7] F. Verhulst. *Nonlinear Differential Equations and Dynamical Systems.* Springer-Verlag, Berlin, 1990.

[8] A. M. Lyapunov. The general problem of the stability of motion. *Int. J. Control* **55** (3) (2092) 531–773.

[9] D. R. Anderson and D. J. Ulness. *Results for conformable differential equations.* Working Paper, January, 2016.

[10] T. Yoshizawa. *Stability Theory by Liapunov's Second Method*. The Math. Sci. of Japan, Tokyo, 1966.

[11] V. I. Zubov. *Mathematical Methods for the Study of Automatic Control Systems*. Pergamon Press, London, 1962.

[12] V. G. Veretennikov. *Stability and Oscillations of Nonlinear Systems*. Provided by the SAO/NASA Astrophysics Data System, 1984.

[13] J. La Salle and S. Lefschetz. *Stability by Liapunov's Direct Method with Applications.* Academic Press, New York, 1961.