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Theoretical and Numerical Results for Nonlinear Optimization Problems

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Abstract: In this work, we develop a new approach for solving a large class of programming optimization problems by employing a logarithmic barrier interior point method, leveraging a vector $\rho \in \mathbb{R}^n_+$ as the penalty term based on some new minorant function. Firstly, we compute the direction by Newton's method. Then, we propose a new alternative way to determine the step length along the direction, our proposed strategy enables easy and quick computation of the step length. Finally, we illustrate the out-performance of our new minorant functions with respect to the line search one through a numerical experiment on numerous collections of test problems.

Keywords: *nonlinear optimization; logarithmic barrier method; applications; minorant function; secant technique; step length.*

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1 Introduction

We consider the nonlinear constrained optimization problem

$$\{\min f(x) : \quad x \in \mathcal{L}\}, \tag{1}$$

where f is a convex and twice continuously differentiable function on $\mathcal{L}$ and $\mathcal{L} = \{x \in \mathbb{R}^n : x \geq 0, Ax = c\}$, with $c \in \mathbb{R}^m$ and $A \in \mathbb{R}^{m\times n}$ being a matrix.
Nonlinear optimization is crucial in various fields such as engineering, economics, machine learning, and nonlinear dynamics and systems (see [3, 8]) for finding optimal solutions under complex constraints.

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The paper highlights the evolution of interior point methods, a significant approach to solving such problems. These methods date back to 1955, see K. R. Frisch [6], and were significantly developed in the contributions of P. Huard [7] in 1967, and A. V. Fiacco and G. P. McCormick [5] in 1968. Notably, the development of logarithmic barrier methods, which replace non-negativity constraints with penalty terms, transforms constrained problems into unconstrained ones, allowing for the use of majorant or minorant functions. Based on this concept, various logarithmic barrier interior point methods leveraging the majorant or minorant functions have been introduced. Crouzeix and Merikhi [2] were the first to develop a logarithmic barrier algorithm based on majorant functions explicitly designed for semidefinite programming. Fellahi and Merikhi [4] introduced new majorant functions for nonlinear programming. Recently, Leulmi et al. [10] focused on devising minorant functions applicable to semidefinite programming while in [9] and [11], they explored the application of minorant functions in the contexts of linear and nonlinear programming, respectively.

Inspired by the methods mentioned above, we propose a novel approach centered on determining the step length in a straightforward manner through the utilization of a minorant function technique. The remainder of this paper is organized as follows. In the next Section 2, we present some useful inequalities that will be utilized throughout the paper. In Section 3, we replace the original problem with a perturbed problem. In Section 4, we study the existence and uniqueness of the optimal solution to the perturbed problem and analyze its convergence. In Section 5, we choose a descent direction and propose new minorant functions to calculate the step length. In Section 6, we describe the algorithm in detail. In the last section, numerical results are reported and some conclusions are drawn.

2 Some Useful Inequalities

In this section, we present some useful inequalities that will be used throughout the paper.

Proposition 2.1 *[14]*

$$\begin{aligned} \bar{z} - \sigma_z\sqrt{n-1} &\leq \min_i z_i \leq \bar{z} - \frac{\sigma_z}{\sqrt{n-1}}, \\ \bar{z} + \frac{\sigma_z}{\sqrt{n-1}} &\leq \max_i z_i \leq \bar{z} + \sigma_z\sqrt{n-1}. \end{aligned}$$

Theorem 2.1 *[2]*

$$n\ln\left(\bar{z} - \sigma_z\sqrt{n-1}\right) \leq A \leq \sum_{i=1}^{n}\ln(z_i) \leq B \leq n\ln(\bar{z}), \tag{2}$$

where

$$\begin{aligned} A &= (n-1)\ln\left(\bar{z} + \frac{\sigma_z}{\sqrt{n-1}}\right) + \ln\left(\bar{z} - \sigma_z\sqrt{n-1}\right), \\ B &= \ln\left(\bar{z} + \sigma_z\sqrt{n-1}\right) + (n-1)\ln\left(\bar{z} - \frac{\sigma_z}{\sqrt{n-1}}\right). \end{aligned}$$

$\bar{z}$ and σ_z are, respectively, the mean and the standard deviation of a statistical series of n real numbers,

$$\bar{z} = \frac{1}{n}\sum_{i=1}^{n} z_i \quad \text{and} \quad \sigma_z^2 = \frac{1}{n}\sum_{i=1}^{n} z_i^2 - \bar{z}^2 = \frac{1}{n}\sum_{i=1}^{n}(z_i - \bar{z})^2.$$

3 Penalization

In this section, we approximated the problem (1) by a perturbed problem where the parameter of penalization is a vector $\rho \in \mathbb{R}^n_+$. Let our problem (1) be considered with the following mild assumptions:

1. A is an $(m \times n)$ full-rank matrix and $c \in \mathbb{R}^m (m < n)$.
2. The optimal solution set of the problem (1) is nonempty and bounded.
3. $\exists x_0 > 0$ such that $Ax_0 = c$.

We have from the optimality conditions that x^* is an optimal solution of (1) if and only if there exists $u^* \in \mathbb{R}^m$ and $v^* \in \mathbb{R}^n_+$,

$$\nabla f(x^*) + A^t u^* - v^* = 0, \quad Ax^* = c, \quad < v^*, x >= 0. \tag{3}$$

3.1 The perturbed problem

Let us present the function $\varphi : \mathbb{R}^n_+ \times \mathbb{R}^n \to \mathbb{R}^n \cup \{+\infty\}$ by

$$\varphi(\rho, x) = \begin{cases} f(x) + \vartheta(\rho, x) & \text{if } \ x, \rho \geq 0, Ax = c. \\ +\infty & \text{otherwise,} \end{cases}$$

where the function $\vartheta : \mathbb{R}^n_+ \times \mathbb{R}^n \to \mathbb{R}^n \cup \{+\infty\}$ is given by

$$\vartheta(\rho, x) = \begin{cases} \sum_{i=1}^{n} \rho_i \ln(\rho_i) - \sum_{i=1}^{n} \rho_i \ln(x_i) & \textit{if} \quad x, \rho > 0, \\ 0 & \textit{if} \quad x \geq 0 \ \textit{and} \ \rho = 0, \\ +\infty & \textit{otherwise.} \end{cases}$$

The two functions are convex, lower semicountinuous and proper. Let us now provide the convex function Ω defined by

$$\Omega(\rho) = \inf_x [\varphi_\rho(x) := \varphi(\rho, x) : x \in \mathbb{R}]. \tag{4}$$

Remark 3.1 The problems (1) and (4) coincide when $\|\rho\|$ tents to 0.

4 Theoretical Aspects of the Perturbed Problem

4.1 Existence and uniqueness of optimal solution

The next lemma deals with the issue of the existence and uniqueness of optimal solution of (4).

Lemma 4.1 *The perturbed problem* (4) *possesses a unique optimal solution if and only if the recession cone* $C_d(\varphi_\rho)$ *reduces to the origin, i.e.,*

$$C_d(\varphi_\rho) = \{d \in \mathbb{R}^n : [\varphi_\rho]_\infty(d) \leq 0, \quad Ad = 0, d \geq 0\} = \{0\}. \tag{5}$$

Proof. For the proof, see [4].

The strictly convex problem (4) possesses a unique optimal solution $x(\rho)$ in its feasible set.

4.2 The convergence analysis

We are now prepared to state the convergence result of the perturbed problem (4).

Lemma 4.2 *Let* $x(\rho)$ *be the optimal solution of the perturbed problem* (4), *and* x^* *be the optimal solution of the the original problem* (1),

$$\text{if} \quad \|\rho\| \to 0, \quad \text{then} \quad x_\rho \to x.$$

Proof. *For the proof, see [4]*

Note: If either the original problem (1) or its perturbed version (4) possesses an optimal solution with finite and equal objective function values, then the other problem also has an optimal solution.

5 The Numerical Aspects of the Perturbed Problem

This section focuses on numerically solving problem (4). Our study starts with computing the descent direction and determining the step length, employing an innovative technique involving minorant functions.

5.1 The descent direction

When x belongs to the feasible set $\mathcal{L}$, the Newton descent direction d is obtained by solving the following convex quadratic optimization problem:

$$\begin{cases} \min_d \left\{ \frac{1}{2}\langle \nabla^2 \varphi_\rho(x)d, d\rangle + \langle \nabla \varphi_\rho(x), d\rangle \right\}, \\ Ad = 0. \end{cases}$$

From the necessary and sufficient optimality conditions, there exists $v \in \mathbb{R}$ such that

$$\begin{cases} \nabla^2 \varphi_\rho(x)d + \nabla \varphi_\rho(x) + Av = 0, \\ Ad = 0 \end{cases}$$

is equal to

$$\begin{pmatrix} \nabla^2 f(x) + PX^{-2} & A^t \\ A & 0 \end{pmatrix} \begin{pmatrix} d \\ v \end{pmatrix} = \begin{pmatrix} X^{-1}\rho - \nabla f(x) \\ 0 \end{pmatrix},$$

where X, P are diagonal matrices with $X_{ii} = x_i$ and $P_{ii} = \rho_i, i = \overline{1, n}$. Then we obtain

$$\begin{pmatrix} d^t & 0 \end{pmatrix} \begin{pmatrix} \nabla^2 f(x) + PX^{-2} & A^t \\ A & 0 \end{pmatrix} \begin{pmatrix} d \\ v \end{pmatrix} = \begin{pmatrix} d^t & 0 \end{pmatrix} \begin{pmatrix} X^{-1}\rho - \nabla f(x) \\ 0 \end{pmatrix}.$$

Then

$$\langle \nabla^2 f(x)d, d\rangle + \langle \nabla f(x), d\rangle = \langle \rho, X^{-1}d\rangle - \langle PX^{-1}d, X^{-1}d\rangle \tag{6}$$

is equivalent to

$$\begin{pmatrix} X\nabla^2 f(x)X + P & XA^t \\ AX & 0 \end{pmatrix} \begin{pmatrix} X^{-1}d \\ v \end{pmatrix} = \begin{pmatrix} \rho - X\nabla f(x) \\ 0 \end{pmatrix}.$$

The descent direction being acquired.

5.2 The proposed calculation of the step length

Typically, the most commonly employed approaches within line search techniques involve traditional iterative methods, but these methods might involve computationally expensive procedures, especially when function evaluations are costly. Hence, Leulmi et al. [2] introduced the minorant function concept, approximating the function $G(\alpha)$, to provide a simple step length for linear semidefinite programming and linear programming, respectively. Building upon this concept, we propose in this study to approximate the function $G(\alpha)$ using a minorant function. This function offers, at each iteration k, a step length in an easy and much simpler way than the line search methods. Let the function G_0 be as

$$G_0(\alpha) = \varphi_\rho(x + \alpha d) - \varphi_\rho(x) = f(x + \alpha d) - f(x) - \sum_{i=1}^{n} \rho_i \ln(1 + \alpha y_i),$$

where $y = X^{-1}d$ and $\alpha \in]0, \widehat{\alpha}_0[$, $\widehat{\alpha}_0 = \max\{\alpha : 1 + \alpha y_i > 0\}$. From Proposition 2.1, we have $\rho_i \leq \max_i \rho_i \leq \bar{\rho} + \sigma_\rho \sqrt{n-1}$ for all $i = 1, ..., n$.

For $\tau = \bar{\rho} + \sigma_\rho \sqrt{n-1}$ and for all $\alpha \in]0, \widehat{\alpha}_0[$, we obtain

$$G(\alpha) \geq G_1(\alpha) = \frac{1}{\tau}(f(x + \alpha d) - f(x)) - \sum_{i=1}^{n} \ln(1 + \alpha y_i), \tag{7}$$

where $G(\alpha) = \frac{G_0(\alpha)}{\tau}$. It easy to show that

$$G'(\alpha) = \frac{1}{\tau}\left(\langle \nabla f(x + \alpha d), d\rangle - \sum_{i=1}^{n} \rho_i \frac{y_i}{1 + \alpha y_i} \right),$$

$$G''(\alpha) = \frac{1}{\tau}\left(\langle \nabla^2 f(x + \alpha d)d, d\rangle - \sum_{i=1}^{n} \rho_i \frac{y_i^2}{(1 + \alpha y_i)^2} \right),$$

and

$$G_1'(\alpha) = \frac{1}{\tau}\left(\langle \nabla f(x + \alpha d), d\rangle - \sum_{i=1}^{n} \frac{y_i}{1 + \alpha y_i} \right),$$

$$G_1''(\alpha) = \frac{1}{\tau}\left(\langle \nabla^2 f(x + \alpha d)d, d\rangle - \sum_{i=1}^{n} \frac{y_i^2}{(1 + \alpha y_i)^2} \right).$$

G_1 verifies the significant decrease because

(i) From(6) : $G'(0) + G''(0) = 0$ and $G''(0) \geq 0$, then we deduce that $G'(0) \leq 0$.

(ii) $G_1'(0) \leq 0$ if

  1. $y_i \geq 0$.

  2. $y_i < 0$, we have from (6): $G_1'(0) + G_1''(0) = 0$ and $G_1''(0) \geq 0$.

5.3 The first minorant function

This approach involves minimizing the minorant approximation, denoted as G_2, of G within the interval $[0, \widehat{\alpha}]$. For effectiveness, this approximation necessitates simplicity while maintaining a close proximity to G_1. In our case, it requires that

$$G(0) = G_1(0) = G_2(0) = 0, \;\; G_1'(0) = G_2'(0), \;\; G_1''(0) = G_2''(0) > 0. \qquad (8)$$

In what follows, we take $x_i = 1 + \alpha y_i$, $\bar{x}_i = 1 + \alpha \bar{y}_i$ and $\sigma_x = \alpha\sigma_y$.

From Theorem 2.1, we have $\sum_{i=1}^{n} \ln(x_i) \leq B$, after a simple calculation, we obtain $G_2(\alpha) \leq G_1(\alpha)$, in which

$$G_2(\alpha) = \frac{1}{\tau}(f(x + \alpha d) - f(x)) - (n-1)\ln(1 + \alpha\gamma) - \ln(1 + \alpha\beta),$$

where $\alpha \in]0, \widehat{\alpha}[$, $\gamma = \bar{y} - \frac{\sigma_y}{\sqrt{n-1}}$, $\beta = \bar{y} + \sigma_y\sqrt{n-1}$ and $\widehat{\alpha} = \min\{\widehat{\alpha}_0, \max\{\alpha : 1 + \alpha\gamma > 0\}\}$. It is clear that

$$\begin{aligned} G_2'(\alpha) &= \frac{1}{\tau}\langle \nabla f(x + \alpha d), d\rangle - (n-1)\frac{\gamma}{1 + \alpha\gamma} - \frac{\beta}{1 + \alpha\beta}, \\ G_2''(\alpha) &= \frac{1}{\tau}\langle \nabla^2 f(x + \alpha d)d, d\rangle + (n-1)\frac{\gamma^2}{(1 + \alpha\gamma)^2} + \frac{\beta^2}{(1 + \alpha\beta)^2}. \end{aligned}$$

G_2 verifies the conditions (8), hence the strictly convex function G_2 is a good approximation of G_1 in $]0, \hat{\alpha}[$. Moreover, the unique minimum α^* of G_2 verifies

$$G_2(\alpha^*) \leq G_1(\alpha^*) \leq G(\alpha^*).$$

5.4 The auxiliary function ζ

5.4.1 If the objective function f is linear

We take $f(x) = c^t x$, where $c, x \in \mathbb{R}^n$, the auxiliary function is defined by

$$\zeta(\alpha) = n\eta\alpha - (n-1)\ln(1 + \alpha\gamma) - \ln(1 + \alpha\beta), \quad \eta = \frac{1}{\tau n}c^t d.$$

The two functions ζ and G_2 coincide. The unique solution of $\zeta'(\alpha) = 0$ is the same unique root of $G_2'(\alpha) = 0$ and the unique $\bar{\alpha}$ verifies $G_2(\bar{\alpha}) \leq G_1(\bar{\alpha}) \leq G(\bar{\alpha})$.

5.4.2 If the objective function f is only convex

Under these circumstances, the equation $G_2'(\bar{\alpha}) = 0$ no longer simplifies to a second-degree equation. To address this, we explore an alternative function better than G_2, employing the secant technique. Let $\bar{\alpha} \in]0, \widehat{\alpha}[$ for all $\alpha \in]\bar{\alpha}, \widehat{\alpha}[$, then we have

$$\frac{\alpha}{\tau\bar{\alpha}}(h(x + \bar{\alpha}d) - h(x)) \leq \frac{1}{\tau}(h(x + \alpha d) - h(x)).$$

So, the auxiliary function is given by

$$\zeta(\alpha) = n\eta\alpha - (n-1)\ln(1 + \alpha\gamma) - \ln(1 + \alpha\beta), \quad \textit{where} \quad \eta = \frac{1}{n\tau\bar{\alpha}}(f(x + \bar{\alpha}d) - f(x)),$$

then we have the following results:

(i) If $\bar{\alpha} = 1$ and $\widehat{\alpha} > 1$, then $\zeta'(\bar{\alpha}) = 0$.

(ii) If $\bar{\alpha} \neq 1$, then

1. If $\alpha^* \leq \bar{\alpha}$, we must take another $\bar{\alpha} \in]\alpha^*, \widehat{\alpha}[$, for example, we choose $\bar{\alpha} = \alpha^* + \xi(\alpha^* - \widehat{\alpha}), \;\; \xi \in [0, 1]$.

2. If $\alpha^* \geq \bar{\alpha}$, we have $\zeta(\alpha^*) \leq G_2(\alpha^*) \leq G_1(\alpha^*) \leq G(\alpha^*)$.

5.4.3 Minimization of the auxiliary function

We have

$$\zeta(\alpha) = n\eta\alpha - (n-1)\ln(1+\alpha\gamma) - \ln(1+\alpha\beta).$$

We obtain

$$\zeta'(\alpha) = n\eta - (n-1)\frac{\gamma}{1+\alpha\gamma} - \frac{\beta}{1+\alpha\beta}, \quad \zeta''(\alpha) = (n-1)\frac{\gamma^2}{(1+\alpha\gamma)^2} - \frac{\beta^2}{(1+\alpha\beta)^2}.$$

We note that $\zeta(0) = 0$, $\zeta'(0) = n(\eta - \bar{y})$, $\zeta''(0) = n(\bar{y}^2 + \sigma_y^2) = \|y\|^2$ and we impose that $\zeta'(0) \leq 0$ and $\zeta''(0) \geq 0$. $\zeta'(\alpha) = 0$ is similar to $\eta\gamma\beta\alpha^2 + (\eta(\alpha+\beta) - \gamma\beta)\alpha + \eta - \bar{y} = 0$.

We obtain

$$\alpha^* = \begin{cases} \frac{-\bar{y}}{\gamma\beta} & \text{if } \eta = 0, \\ \frac{\bar{y}-\eta}{\eta\beta} & \text{if } \gamma = 0, \\ \frac{\bar{y}-\eta}{\eta\gamma} & \text{if } \beta = 0. \end{cases} \tag{9}$$

In the case of $\eta\gamma\beta \neq 0$, we have two solutions, we choose only the root α^* that belongs to the domain of ζ, we have

$$\alpha_1^* = \frac{1}{2}\left(\frac{1}{\eta} - \frac{1}{\gamma} - \frac{1}{\beta} - \sqrt{\Delta}\right), \quad \alpha_2^* = \frac{1}{2}\left(\frac{1}{\eta} - \frac{1}{\gamma} - \frac{1}{\beta} + \sqrt{\Delta}\right),$$

where

$$\Delta = \frac{1}{\eta^2} + \frac{1}{\gamma^2} + \frac{1}{\beta^2} - \frac{2}{\gamma\beta} - \left(\frac{2n-4}{n\eta}\right)\left(\frac{1}{\gamma} - \frac{1}{\beta}\right).$$

We take $\alpha^* \in]0, \widehat{\alpha} - \epsilon[$, where $\epsilon > 0$.

Remark 5.1 The computation of α^* is conducted using a dichotomous procedure under the conditions where α^* is not within the interval $]0, \widehat{\alpha} - \epsilon[$ and $G'(\alpha^*) > 0$.
Take $a = 0$, $b = \widehat{\alpha} - \epsilon$.
While $|b - a| > \epsilon$ do
- $c = \frac{a+b}{2}$.
- If $G'(c) < 0$, then $b = c$ else $a = c$.

This computation ensures an improved approximation of the minimum of $G(\alpha)$ while maintaining adherence to the domain of G.

5.5 The second minorant function

In this context, our aim is to discover another function simpler than G_1, hence we used the following inequality:

$$(\|y\| - n\bar{y})\alpha - \ln(1 + \alpha\|y\|) \leq -\sum_{i=1}^{n} ln(1 + \alpha y_i). \tag{10}$$

From (7) and (10), we take

$$G_3(t) = \frac{1}{\tau}(f(x+\alpha d) - f(x)) + (\|y\| - n\bar{y})\alpha - \ln(1+\alpha\|y\|),$$

where $\alpha \in [0, \widehat{\alpha}[$, it is clear that

$$G_3'(\alpha) = \frac{1}{\tau}\langle \nabla f(x+\alpha d), d\rangle + \|y\| - n\bar{y} - \frac{\|y\|}{1+\alpha\|y\|},\ G_3''(\alpha) = \frac{1}{\tau}\langle \nabla^2 f(x+\alpha d)d, d\rangle + \frac{\|y\|^2}{(1+\alpha\gamma)^2}.$$

The strictly convex function G_3 is a good approximation of G_1 in $]0, \widehat{\alpha}[$ and the unique root $\alpha^*$of $G_3'(\alpha) = 0$ verifies $G_3(\alpha^*) \leq G_2(\alpha^*) \leq G_1(\alpha^*)$.

Because we have $G(0) = G_1(0) = G_2(0) = G_3(0) = 0,\ G_3'(0) = G_1'(0) = G_2'(0) < 0$ and $G_3''(0) = G_1''(0) = G_2''(0) > 0$.

5.5.1 Minimization of an auxiliary function

Let the auxiliary function ζ_2 be defined as

$$\zeta_2(\alpha) = n\eta\alpha + (\|y\| - n\bar{y})\alpha - \ln(1+\alpha\|y\|),$$

we have

$$\zeta_2'(\alpha) = n\eta + \|y\| - n\bar{y} - \frac{\|y\|}{1+\alpha\|y\|},\quad \zeta_2''(\alpha) = \frac{\|y\|^2}{(1+\alpha\|y\|)^2}.$$

We note that $\zeta_2(0) = 0,\ \zeta_2'(0) = n(\eta - \bar{y}),\ \zeta_2''(0) = \|y\|^2$ and we impose that $\zeta_2'(0) \leq 0$ and $\zeta_2''(0) \geq 0$. The minimum α^* is the root of $\zeta_2'(\alpha) = 0$.

We take $\alpha^* \in [0, \widehat{\alpha} - \epsilon[$, where $G_1'(\alpha^*) < 0$. We use Remark 5.1 to obtain a good approximation for the minimum of $G_1'(\alpha)$ while $\alpha^* \notin [0, \widehat{\alpha} - \epsilon[$.

6 The Algorithm

The algorithm below summarizes the main steps of the proposed method.

Algorithm.

Step 0: (Initialization) Select $x_0 \in \mathcal{L}$, $X_{ii} = (x_0)_i$ and the parameters $\rho_s,\ \rho \in \mathbb{R}^n_+$, $b \in [0,1]^n$ *and* $\epsilon > 0$.

Step 1: Calculate d and $y = X^{-1}d$.

Step 2:

- If $\|y\| \leq \epsilon$, then we have a good approximation of $\Omega(\rho)$, So if $\|\rho\| \leq \rho_s$, then stop; (We have a good approximation of the optimal solution), otherwise we put $\rho = b \times \rho$ with $b \times \rho = (b_1 \times \rho_1, ..., b_n \times \rho_n)$ and go to the **Step 1.**

- If $\|y\| \geq \epsilon$,

* Compute η, γ, β and calculate $\alpha^* > 0$ using the equation $\zeta'(\alpha) = 0$.

*Put $x = x + \alpha^* d$ and return to the **Step 1**.

End Algorithm.

7 Numerical Experiments

To evaluate the enhanced performance and accuracy of our algorithm, which leverages minorant functions, we conduct numerical tests to compare our new approach with the traditional line search method. In this section, we present comparative numerical tests using various examples from the literature [12, 13]. The results were obtained by implementing the algorithm in MATLAB on an Intel Core i7-7700HQ (2.80 GHz) machine with 16 GB of RAM. In the following tables, we take $\epsilon = 10^{-4}$. We also note that

- (iter) is the number of iterations.
- (time) is the computational time in seconds (s).
- $(\text{sti})_{i=1,2}$ represents the strategy of approximate functions introduced in this paper.
- (LS) represents the classical line search method.

- **Example 1:** (Erikson's problem [13]).

We consider the following quadratic problem, with $n = 2m$:

$$\zeta = \min \left[f(x) = \sum_{i=1}^{n} x_i \ln \left(\frac{x_i}{a_i} \right) : x_i + x_{i+m} = b, x \geq 0 \right],$$

where $a_i > 0$ and $b \in \mathbb{R}^m$ are fixed.

We test this example for the different values of n, a_i and b_i.

The following table resumes the obtained result in the case $(a_i = 2, \forall i = 1, ..., n, b_i = 4, \forall i = 1, ..., m)$.

ex(m,n)	st1		st2		LS	
	iter	time	iter	time	iter	time
30×60	1	0.0009	1	0.0013	3	0.023
150×300	1	0.0035	2	0.0091	4	0.0645
300×600	2	0.0035	3	0.0212	5	3.199
500×1000	2	0.1112	3	0.0987	5	5.324

Table 1: Example of Erikson's problem (with variable size (Example 1)).

- **Example 2:** Quadratic case [12].
  We consider the following quadratic problem with $n = m + 2$:

$$f^* = \min\{f(x) : Ax = c, x \geq 0\},$$

with $f(x) = \frac{1}{2}\langle x, Qx \rangle$, $Q[i,j] = \begin{cases} 2 & \text{if } i = j = 1 \text{ or } i = j = m, \\ 4 & \text{if } i = j \text{ and } i \neq \{1, m\}, \\ 2 & \text{if } i = j - 1 \text{ or } i = j + 1, \\ 0 & \text{otherwise,} \end{cases}$

and $A[i,j] = \begin{cases} 1 & \text{if } i = j, \\ 2 & \text{if } i = j - 1, \\ 3 & \text{if } i = j - 2, \\ 0 & \text{otherwise.} \end{cases}$

$f_i = 1 \ \forall i, j = 1, .., n$. We test this example for the different values of n.

ex(n)	st1		st2		LS	
	iter	time	iter	time	iter	time
30	5	0.0041	3	0.0053	26	18.3244
400	3	0.0985	1	0.0121	35	60.1003
600	3	9.6544	1	7.0129	23	79.0024
1000	5	11.9912	3	9.0473	33	91.3358

Table 2: Example of Quadratic case (with variable size (Example 2)).

8 Conclusion

In this paper, we introduced a logarithmic barrier method for solving nonlinear programming. Based on some new minorant functions and secant technique, this method calculates the step length in a straightforward manner. The numerical simulations demonstrate the efficacy of our approach as a significant alternative, yielding promising outcomes when compared to traditional line search methods. Exploring new approximate functions presents a promising direction for future research in various classes of optimization. We plan to apply our key findings to a range of issues in nonlinear dynamics problems.

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