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Φ-Hilfer Proportional Fractional Differential System: Uniqueness and Stability Result

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Abstract: This work derives uniqueness and Ulam-Hyers (UH) stability results for a coupled system with the Φ-Hilfer proportional fractional derivative. We first construct a new Bielecki-type vector-valued norm in weighted space and then use the fixed-point argument to achieve a new uniqueness criterion. Secondly, the UH and the generalized Ulam-Hyers (GUH) stability is established. To verify the obtained result, an example is provided.

Keywords: *Φ-Hilfer proportional fractional derivative; uniqueness; Ulam-Hyers stability; fixed point theorem.*

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1 Introduction

Fractional calculus stands as a cornerstone in applied mathematics owing to its profound implications across various scientific and technological domains. The prevalence of fractional differential equations (FDEs) in modeling diverse natural and engineered phenomena underscores its significance [1, 9, 13, 16, 18, 22, 28]. Notably, the exploration of coupled systems featuring fractional derivatives, in their assorted manifestations, has been the subject of extensive inquiry by numerous researchers [7, 15, 29, 30].

This pervasive utility instigates a scholarly endeavor towards the exploration of novel fractional operators, to enrich our understanding and refine the accuracy in modeling real-world phenomena [3, 12, 23–25].

A recent milestone in this realm is the introduction by Jarad et al. [10]. of a new definition of generalized fractional derivatives, as elucidated within the frameworks of Caputo and Riemann-Liouville calculus. This innovation, utilizing a specialized instantiation of proportional derivatives, offers a nuanced perspective [10]. Furthermore, by leveraging the notion of proportional derivatives within the context of functional analysis, the research presented in [11] extends and generalizes prior investigations [10]. Notably, Ahmed et al. [2] present the Hilfer proportional fractional derivative (PFD) by amalgamating operators delineated in prior works [2]. Subsequently, in [17], the authors offer a further refinement, denoted as the Φ-Hilfer PFD, accompanied by a comprehensive examination of its properties.

Concurrently, the exploration of the Ulam-Hyers (UH) stability for fractional differential systems has emerged as a focal point of research interest [14]. This pursuit, characterized by its aim to approximate solutions with minimal error vis-à-vis exact counterparts, has engendered a manifold of investigations [1].

This work, in alignment with this scholarly discourse, embarks on a rigorous examination of a coupled system involving the Φ-Hilfer PFD, seeking to illuminate its dynamics and ramifications.

$$\begin{cases} \mathcal{D}_{\mathtt{a}^+}^{\alpha,\beta,\sigma,\Phi}\mathtt{u}_1(z) = \xi_1\mathtt{u}_1(z) + \mathtt{g}_1(z,\mathtt{u}_1(z),\mathtt{u}_2(z)), \quad z \in \mathfrak{J}' := (\mathtt{a},\mathtt{b}], \\ \mathcal{D}_{\mathtt{a}^+}^{\alpha,\beta,\sigma,\Phi}\mathtt{u}_2(z) = \xi_2\mathtt{u}_2(z) + \mathtt{g}_2(z,\mathtt{u}_1(z),\mathtt{u}_2(z)), \quad z \in \mathfrak{J}' := (\mathtt{a},\mathtt{b}], \\ \left(\left(\mathcal{I}_{\mathtt{a}^+}^{1-\delta,\sigma,\Phi}\mathtt{u}_1\right)(\mathtt{a}^+), \left(\mathcal{I}_{\mathtt{a}^+}^{1-\delta,\sigma,\Phi}\mathtt{u}_2\right)(\mathtt{a}^+)\right) = (\zeta_1,\zeta_2), \end{cases} \tag{1}$$

where $\mathcal{D}_{\mathtt{a}^+}^{\alpha,\beta,\sigma,\Phi}$ are the Φ-Hilfer PFDs of order $0 < \alpha < 1$, type $\beta \in [0,1]$ and index $\sigma \in (0,1]$, $\mathcal{I}_{\mathtt{a}^+}^{1-\delta,\sigma,\Phi}$ is the fractional proportional integral (FPI) of order $1-\delta$, where $0 < \delta = \alpha + \beta(1-\alpha) < 1$ and index σ, $\mathtt{g}_i : \mathfrak{J} \times \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ are given appropriate functions specified later, $\zeta_i \in \mathbb{R}^n$ and $\xi_i \in \mathbb{R}^{n\times n}$.

System (1) applies to a vast range of models, necessitating further investigation. For instance, the authors in [6] employed it to study the dynamics of a measles epidemic model. Batiha et al. [31] use it to understand the relationship between pharmacological effects and drugs in anesthesia modeling. Therefore, exploring coupled systems within the Φ-Hilfer PFD with specific configurations becomes essential. We base our study on the fixed-point argument with the vector-valued norm theory in the weighted space of continuous functions. Finally, it is important to highlight that the findings obtained in this study build upon and directly extend the results presented in the related literature, see [2, 4, 5, 17].

The work is divided into three sections. Some necessary background materials are provided in Section 2. The main result of the work is given in Section 3 by using Perov's fixed point principle associated with the Bielecki vector-valued norm. The UH stability of solutions to problem (1) is established in Section 4. An example is given in the last section to illustrate the applicability of our result.

2 Preliminaries

This section presents some background material that will be used throughout this work.

$C(\mathfrak{J},\mathbb{R}^n)$ is the space of $\mathbb{R}^n$–valued continuous functions on $\mathfrak{J}$ equipped with

$$\|\mathtt{u}\|_{\mathfrak{J}} = \sup_{z\in\mathfrak{J}} \|\mathtt{u}(z)\|.$$

$C^n(\mathfrak{J},\mathbb{R}^n)$ denotes the space of n-times continuously differentiable functions from $\mathfrak{J}$ into $\mathbb{R}^n$.

$L^\infty(\mathfrak{J},\mathbb{R}^n)$ is the set of all equivalence classes of measurable functions which are essentially bounded on $\mathfrak{J}$ equipped with

$$\|\mathtt{u}\|_\infty = \sup_{z\in\mathfrak{J}} \|\mathtt{u}(z)\| = \inf\{M>0;\ \|\mathtt{u}(z)\| \le M \text{ for almost every } \mathtt{u}\in\mathfrak{J}$$

For our convenience, define the set

$$\mathbb{S}^1_+(\mathfrak{J},\mathbb{R}) = \{\Phi \ : \ \Phi\in C(\mathfrak{J},\mathbb{R}) \ \text{ and } \ \Phi'(z)>0 \ \text{ for all } \ z\in\mathfrak{J}.\}$$

For $\Phi\in\mathbb{S}^1_+(\mathfrak{J},\mathbb{R})$ and $z,s\in\mathfrak{J}$, $(z>s)$, we set

$$\Psi(z,s) = \Phi(z)-\Phi(s) \ \text{ and } \ \Psi(z,s)^\alpha = (\Phi(z)-\Phi(s))^\alpha.$$

We endow the weighted space $\mathfrak{C}_{\delta,\Phi}(\mathfrak{J},\mathbb{R}^n)$ defined by

$$\mathfrak{C}_{\delta,\Phi}(\mathfrak{J},\mathbb{R}^n) = \left\{\mathtt{u}:\mathfrak{J}\to\mathbb{R}^n \ : \ \Psi(\cdot,\mathtt{a})^{1-\delta}\mathtt{u}(\cdot)\in C(\mathfrak{J},\mathbb{R}^n)\right\},$$

with the norm

$$\|\mathtt{u}\|_\delta = \sup_{z\in\mathfrak{J}} \Psi(z,\mathtt{a})^{1-\delta}\|\mathtt{u}(z)\|. \tag{2}$$

Definition 2.1 [8] The Mittag-Leffler function $\mathbb{E}_{\mu,\delta}(\cdot)$ is given by

$$\mathbb{E}_{\mu,\kappa}(\mathtt{u}) = \sum_{j=0}^{\infty} \frac{\mathtt{u}^j}{\Gamma(j\mu+\kappa)}, \quad \mathtt{u}\in\mathbb{R}, \ \text{ and } \ \mu,\kappa>0,$$

where $\Gamma(\cdot)$ is the Gamma function.

Definition 2.2 [11,12] Let $\sigma\in(0,1]$, $\alpha>0$ and $\Phi\in\mathbb{S}^1_+(\mathfrak{J},\mathbb{R})$. The left-sided FPI of order α and index σ of the function $\mathtt{u}$ w.r.t Φ is given by

$$\mathcal{I}^{\alpha,\sigma,\Phi}_{\mathtt{a}^+}\mathtt{u}(z) = \frac{1}{\sigma^\alpha\Gamma(\alpha)}\int_{\mathtt{a}}^{z} \mathfrak{L}^{\alpha-1}_{\sigma,\Psi}(z,s)\Phi'(s)\mathtt{u}(s)ds,$$

where $\mathfrak{L}^{\alpha-1}_{\sigma,\Psi}(z,s) = e^{\frac{\sigma-1}{\sigma}\Psi(z,s)}\Psi(z,s)^{\alpha-1}$.

Definition 2.3 [11, 12] Let $\sigma \in (0,1]$, $0 < \alpha < 1$ and $\Phi \in \mathbb{S}^1_+(\mathfrak{J}, \mathbb{R})$, the FPD of order α and index σ of the function $\mathtt{u}$ w.r.t Φ is given by

$$\mathcal{D}^{\alpha,\sigma,\Phi}_{\mathtt{a}^+}\mathtt{u}(z) = \mathcal{D}^{1,\sigma,\Phi}_{z}\left(\mathcal{I}^{1-\alpha,\sigma,\Phi}_{\mathtt{a}^+}\mathtt{u}(z)\right),$$

where $\mathcal{D}^{1,\sigma,\Phi}_{z}\mathtt{u}(z) = \mathcal{D}^{\sigma,\Phi}_{z}\mathtt{u}(z) = \sigma\frac{\mathtt{u}'(z)}{\Phi'(z)} + (1-\sigma)\mathtt{u}(z)$.

Definition 2.4 [17] Let $\mathtt{u} \in C^1(\mathfrak{J}, \mathbb{R}^n)$ and $\Phi \in \mathbb{S}^1_+(\mathfrak{J}, \mathbb{R})$. The Φ-Hilfer FPD (right-sided/left-sided) of order $0 < \alpha < 1$, type $0 \le \beta \le 1$ and index $\sigma \in (0,1]$ of $\mathtt{u}$ w.r.t Φ is given by

$$\mathcal{D}^{\alpha,\beta,\sigma,\Phi}_{\mathtt{a}^\pm}\mathtt{u}(z) = \left(\mathcal{I}^{\beta(1-\alpha),\sigma,\Phi}_{\mathtt{a}^\pm}(\mathcal{D}^{1,\sigma,\Phi}_{z})\mathcal{I}^{(1-\beta)(1-\alpha),\sigma,\Phi}_{\mathtt{a}^\pm}\mathtt{u}\right)(z).$$

Lemma 2.1 [12, 17] *If $\alpha \ge 0$ and $\beta > 0$, then for any $\sigma > 0$, we have*

$$\mathcal{I}^{\alpha,\sigma,\Phi}_{\mathtt{a}^+}\mathfrak{L}^{\beta-1}_{\sigma,\Psi}(z,\mathtt{a}) = \frac{\Gamma(\beta)}{\sigma^\alpha\Gamma(\alpha+\beta)}\mathfrak{L}^{\alpha+\beta-1}_{\sigma,\Psi}(z,\mathtt{a}). \tag{3}$$

Remark 2.1 If $\alpha \ge 0$, $\beta > 0$ and $\sigma \in (0,1]$, then by the fact that $e^{\frac{\sigma-1}{\sigma}\Psi(z,s)} \le 1$, for all $a \le s < z \le b$, we get

$$\mathcal{I}^{\alpha,\sigma,\Phi}_{\mathtt{a}^+}\Psi(z,a)^{\beta-1} \le \sigma^{-\alpha}\mathcal{J}^{\alpha;\Phi}_{\mathtt{a}^+}\Psi(z,\mathtt{a})^{\beta-1},$$

where $\mathcal{J}^{\alpha;\Phi}_{\mathtt{a}^+}(\cdot)$ is the integral defined in [25]. On the order hand, by Lemma 2 from [25], we conclude that

$$\mathcal{I}^{\alpha,\sigma,\Phi}_{\mathtt{a}^+}\Psi(z,\mathtt{a})^{\beta-1} \le \frac{\Gamma(\beta)}{\sigma^\alpha\Gamma(\alpha+\beta)}\Psi(z,\mathtt{a})^{\alpha+\beta-1}. \tag{4}$$

Remark 2.2 Let $\alpha \ge 0$ and $\sigma > 0$, then, by Definition 2.4 and Lemma 2.1, we get

$$\mathcal{D}^{\alpha,\sigma,\Phi}_{\mathtt{a}^+}\mathfrak{L}^{\alpha-1}_{\sigma,\Psi}(z,\mathtt{a}) = 0.$$

Suppose the parameters α, β, δ fulfill the relations

$$\delta = \alpha + \beta(1-\alpha), \quad 0 < \alpha \le \delta \le 1, \quad 1 \geqslant \beta \geqslant 0, \quad \beta < \delta, \quad 1-\delta < 1-\beta(1-\alpha).$$

Therefore, we consider the following spaces:

$$\mathfrak{C}^{\delta}_{\delta,\Phi}(\mathfrak{J}, \mathbb{R}^n) = \left\{\mathtt{u} \in \mathfrak{C}_{\delta,\Phi}(\mathfrak{J}, \mathbb{R}^n) : \mathcal{D}^{\delta,\sigma,\Phi}_{\mathtt{a}^+}\mathtt{u} \in \mathfrak{C}_{\delta,\Phi}(\mathfrak{J}, \mathbb{R}^n)\right\},$$

$$\mathfrak{C}^{\alpha,\beta}_{\delta,\Phi}(\mathfrak{J}, \mathbb{R}^n) = \left\{\mathtt{u} \in \mathfrak{C}_{\delta,\Phi}(\mathfrak{J}, \mathbb{R}^n) : \mathcal{D}^{\alpha,\beta,\sigma,\Phi}_{\mathtt{a}^+}\mathtt{u} \in \mathfrak{C}_{\delta,\Phi}(\mathfrak{J}, \mathbb{R}^n)\right\}.$$

Since $\mathcal{D}^{\alpha,\beta,\sigma,\Phi}_{\mathtt{a}^+}\mathtt{u} = \mathcal{I}^{\beta(1-\alpha),\sigma,\Phi}_{\mathtt{a}^+}\mathcal{D}^{\delta,\sigma,\Phi}_{\mathtt{a}^+}\mathtt{u}$, it follows that

$$\mathfrak{C}^{\delta}_{\delta,\Phi}(\mathfrak{J}, \mathbb{R}^n) \subset \mathfrak{C}^{\alpha,\beta}_{\delta,\Phi}(\mathfrak{J}, \mathbb{R}^n).$$

Lemma 2.2 [17] *Let $\delta = \beta(1-\alpha) + \alpha$ with $0 < \alpha < 1$, $0 < \sigma \leqslant 1$ and $0 \leqslant \beta \leqslant 1$. If $f \in \mathfrak{C}^{\delta}_{\delta,\Phi}(\mathfrak{J}, \mathbb{R}^n)$, then*

1. $\mathcal{D}^{\delta,\sigma,\Phi}_{\mathtt{a}^+}\mathcal{I}^{\alpha,\sigma,\Phi}_{\mathtt{a}^+}\mathtt{u} = \mathcal{D}^{\beta(1-\alpha),\sigma,\Phi}_{\mathtt{a}^+}\mathtt{u}$,

2. $\mathcal{I}_{\mathtt{a}^+}^{\delta,\sigma,\Phi}\mathcal{D}_{\mathtt{a}^+}^{\delta,\sigma,\Phi}\mathtt{u} = \mathcal{I}_{\mathtt{a}^+}^{\alpha,\sigma,\Phi}\mathcal{D}_{\mathtt{a}^+}^{\alpha,\beta,\sigma,\Phi}\mathtt{u}$.

Let $u, v \in \mathbb{R}^n$, $u = (\mathtt{u}_1, \dots, \mathtt{u}_n)$, $v = (\mathtt{v}_1, \dots, \mathtt{v}_n)$, by $u \le v$ we mean $\mathtt{u}_l \le \mathtt{v}_l$ for all $l = 1, \dots, n$. Also $|u| = (|\mathtt{u}_1|, \dots, |\mathtt{u}_n|)$, $\max(u, v) = (\max(\mathtt{u}_1, \mathtt{v}_1), \dots,$ $\max(\mathtt{u}_n, \mathtt{v}_n))$ and $\mathbb{R}^n_+ = \{u \in \mathbb{R}^n : \ \mathtt{u}_l > 0\}$. If $c \in \mathbb{R}$, then $u \le c$ means $\mathtt{u}_l \le c$ for each $l = 1, \dots, n$.

Definition 2.5 [20] Let for a nonempty set $\mathbb{X}$, a map $d : \mathbb{X} \times \mathbb{X} \to \mathbb{R}^n_+$ be called the vector-valued metric on $\mathbb{X}$ if for all $x, y, z \in \mathbb{X}$, the following properties hold:

(a) $d(x, y) = 0$, then $x = y$,

(b) $d(x, y) = d(y, x)$,

(c) $d(x, y) \le d(x, z) + d(z, y)$.

$(\mathbb{X}, d)$ is called a generalized metric space with
$d(x, y) := (d_1(x, y), \cdots, d_n(x, y))$.

Definition 2.6 [27] We said that a matrix $\mathsf{M} \in \mathsf{M}_{n\times n}(\mathbb{R})$ converges to zero if the spectral radius $\rho(\mathsf{M}) < 1$.

Theorem 2.1 [27] *For any positive matrix $\mathsf{M} \in \mathsf{M}_{n\times n}(\mathbb{R})$, the following assertions are equivalent:*

1. $\rho(\mathsf{M}) < 1$;

2. M *is convergent to zero;*

3. $I - \mathsf{M}$ *is nonsingular and* $(I - \mathsf{M})^{-1}$ *is a nonnegative matrix.*

Lemma 2.3 *Let $\alpha, \delta, \varpi > 0$ and $\sigma \in (0, 1]$, then*

$$\mathcal{I}_{\mathtt{a}^+}^{\alpha,\sigma,\Phi}\mathfrak{L}_{\sigma,\Psi}^{\delta-1}(z,\mathtt{a})\mathbb{E}_{\alpha,\delta}(\varpi\Psi(z,\mathtt{a})^{\alpha}) \quad = \frac{\mathfrak{L}_{\sigma,\Psi}^{\delta-1}(z,\mathtt{a})}{\varpi\sigma^{\alpha}}\left(\mathbb{E}_{\alpha,\delta}(\varpi\Psi(z,\mathtt{a}))^{\alpha}) - \frac{1}{\Gamma(\delta)}\right).$$

Proof. First, we can write

$$\begin{aligned}\mathcal{O} = \mathcal{I}_{\mathtt{a}^+}^{\alpha,\sigma,\Phi}\mathfrak{L}_{\sigma,\Psi}^{\delta-1}(z,\mathtt{a})\mathbb{E}_{\alpha,\delta}(\varpi\Psi(z,\mathtt{a})^{\alpha}) &= \mathcal{I}_{\mathtt{a}^+}^{\alpha,\sigma,\Phi}\left(\mathfrak{L}_{\sigma,\Psi}^{\delta-1}(z,\mathtt{a})\sum_{j=0}^{\infty}\frac{(\varpi\Psi(z,\mathtt{a})^{\alpha})^j}{\Gamma(j\alpha+\delta)}\right)\\ &= \sum_{j=0}^{\infty}\frac{\varpi^j}{\Gamma(j\alpha+\delta)}\ \mathcal{I}_{\mathtt{a}^+}^{\alpha,\sigma,\Phi}\ \mathfrak{L}_{\sigma,\Psi}^{j\alpha+\delta-1}(z,\mathtt{a}).\end{aligned}$$

From Lemma 2.1, one gets

$$\mathcal{O} \quad = \textstyle\sum_{j=0}^{\infty}\frac{\varpi^{j+1}\mathfrak{L}_{\sigma,\Psi}^{\alpha(j+1)+\delta-1}(z,\mathtt{a})}{\sigma^{\alpha}\varpi\Gamma(\alpha(j+1)+\delta)} = \frac{\mathfrak{L}_{\sigma,\Psi}^{\delta-1}(z,\mathtt{a})}{\sigma^{\alpha}\varpi}\ \textstyle\sum_{j=0}^{\infty}\frac{(\varpi\Psi(z,\mathtt{a})^{\alpha})^{j+1}}{\Gamma(\alpha(j+1)+\delta)}.$$

Replace $j+1$ by j', we obtain

$$\mathcal{O} \quad = \quad \frac{\mathfrak{L}_{\sigma,\Psi}^{\delta-1}(z,\mathtt{a})}{\sigma^{\alpha}\varpi}\ \textstyle\sum_{j=1}^{\infty}\frac{(\varpi\Psi(z,\mathtt{a})^{\alpha})^{j'}}{\Gamma(\alpha j'+\delta)} = \frac{\mathfrak{L}_{\sigma,\Psi}^{\delta-1}(z,\mathtt{a})}{\sigma^{\alpha}\varpi}\ \left[\textstyle\sum_{j'=0}^{\infty}\frac{(\varpi\Psi(z,\mathtt{a})^{\alpha})^{j'}}{\Gamma(\alpha j'+\delta)} - \frac{1}{\Gamma(\delta)}\right],$$

and this completes the proof.

Lemma 2.4 *We define a Bielecki-type norm, denoted by $\|\cdot\|_{\sigma,B}$, on the Banach space $\mathfrak{C}_{\delta,\Phi}(\mathfrak{J},\mathbb{R}^n)$ as follows:*

$$\|\mathtt{u}\|_{\sigma,B} = \sup_{z\in\mathfrak{J}} \frac{\|\mathtt{u}(z)\|}{\mathfrak{L}^{\delta-1}_{\sigma,\Psi}(z,\mathtt{a})\mathbb{E}_{\alpha,\delta}(\varpi\Psi(z,\mathtt{a})^{\alpha})}. \tag{5}$$

Then the norms $\|\cdot\|_{\sigma,B}$ and $\|\cdot\|_{\delta}$ defined by (2) are equivalent.

Proof. Since $\mathbb{E}_{\alpha,\delta}(\varpi\Psi(z,\mathtt{a})^{\alpha}) > 0$ and $e^{\frac{\sigma-1}{\sigma}\Psi(\mathtt{b},\mathtt{a})} \le e^{\frac{\sigma-1}{\sigma}\Psi(z,\mathtt{a})} \le 1$, for $z\in\mathfrak{J}$, one obtains

$$\frac{1}{\mathbb{E}_{\alpha,\delta}(\varpi\Psi(z,\mathtt{a})^{\alpha})} \le \frac{e^{\frac{1-\sigma}{\sigma}\Psi(z,\mathtt{a})}}{\mathbb{E}_{\alpha,\delta}(\varpi\Psi(z,\mathtt{a})^{\alpha})} \le \frac{e^{\frac{1-\sigma}{\sigma}\Psi(\mathtt{b},\mathtt{a})}}{\mathbb{E}_{\alpha,\delta}(\varpi\Psi(z,\mathtt{a})^{\alpha})}.$$

Moreover, the function $\mathbb{E}_{\alpha,\delta}(\varpi\Psi(z,\mathtt{a})^{\alpha})$ is continuous on $\mathfrak{J}$, then there exists $c^*, c_* > 0$ so that $c^* = \sup_{z\in\mathfrak{J}}\mathbb{E}_{\alpha,\delta}(\varpi\Psi(z,\mathtt{a})^{\alpha})$ and $c_* = \inf_{z\in\mathfrak{J}}\mathbb{E}_{\alpha,\delta}(\varpi\Psi(z,\mathtt{a})^{\alpha})$. Accordingly,

$$\frac{1}{c^*}\|u\|_{\mathfrak{C}_{\delta,\Phi}} \le \|u\|_{\sigma,B} \le \frac{\|u\|_{\mathfrak{C}_{\delta,\Phi}}}{c_* e^{\frac{\sigma-1}{\sigma}\Psi(\mathtt{b},\mathtt{a})}},$$

which means that $\|\cdot\|_{\sigma,B}$ and $\|\cdot\|_{\delta}$ are equivalent.

3 Uniqueness of the Solution

This section introduces sufficient conditions to prove the uniqueness of solutions of the system (1). Firstly, consider the Banach space $\mathbf{E} := \mathfrak{C}_{\delta,\Phi}(\mathfrak{J},\mathbb{R}^n)\times\mathfrak{C}_{\delta,\Phi}(\mathfrak{J},\mathbb{R}^n)$ equipped with the Bielecki vector-valued norm

$$\|(\mathtt{u}_1,\mathtt{u}_2)\|_{\mathbf{E},B} = \begin{pmatrix}\|\mathtt{u}_1\|_{\sigma,B}\\ \|\mathtt{u}_2\|_{\sigma,B}\end{pmatrix}.$$

Next, we need the following hypotheses:

$(\mathcal{H}_1)$ $\mathtt{g}_i : \mathfrak{J}\times\mathbb{R}^n\times\mathbb{R}^n \longrightarrow \mathbb{R}^n$ $(i=1,2)$ are functions and

$$\mathtt{g}_i(z,\mathtt{u}_1(\cdot),\mathtt{u}_2(\cdot)) \in \mathfrak{C}^{\beta(1-\alpha)}_{\delta,\Phi}(\mathfrak{J},\mathbb{R}^n) \quad \text{for any} \quad \mathtt{u}_1,\mathtt{u}_2 \in \mathfrak{C}^{\beta(1-\alpha)}_{\delta,\Phi}(\mathfrak{J},\mathbb{R}^n),$$

$(\mathcal{H}_2)$ There exist functions $\chi_i, \widehat{\chi}_i \in L^{\infty}(\mathfrak{J},\mathbb{R}^+)$ such that

$$\|\mathtt{g}_i(z,\mathtt{u}_1,\mathtt{u}_2) - \mathtt{g}_i(z,\mathtt{v}_1,\mathtt{v}_2)\| \le \chi_i(z)\|\mathtt{u}_1-\mathtt{v}_1\| + \widehat{\chi}_i(z)\|\mathtt{u}_2-\mathtt{v}_2\|$$

for all $z\in\mathfrak{J}$ and each $\mathtt{u}_1,\mathtt{u}_2,\mathtt{v}_1,\mathtt{v}_2\in\mathbb{R}^n$.

For the sake of brevity, we will use the notation

$$N_{\sigma,\Phi}(\alpha,z) = \frac{\Psi(\mathtt{z},\mathtt{a})^{\alpha}}{\sigma^{\alpha}\Gamma(\alpha+1)} \quad \text{and} \quad N_{\sigma,\Phi}(\alpha,\mathtt{b},c_*) = \frac{\Psi(\mathtt{b},\mathtt{a})^{1-\beta(1-\alpha)}}{\sigma^{\alpha}\Gamma(\alpha+1)e^{\frac{\sigma-1}{\sigma}\Psi(\mathtt{b},\mathtt{a})}c_*}.$$

Theorem 3.1 *Assume that the conditions $(\mathcal{H}_1)-(\mathcal{H}_2)$ hold. Then system (1) has a unique global solution in $\mathbf{F} := \mathfrak{C}^{\delta}_{\delta,\Phi}(\mathfrak{J},\mathbb{R}^n)\times\mathfrak{C}^{\delta}_{\delta,\Phi}(\mathfrak{J},\mathbb{R}^n)$.*

Proof. By Lemma 3.9 from [17], the solutions of the system (1) are the solutions of the following coupled integral equations:

$$\begin{cases} \mathtt{u}_1(z) &= \frac{\zeta_1}{\sigma^{\delta-1}\Gamma(\delta)}\mathfrak{L}^{\delta-1}_{\sigma,\Psi}(z,\mathtt{a}) + \mathcal{I}^{\alpha,\sigma,\Phi}_{\mathtt{a}^+}\left(\xi_1\mathtt{u}_1(z) + \mathtt{g}_1(z,\mathtt{u}_1(z),\mathtt{u}_2(z))\right), \\ \mathtt{u}_2(z) &= \frac{\zeta_2}{\sigma^{\delta-1}\Gamma(\delta)}\mathfrak{L}^{\delta-1}_{\sigma,\Psi}(z,\mathtt{a}) + \mathcal{I}^{\alpha,\sigma,\Phi}_{\mathtt{a}^+}\left(\xi_2\mathtt{u}_2(z) + \mathtt{g}_2(z,\mathtt{u}_1(z),\mathtt{u}_2(z))\right). \end{cases} \tag{6}$$

We define the operator $\mathcal{L} = (\mathcal{L}_1, \mathcal{L}_2) : \mathbf{E} \longrightarrow \mathbf{E}$ by

$$\mathcal{L}(\mathtt{u}_1,\mathtt{u}_2) = (\mathcal{L}_1(\mathtt{u}_1,\mathtt{u}_2), \mathcal{L}_2(\mathtt{u}_1,\mathtt{u}_2)), \tag{7}$$

where for $i = 1, 2$,

$$\mathcal{L}_i(\mathtt{u}_1,\mathtt{u}_2)(z) = \frac{\zeta_i}{\sigma^{\delta-1}\Gamma(\delta)}\mathfrak{L}^{\delta-1}_{\sigma,\Psi}(z,\mathtt{a}) + \mathcal{I}^{\alpha,\sigma,\Phi}_{\mathtt{a}^+}\left(\xi_i\mathtt{u}_i(z) + \mathtt{g}_i(z,\mathtt{u}_1(z),\mathtt{u}_2(z))\right). \tag{8}$$

Now, let $(\mathtt{u}_1,\mathtt{u}_2),(\mathtt{v}_1,\mathtt{v}_2) \in \mathbf{E}$ and $z \in \mathfrak{J}$, using $(\mathcal{H}_2)$, one gets

$$\begin{aligned} &\|\mathcal{L}_1(\mathtt{u}_1,\mathtt{u}_2)(z) - \mathcal{L}_1(\mathtt{v}_1,\mathtt{v}_2)(z)\| \\ &\leq \|\xi_1\|\mathcal{I}^{\alpha,\sigma,\Phi}_{\mathtt{a}^+}\left(\|\mathtt{u}_1(z) - \mathtt{v}_1(z)\|\right) + \mathcal{I}^{\alpha,\sigma,\Phi}_{\mathtt{a}^+}\left(\|\mathtt{g}_1(z,\mathtt{u}_1(z),\mathtt{u}_2(z)) - \mathtt{g}_1(z,\mathtt{v}_1(z),\mathtt{v}_2(z))\|\right), \\ &\leq \|\xi_1\|\mathcal{I}^{\alpha,\sigma,\Phi}_{\mathtt{a}^+}\left(\|\mathtt{u}_1(z) - \mathtt{v}_1(z)\|\right) + \mathcal{I}^{\alpha,\sigma,\Phi}_{\mathtt{a}^+}\left(\chi_1(z)\|\mathtt{u}_1(z) - \mathtt{v}_1(z)\| + \widehat{\chi}_1(z)\|\mathtt{u}_2(z) - \mathtt{v}_2(z)\|\right), \\ &\leq \|\xi_1\|\|\mathtt{u}_1 - \mathtt{v}_1\|_{\sigma,B}\mathcal{I}^{\alpha,\sigma,\Phi}_{\mathtt{a}^+}\left(\mathfrak{L}^{\delta-1}_{\sigma,\Psi}(z,\mathtt{a})\mathbb{E}_{\alpha,\delta}(\varpi\Psi(z,\mathtt{a})^\alpha)\right), \\ &\quad + \left(\|\chi_1\|_\infty\|\mathtt{u}_1 - \mathtt{v}_1\|_{\sigma,B} + \|\widehat{\chi}_1\|_\infty\|\mathtt{u}_2 - \mathtt{v}_2\|_{\sigma,B}\right)\mathcal{I}^{\alpha,\sigma,\Phi}_{\mathtt{a}^+}\left(\mathfrak{L}^{\delta-1}_{\sigma,\Psi}(z,\mathtt{a})\mathbb{E}_{\alpha,\delta}(\varpi\Psi(z,\mathtt{a})^\alpha)\right). \end{aligned}$$

By Lemma 2.3, we have

$$\begin{aligned} &\|\mathcal{L}_1(\mathtt{u}_1,\mathtt{u}_2)(z) - \mathcal{L}_1(\mathtt{v}_1,\mathtt{v}_2)(z)\| \\ &\leq \Big(\left(\|\xi_1\| + \|\chi_1\|_\infty\right)\|\mathtt{u}_1 - \mathtt{v}_1\|_{\sigma,B} + \|\widehat{\chi}_1\|_\infty\|\mathtt{u}_2 - \mathtt{v}_2\|_{\sigma,B}\Big)\times \\ &\frac{\mathfrak{L}^{\delta-1}_{\sigma,\Psi}(z,\mathtt{a})}{\varpi\sigma^\alpha}\left(\mathbb{E}_{\alpha,\delta}(\varpi\Psi(z,\mathtt{a})^\alpha) - \frac{1}{\Gamma(\delta)}\right). \end{aligned}$$

Hence

$$\begin{aligned} \|\mathcal{L}_1(\mathtt{u}_1,\mathtt{u}_2) - \mathcal{L}_1(\mathtt{v}_1,\mathtt{v}_2)\|_{\sigma,B} &\leq \Big(\left(\|\xi_1\| + \|\chi_1\|_\infty\right)\|\mathtt{u}_1 - \mathtt{v}_1\|_{\sigma,B} + \|\widehat{\chi}_1\|_\infty\|\mathtt{u}_2 - \mathtt{v}_2\|_{\sigma,B}\Big)\times \\ &\quad \frac{1}{\varpi\sigma^\alpha}\left(1 - \frac{1}{\Gamma(\delta)\mathbb{E}_{\alpha,\delta}(\varpi\Psi(z,\mathtt{a})^\alpha)}\right) \\ &\leq \frac{\|\xi_1\| + \|\chi_1\|_\infty}{\varpi\sigma^\alpha}\|\mathtt{u}_1 - \mathtt{v}_1\|_{\sigma,B} + \frac{\|\widehat{\chi}_1\|_\infty}{\varpi\sigma^\alpha}\|\mathtt{u}_2 - \mathtt{v}_2\|_{\sigma,B}. \end{aligned}$$

By the same technique, one obtains

$$\|\mathcal{L}_2(\mathtt{u}_1,\mathtt{u}_2) - \mathcal{L}_2(\mathtt{v}_1,\mathtt{v}_2)\|_{\sigma,B} \leq \frac{\|\chi_2\|_\infty}{\varpi\sigma^\alpha}\|\mathtt{u}_1 - \mathtt{v}_1\|_{\sigma,B} + \frac{\|\xi_2\| + \|\widehat{\chi}_2\|_\infty}{\varpi\sigma^\alpha}\|\mathtt{u}_2 - \mathtt{v}_2\|_{\sigma,B}.$$

This implies that

$$\|\mathcal{L}(\mathtt{u}_1,\mathtt{u}_2) - \mathcal{L}(\mathtt{v}_1,\mathtt{v}_2)\|_{\mathbf{E},B} \leq \mathsf{A}_\varpi\|(\mathtt{u}_1,\mathtt{u}_2) - (\mathtt{v}_1,\mathtt{v}_2)\|_{\mathbf{E},B},$$

where

$$\mathsf{A}_{\varpi} = \frac{1}{\varpi\sigma^{\alpha}} \begin{pmatrix} \|\xi_1\| + \|\chi_1\|_\infty & \|\widehat{\chi}_1\|_\infty \\ \\ \|\chi_2\|_\infty & \|\xi_2\| + \|\widehat{\chi}_2\|_\infty \end{pmatrix}. \tag{9}$$

When taking ϖ large enough, the matrix A_ϖ converges to zero. Therefore, by applying Perov's theorem [19], we show that $\mathcal{L}$ has a unique fixed point $(\mathtt{u}_1, \mathtt{u}_2) \in \mathbf{E}$. Therefore, the system (1) has a unique global solution.

Next, we show that such a fixed point $(\mathtt{u}_1, \mathtt{u}_2) \in \mathbf{E}$ is actually in $\mathbf{F}$. Since $\mathtt{y}_1$ and $\mathtt{y}_2$ are the unique fixed points of $\mathcal{L}_1$ and $\mathcal{L}_2$ in $\mathbf{E}$, respectively, for $z \in (a, b]$, we have

$$\begin{cases} \mathtt{y}_1(z) &= \frac{\zeta_1}{\sigma^{\delta-1}\Gamma(\delta)} \mathfrak{L}^{\delta-1}_{\sigma,\Psi}(z, \mathtt{a}) + \mathcal{I}^{\alpha,\sigma,\Phi}_{\mathtt{a}^+} \left(\xi_1 \mathtt{y}_1(z) + \mathtt{g}_1(z, \mathtt{y}_1(z), \mathtt{y}_2(z))\right), \\ \\ \mathtt{y}_2(z) &= \frac{\zeta_2}{\sigma^{\delta-1}\Gamma(\delta)} \mathfrak{L}^{\delta-1}_{\sigma,\Psi}(z, \mathtt{a}) + \mathcal{I}^{\alpha,\sigma,\Phi}_{\mathtt{a}^+} \left(\xi_2 \mathtt{y}_2(z) + \mathtt{g}_2(z, \mathtt{y}_1(z), \mathtt{y}_2(z))\right). \end{cases}$$

After multiplying both sides of the last system by $\mathcal{D}^{\delta,\sigma,\Phi}_{\mathtt{a}^+}$, it follows from Remark 2.2 and Lemma 2.2 that

$$\begin{cases} \mathcal{D}^{\delta,\sigma,\Phi}_{\mathtt{a}^+} \mathtt{y}_1(z) &= \mathcal{D}^{\beta(1-\alpha),\sigma,\Phi}_{\mathtt{a}^+} \left(\xi_1 \mathtt{y}_1(z) + \mathtt{g}_1(z, \mathtt{y}_1(z), \mathtt{y}_2(z))\right), \\ \\ \mathcal{D}^{\delta,\sigma,\Phi}_{\mathtt{a}^+} \mathtt{y}_2(z) &= \mathcal{D}^{\beta(1-\alpha),\sigma,\Phi}_{\mathtt{a}^+} \left(\xi_2 \mathtt{y}_2(z) + \mathtt{g}_2(z, \mathtt{y}_1(z), \mathtt{y}_2(z))\right). \end{cases}$$

Since $\delta \geq \alpha$ and by $(\mathcal{H}_1)$, we get

$$\begin{cases} \mathcal{D}^{\beta(1-\alpha),\sigma,\Phi}_{\mathtt{a}^+} \left(\xi_1 \mathtt{y}_1(z) + \mathtt{g}_1(z, \mathtt{y}_1(z), \mathtt{y}_2(z))\right) \in \mathfrak{C}_{\delta,\Phi}(\mathfrak{J}, \mathbb{R}^n), \\ \\ \mathcal{D}^{\beta(1-\alpha),\sigma,\Phi}_{\mathtt{a}^+} \left(\xi_2 \mathtt{y}_2(z) + \mathtt{g}_2(z, \mathtt{y}_1(z), \mathtt{y}_2(z))\right) \in \mathfrak{C}_{\delta,\Phi}(\mathfrak{J}, \mathbb{R}^n). \end{cases}$$

Hence $\mathcal{D}^{\delta,\sigma,\Phi}_{\mathtt{a}^+} \mathtt{y}_i \in \mathfrak{C}_{\delta,\Phi}(\mathfrak{J}, \mathbb{R}^n)$, $i = 1, 2$, it follows from the definition of the space $\mathfrak{C}^{\beta(1-\alpha)}_{\delta,\Phi}(\mathfrak{J}, \mathbb{R}^n)$ that $\mathtt{y}_i \in \mathfrak{C}^{\delta}_{\delta,\Phi}(\mathfrak{J}, \mathbb{R}^n)$, $i = 1, 2$. As a sequel to the steps outlined above, we infer that the coupled system (1) has a unique solution in $\mathbf{F}$.

4 Stability Results

This section discusses the stability in the UH and GUH sense of the coupled system (1).

Let $\eta_1, \eta_2 > 0$. We consider the inequalities

$$\begin{cases} \|\mathcal{D}^{\alpha,\beta,\sigma,\Phi}_{\mathtt{a}^+} \mathtt{u}_1(z) - \xi_1 \mathtt{u}_1(z) - \mathtt{g}_1(z, \mathtt{u}_1(z), \mathtt{u}_2(z))\| \leq \eta_1, \\ \\ \|\mathcal{D}^{\alpha,\beta,\sigma,\Phi}_{\mathtt{a}^+} \mathtt{u}_2(z) - \xi_2 \mathtt{u}_2(z) - \mathtt{g}_2(z, \mathtt{u}_1(z), \mathtt{u}_2(z))\| \leq \eta_2, \end{cases} \quad z \in \mathfrak{J}. \tag{10}$$

Definition 4.1 [14, 26] The system (1) is stable in the UH sense if we can find $\Omega_i \geqslant 0$, $i = \overline{1, 4}$, such that for any $\eta_1, \eta_1 > 0$ and any solution $(\mathtt{v}_1, \mathtt{v}_2) \in \mathbf{E}$ of inequalities (10), there exists a solution $(\mathtt{u}_1, \mathtt{u}_2) \in \mathbf{E}$ of the problem (1) with $\zeta_i = \left(\mathcal{I}^{1-\delta,\sigma,\Phi}_{\mathtt{a}^+} \mathtt{v}_i\right)(\mathtt{a}^+)$, $i = 1, 2$, such that

$$\begin{cases} \|\mathtt{v}_1 - \mathtt{u}_1\|_{\sigma,B} \leq \Omega_1 \eta_1 + \Omega_2 \eta_2, \\ \\ \|\mathtt{v}_2 - \mathtt{u}_2\|_{\sigma,B} \leq \Omega_3 \eta_1 + \Omega_4 \eta_2. \end{cases} \tag{11}$$

Definition 4.2 The system (1) is stable in the GUH sence if we can find $\mathcal{N} = (\mathcal{N}_1, \mathcal{N}_2) \in C(\mathbb{R}_+, \mathbb{R}_+) \times C(\mathbb{R}_+, \mathbb{R}_+)$ with $\mathcal{N}(0) = (\mathcal{N}_1(0), \mathcal{N}_2(0)) = (0,0)$ such that for any $\eta = (\eta_1, \eta_2) > 0$ and any solution $(\mathtt{v}_1, \mathtt{v}_2) \in \mathbf{E}$ of inequalities (10), there exists a solution $(\mathtt{u}_1, \mathtt{u}_2) \in \mathbf{E}$ of the problem (1) with $\zeta_i = \left(\mathcal{I}_{\mathtt{a}^+}^{1-\delta,\sigma,\Phi} \mathtt{v}_i\right)(\mathtt{a}^+)$, $i = 1, 2$, complying with

$$\|(\mathtt{v}_1, \mathtt{v}_2) - (\mathtt{u}_1, \mathtt{u}_2)\|_{\mathbf{E},B} \leq \mathcal{N}(\eta).$$

Theorem 4.1 (Stability results) *Suppose that*

1. *All hypotheses of Theorem 3.1 are verified,*
2. *For any $\eta_1, \eta_2 > 0$, the inequalities (10) have at least one solution.*

Then the problem (1) is UH and GUH stable w.r.t to the Bielecki vector-valued norm.

Proof. Let $\eta_1, \eta_2 > 0$ be arbitrary numbers and let $(\mathtt{v}_1, \mathtt{v}_2) \in \mathbf{E}$ be a solution of the inequalities (10). According to Theorem 3.1, there exists a solution $(\mathtt{u}_1, \mathtt{u}_2)$ of the system (1) on $\mathfrak{J}$ with

$$\zeta_i = \left(\mathcal{I}_{\mathtt{a}^+}^{1-\delta,\sigma,\Phi} \mathtt{v}_i\right)(\mathtt{a}^+), \quad i = 1, 2,$$

and it is a fixed point of the operator $\mathcal{L}$ defined by (7).

On the other hand, applying operator $\mathcal{I}_{\mathtt{a}^+}^{\alpha,\sigma,\Phi}$ on both sides of (10), we get

$$\|\mathcal{I}_{\mathtt{a}^+}^{\alpha,\sigma,\Phi} \mathcal{D}_{\mathtt{a}^+}^{\alpha,\beta,\sigma,\Phi} \mathtt{v}_i(z) - \mathcal{I}_{\mathtt{a}^+}^{\alpha,\sigma,\Phi} \left(\xi_i \mathtt{v}_i(z) + \mathtt{g}_i(z, \mathtt{v}_1(z), \mathtt{v}_2(z))\right)\| \leq \mathcal{I}_{\mathtt{a}^+}^{\alpha,\sigma,\Phi} \eta_i, \quad i = 1, 2.$$

Using Lemma 3.8 from [17] yields

$$\|\mathtt{v}_i(z) - \mathcal{L}_i(\mathtt{v}_1, \mathtt{v}_2)(z)\| \leq \eta_i \left(\mathcal{I}_{\mathtt{a}^+}^{\alpha,\sigma,\Phi} 1\right)(z), \quad i = 1, 2.$$

Hence, we have by Remark 2.1,

$$\|\mathtt{v}_i(z) - \mathcal{L}_i(\mathtt{v}_1, \mathtt{v}_2)(z)\| \leq \frac{\eta_i \Psi(z, \mathtt{a})^\alpha}{\sigma^\alpha \Gamma(\alpha+1)} := \eta_i N_{\sigma,\Phi}(\alpha, z), \quad i = 1, 2. \tag{12}$$

Now, by $(\mathcal{H}_2)$ and using (12) and Lemma 2.3, we have

$$\begin{aligned}
&\|\mathtt{v}_1(z) - \mathtt{u}_1(z)\| \\
&= \|\mathtt{v}_1(z) - \mathcal{L}_1(\mathtt{v}_1, \mathtt{v}_2)(z) + \mathcal{L}_1(\mathtt{v}_1, \mathtt{v}_2)(z) - \mathtt{u}_1(z)\|, \\
&\leq \|\mathtt{v}_1(z) - \mathcal{L}_1(\mathtt{v}_1, \mathtt{v}_2)(z)\| + \|\mathcal{L}_1(\mathtt{v}_1, \mathtt{v}_2)(z) - \mathcal{L}_1(\mathtt{u}_1, \mathtt{u}_2)(z)\|, \\
&\leq \eta_1 N_{\sigma,\Phi}(\alpha, z) + \left(\left(\|\xi_1\| + \|\chi_1\|_\infty\right) \|\mathtt{v}_1 - \mathtt{u}_1\|_{\sigma,B} + \|\widehat{\chi}_1\|_\infty \|\mathtt{v}_2 - \mathtt{u}_2\|_{\sigma,B}\right) \\
&\quad \times \frac{\mathfrak{L}_{\sigma,\Psi}^{\delta-1}(z,\mathtt{a})}{\varpi \sigma^\alpha} \left(\mathbb{E}_{\alpha,\delta}(\varpi \Psi(z, \mathtt{a})^\alpha) - \frac{1}{\Gamma(\delta)}\right).
\end{aligned}$$

Hence, we get

$$\|\mathtt{v}_1 - \mathtt{u}_1\|_{\sigma,B} \leq N_{\sigma,\Phi}(\alpha, \mathtt{b}, c_*)\eta_1 + \frac{\|\xi_1\| + \|\chi_1\|_\infty}{\varpi \sigma^\alpha} \|\mathtt{v}_1 - \mathtt{u}_1\|_{\sigma,B} + \frac{\|\widehat{\chi}_1\|_\infty}{\varpi \sigma^\alpha} \|\mathtt{v}_2 - \mathtt{u}_2\|_{\sigma,B}. \tag{13}$$

Similarly, we have

$$\|\mathtt{v}_2-\mathtt{u}_2\|_{\sigma,B}\le N_{\sigma,\Phi}(\alpha,\mathtt{b},c_*)\eta_2+\frac{\|\chi_2\|_\infty}{\varpi\sigma^\alpha}\|\mathtt{v}_1-\mathtt{u}_1\|_{\sigma,B}+\frac{\|\xi_2\|+\|\widehat{\chi}_2\|_\infty}{\varpi\sigma^\alpha}\|\mathtt{v}_2-\mathtt{u}_2\|_{\sigma,B}. \quad (14)$$

Inequalities (13) and (14) yield

$$(\mathsf{I}-\mathsf{A}_\varpi)\begin{pmatrix}\|\mathtt{v}_1-\mathtt{u}_1\|_{\sigma,B}\\ \|\mathtt{v}_2-\mathtt{u}_2\|_{\sigma,B}\end{pmatrix}\le\begin{pmatrix}N_{\sigma,\Phi}(\alpha,\mathtt{b},c_*)\eta_1\\ N_{\sigma,\Phi}(\alpha,\mathtt{b},c_*)\eta_2\end{pmatrix}, \quad (15)$$

where A_ϖ is defined in (9). Hence, A_ϖ converges to zero when taking ϖ large enough, by Theorem 2.1, $(\mathsf{I}-\mathsf{A}_\varpi)^{-1}$ has nonnegative elements since the matrix $(\mathsf{I}-\mathsf{A}_\varpi)$ is non-singular. Accordingly, (15) is equivalent to

$$\begin{pmatrix}\|\mathtt{v}_1-\mathtt{u}_1\|_{\sigma,B}\\ \|\mathtt{v}_2-\mathtt{u}_2\|_{\sigma,B}\end{pmatrix}\le(\mathsf{I}-\mathsf{A}_\varpi)^{-1}\begin{pmatrix}N_{\sigma,\Phi}(\alpha,\mathtt{b},c_*)\eta_1\\ N_{\sigma,\Phi}(\alpha,\mathtt{b},c_*)\eta_2\end{pmatrix}, \quad (16)$$

which means that

$$\begin{cases}\|\mathtt{v}_1-\mathtt{u}_1\|_{\sigma,B} & \le & \Omega_1\eta_1+\Omega_2\eta_2,\\ \|\mathtt{v}_2-\mathtt{u}_2\|_{\sigma,B} & \le & \Omega_3\eta_1+\Omega_4\eta_2,\end{cases} \quad (17)$$

where $\Omega_i=\theta_i N_{\sigma,\Phi}(\alpha,\mathtt{b},c_*)$ and $(\theta_i)_{i=\overline{1,4}}$ are the elements of $(\mathsf{I}-\mathsf{A}_\varpi)^{-1}$. This proves that system (1) is UH stable.

Moreover, let

$$\theta=(\theta_1,\theta_2)=(\Omega_1+\Omega_3,\Omega_2+\Omega_4),$$

we can write inequalities (17) as

$$\|(\mathtt{v}_1,\mathtt{v}_2)-(\mathtt{u}_1,\mathtt{u}_2)\|_{\mathbf{E},B}\le\mathcal{N}(\eta),$$

where $\mathcal{N}(\eta)=\theta\eta$ with $\mathcal{N}(0)=(0)$. This shows that the coupled system (1) is GUH stable.

5 Examples

Here, we provide some illustrative examples to validate the obtained results.

Example 1. We consider a particular case of our systems (1) with $\mathfrak{J}=[0,\mathtt{b}]$ and

$$\xi_1=\begin{pmatrix}1/2 & 1\\ 3/2 & 2\end{pmatrix},\quad \xi_2=\begin{pmatrix}5 & 6\\ 7 & 8\end{pmatrix},$$

and $\mathtt{g}_1,\mathtt{g}_2:\mathfrak{J}\times\mathbb{R}^2\times\mathbb{R}^2\to\mathbb{R}^2$ such that $\mathtt{u}_1=(\mathtt{t}_1,\mathtt{t}_2),\mathtt{u}_2=(\mathtt{r}_1,\mathtt{r}_2)$ with

$$\mathtt{g}_1(z,\mathtt{u}_1(z),\mathtt{u}_2(z))=\begin{pmatrix}\frac{|\mathtt{t}_1(z)|+|\mathtt{t}_2(z)|}{z+5}\\ e^{-2z-1}\log_e(1+|\mathtt{r}_1(z)|+|\mathtt{r}_2(z)|)\end{pmatrix},$$

$$\mathtt{g}_2(z,\mathtt{u}_1(z),\mathtt{u}_2(z))=\frac{1}{e^z+5}\begin{pmatrix}\sin(|\mathtt{r}_1(z)|+|\mathtt{r}_2(z)|)\\ \tan^{-1}(\mathtt{t}_1(z)+\mathtt{t}_2(z))\end{pmatrix}.$$

Clearly, the functions $\mathtt{u}_i$, $i = 1, 2$, are continuous. Furthermore, for all $\mathtt{x}_1, \mathtt{x}_2, \bar{\mathtt{x}}_1, \bar{\mathtt{x}}_2 \in \mathbb{R}^2$ and $z \in \mathfrak{J}$, one has

$$\|\mathtt{g}_1(z, \mathtt{x}_1, \mathtt{x}_2) - \mathtt{g}_1(z, \bar{\mathtt{x}}_1, \bar{\mathtt{x}}_2)\|_1 \leq \frac{1}{z+5}\|\mathtt{x}_1 - \bar{\mathtt{x}}_1\|_1 + \frac{1}{e^{2z+1}}\|\mathtt{x}_2 - \bar{\mathtt{x}}_2\|_1$$

and

$$\|\mathtt{g}_2(z, \mathtt{x}_1, \mathtt{x}_2) - \mathtt{g}_2(z, \bar{\mathtt{x}}_1, \bar{\mathtt{x}}_2)\|_1 \leq \frac{1}{e^z+5}\|\mathtt{x}_1 - \bar{\mathtt{x}}_1\|_1 + \frac{1}{e^z+5}\|\mathtt{x}_2 - \bar{\mathtt{x}}_2\|_1,$$

where $\|\cdot\|_1$ is a norm in $\mathbb{R}^2$ defined as follows:

$$\|(\mathtt{x}_1, \mathtt{x}_2)\|_1 = |\mathtt{x}_1| + |\mathtt{x}_2|.$$

Hence the hypothesis $(\mathcal{H}_2)$ is satisfied with

$$\chi_1(z) = \frac{1}{z+5}, \quad \widehat{\chi}_1(z) = \frac{1}{e^{2z+1}}, \quad \chi_2(z) = \widehat{\chi}_2(z) = \frac{1}{e^z+5}.$$

Obviously,

$$\|\chi_1\|_{L^\infty} = \frac{1}{5}, \quad \|\widehat{\chi}_1\|_{L^\infty} = \frac{1}{e}, \quad \|\chi_2\|_{L^\infty} = \|\widehat{\chi}_2\|_{L^\infty} = \frac{1}{6},$$

and

$$\|\xi_1\|_{\max} = 2, \quad \|\xi_2\|_{\max} = 8,$$

where $\|\cdot\|_{\max}$ is a norm of the matrix $\mathsf{A} = (a_{k,j})$ defined as follows:

$$\|\mathsf{A}\|_{\max} = \max_{k,j} |a_{k,j}|.$$

Moreover, the matrix A_ϖ defined by (9) has the following form:

$$\mathsf{A}_\varpi = \begin{pmatrix} \frac{11}{5\varpi\sigma^\alpha} & \frac{1}{e\varpi\sigma^\alpha} \\ \\ \frac{1}{6\varpi\sigma^\alpha} & \frac{49}{6\varpi\sigma^\alpha} \end{pmatrix}.$$

It converges to zero for ϖ large enough. By Theorem 3.1, system (1) with the above conditions, has a unique solution. System (1) is not only UH stable but also GUH stable, according to Theorem 4.1.

Example 2. Consider the following systems:

$$\begin{cases} \mathcal{D}_{0^+}^{\frac{1}{2},\frac{1}{2},\frac{1}{4},\Phi}\mathtt{u}_1(z) = \frac{2}{3}\mathtt{u}_1(z) + \mathtt{g}_1(z, \mathtt{u}_1(z), \mathtt{u}_2(z)), \\ \mathcal{D}_{0^+}^{\frac{1}{2},\frac{1}{2},\frac{1}{4},\Phi}\mathtt{u}_2(z) = \frac{1}{5}\mathtt{u}_2(z) + \mathtt{g}_2(z, \mathtt{u}_1(z), \mathtt{u}_2(z)), \end{cases} \quad z \in (0, 1] \tag{18}$$

with the initial conditions

$$\begin{cases} \left(\mathcal{I}_{0^+}^{1-\frac{3}{4},\frac{1}{4},\Phi}\mathtt{u}_1\right)(0^+) = \sqrt{2}, \\ \left(\mathcal{I}_{0^+}^{1-\frac{3}{4},\frac{1}{4},\Phi}\mathtt{u}_2\right)(0^+) = \sqrt{3}. \end{cases} \tag{19}$$

In this case, we take

$$\alpha = \beta = \frac{1}{2}, \delta = \frac{3}{4}, \sigma = \frac{1}{4}, \xi_1 = \frac{2}{3}, \xi_2 = \frac{1}{5}, \zeta_1 = \sqrt{2}, \zeta_2 = \sqrt{3}, \mathtt{a} = 0, \mathtt{b} = 1, n = 1,$$

and

$$\begin{aligned} \mathtt{g}_1(z,\mathtt{u}_1(z),\mathtt{u}_2(z)) &= \tfrac{\sin(\mathtt{u}_1(z))}{e^{-z}+8} + (z+\sqrt{2})\ln(|\mathtt{u}_2(z)|+1), \\ \mathtt{g}_2(z,\mathtt{u}_1(z),\mathtt{u}_2(z)) &= \tfrac{1}{e^{z}+8}\left(1+\tfrac{|\mathtt{u}_1(z)|}{1+|\mathtt{u}_1(z)|}\right) + (e^z+1)(\arctan(\mathtt{u}_2(z))+1). \end{aligned}$$

Then

$$\mathfrak{C}_{\delta,\Phi}(\mathfrak{J},\mathbb{R}^n) = \mathfrak{C}_{3/4,\Phi}([0,1],\mathbb{R}) = \left\{\mathtt{u}:[0,1]\to\mathbb{R}:\ \Psi(z,0)^{1/4}\mathtt{u}\in\mathfrak{C}([0,1],\mathbb{R})\right\},$$

$$\mathfrak{C}^{\beta(1-\alpha)}_{\delta,\Phi}(\mathfrak{J},\mathbb{R}^n) = \mathfrak{C}^{1/4}_{3/4,\Phi}([0,1],\mathbb{R}) = \left\{\mathtt{u}\in\mathfrak{C}_{3/4,\Phi}([0,1],\mathbb{R}):\ \mathcal{D}^{1/4,1/4,\Phi}_{0^+}\mathtt{u}\in\mathfrak{C}_{3/4,\Phi}([0,1],\mathbb{R})\right\}.$$

It is clear that the continuous functions $\mathtt{g}_i \in \mathfrak{C}^{1/4}_{3/4,\Phi}([0,1],\mathbb{R})$, $i=1,2$. So, hypothesis $(\mathcal{H}_1)$ is satisfied. And, for any $\mathtt{u}_1,\mathtt{v}_1,\mathtt{u}_2,\mathtt{v}_2\in\mathbb{R}$ and $z\in\mathfrak{J}$, we have

$$|\mathtt{g}_1(z,\mathtt{u}_1(z),\mathtt{u}_2(z)) - \mathtt{g}_1(z,\mathtt{v}_1(z),\mathtt{v}_2(z))| \le \chi_1(z)|\mathtt{u}_1(z)-\mathtt{v}_1(z)| + \widehat{\chi}_1(z)|\mathtt{u}_2(z)-\mathtt{v}_2(z)|,$$

$$|\mathtt{g}_2(z,\mathtt{u}_1(z),\mathtt{u}_2(z)) - \mathtt{g}_2(z,\mathtt{v}_1(z),\mathtt{v}_2(z))| \le \chi_2(z)|\mathtt{u}_1(z)-\mathtt{v}_1(z)| + \widehat{\chi}_2(z)|\mathtt{u}_2(z)-\mathtt{v}_2(z)|,$$

where

$$\chi_1(z) = \frac{1}{e^{-z}+8},\quad \widehat{\chi}_1(z) = z+\sqrt{2},\quad \chi_2(z) = \frac{1}{e^{z}+8},\quad \widehat{\chi}_2(z) = e^z+1.$$

Obviously,

$$\|\chi_1\|_\infty = \frac{1}{e^{-1}+8},\quad \|\widehat{\chi}_1\|_\infty = 1+\sqrt{2},\quad \|\chi_2\|_\infty = \frac{1}{9},\quad \|\widehat{\chi}_2\|_\infty = e+1.$$

Then the matrix A_ϖ has the following representation:

$$\mathsf{A}_\varpi = \frac{2}{\varpi}\begin{pmatrix} \frac{2+19e}{3+24e} & 1+\sqrt{2} \\ \frac{1}{9} & e+\frac{6}{5} \end{pmatrix}.$$

For $\varpi > 0$ suitably chosen, by virtue of Theorem 3.1, the system (18)-(19) has a unique solution in $\mathfrak{C}_{3/4,\Phi}([0,1],\mathbb{R})\times\mathfrak{C}_{3/4,\Phi}([0,1],\mathbb{R})$. Therefore, from Theorem 4.1, coupled system (18)-(19) is UH and GUH stable.

Example 3. In a particular case, for $\alpha=\frac{1}{2}$, $\beta=1$, $\Phi(z)=z$, the coupled system for Φ-Hilfer proportional FDE (1) reduces to the coupled system for the Caputo proportional FDE given by

$$\begin{cases} \mathcal{D}^{\frac{1}{2},\frac{1}{4}}_{0^+}\mathtt{u}_1(z) &= \frac{2\mathtt{u}_1(z)}{3} + \frac{\sin(\mathtt{u}_1(z))}{e^{-z}+8} + (z+\sqrt{2})\ln(|\mathtt{u}_2(z)|+1),\ z\in(0,1], \\ \mathcal{D}^{\frac{1}{2},\frac{1}{4}}_{0^+}\mathtt{u}_2(z) &= \frac{\mathtt{u}_2(z)}{5} + \frac{1}{e^z+8}\left(1+\frac{|\mathtt{u}_1(z)|}{1+|\mathtt{u}_1(z)|}\right) + \frac{\arctan(\mathtt{u}_2(z))+1}{(e^z+1)^{-1}},\ z\in(0,1], \end{cases} \tag{20}$$

and

$$(\mathcal{I}^{0,\frac{1}{2},z}_{0^+}\mathtt{u}_1(0^+), \mathcal{I}^{0,\frac{1}{2},z}_{0^+}\mathtt{u}_2(0^+)) = (0,0). \tag{21}$$

We have

$$\mathfrak{C}_{\delta,\Phi}(\mathfrak{J},\mathbb{R}^n) = \mathfrak{C}_{1,z}([0,1],\mathbb{R})$$

and

$$\mathfrak{C}^{\beta(1-\alpha)}_{\delta,\Phi}(\mathfrak{J},\mathbb{R}^n) = \mathfrak{C}^{1/2}_{1,z}([0,1],\mathbb{R}).$$

As all the hypotheses of Theorem 3.1 and Theorem 4.1 are satisfied, then system (18)-(19) has a unique solution in $\mathfrak{C}_{1,z}([0,1],\mathbb{R})\times\mathfrak{C}_{1,z}([0,1],\mathbb{R})$ and is UH and GUH stable.

6 Conclusion

In this paper, we successfully employed the fixed-point approach in a vector-valued Banach space to establish qualitative results for a coupled system driven by the Φ-Hilfer PFD. Our analysis relies on the convergence to zero of the matrices and introduces a new Bielecki-type vector-valued norm, allowing us to avoid any extra assumptions. We also investigate Ulam's types stability. Consequently, numerous findings in the literature can be recovered through our results. By varying the functions Φ, β, and σ, one can explore several cases of our system (1). More precisely:

1. Proportional Φ-Riemann–Liouville FD : By taking $\beta = 0$.
2. Proportional Φ-Caputo FD : By taking $\beta = 1$.
3. Φ-Hilfer FD : By taking $\sigma = 1$.
4. Hilfer-Hadmard FD : By taking $\Phi(z) = \ln(z)$.
5. Hilfer-Katugampola FD : By taking $\Phi(z) = z^{\mu}$, where $\mu > 0$.

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