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On Neumann Systems with Singularity Applied in Quenching Phenomena in Museilack Spaces

A. Elouardani [1], A. Aberqi [2] and M. Elmassoudi [1*]

[1] *Faculty of Sciences Dhar El Mahraz, Sidi Mohamed Ben Abdellah University, Fez, Morocco.*
[2] *National School of Applied Sciences, Sidi Mohamed Ben Abdellah University, Fez, Morocco.*

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Abstract: In this study, by the uniform control of the singularity and energy estimates, we establish the existence of solutions for the reaction-diffusion model with singularity due to Quenching phenomena in the boundary.

$$\begin{cases} \dfrac{\partial u_1}{\partial t} - \operatorname{div}\Big(a(x,t,\nabla u_1)\Big) = f_1(t,x)h_\gamma(u_1,u_2) & \text{in} \quad Q_T, \\ \dfrac{\partial u_2}{\partial t} - \operatorname{div}\Big(a(x,t,\nabla u_2)\Big) = f_2(t,x)h_\gamma(u_1,u_2) & \text{in} \quad Q_T, \end{cases} \tag{1}$$

where the operator $A(u) = \operatorname{div}\Big(a(x,t,\nabla u_i)\Big)$ is a generalized Leray-Lions operator defined on the inhomogeneous Musielak-Orlicz spaces (the vector field $a(x,t,\nabla u_i)$ has a growth prescribed by a generalized N-function).

Keywords: *quenching phenomena, Museilack-Orlciz spaces, local singularity, nonlinear dynamical systems.*

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* Corresponding author: mailto:elmassoudi09@gmail.com

1 Introduction

Let Ω be an open subset of $\mathbb{R}^N$ $(N \geq 2)$, and let $T > 0$, $Q_T = \Omega \times (0,T)$, $\Gamma_1$and Γ_2 are such that $\Gamma_1 \cup \Gamma_2 = \partial\Omega$ and $\Gamma_1 \cap \Gamma_2 = \emptyset$. The Haussdorff measure of Γ_1 and Γ_2 does not vanish. Here ν denotes the outer normal to $\partial\Omega$.

This paper is concerned with the existence of solutions for a nonlinear parabolic singular reaction-diffusion system of the form (i=1,2)

$$(\mathcal{S}) \begin{cases} \dfrac{\partial u_i}{\partial t} - \operatorname{div}\big(a(x,t,\nabla u_1)\big) = F_i(t,x,u_1,u_2) & \text{in} \quad Q_T, \\ u_i(0,x) = u_{i,0}(x) & \text{in} \quad \Omega. \end{cases} \tag{2}$$

The following boundary conditions:

$$\begin{cases} u_i(t,x) = 0 & \text{on} \quad \Gamma_i \times (0,T), \\ \dfrac{\partial u_1}{\partial \nu} = 0 & \text{on} \quad \Gamma_2 \times (0,T), \\ \dfrac{\partial u_2}{\partial \nu} = 0 & \text{on} \quad \Gamma_1 \times (0,T) \end{cases} \tag{3}$$

hold with

$$F_i(t,x,r,s) = f_i(t,x)h_\gamma(s,r). \tag{4}$$

The functions $f_1, f_2 : Q_T \mapsto R$ satisfy the following conditions:

$$f_1, f_2 \in L^1(Q_T), \quad f_1 \geq 0, \quad \text{and} \quad f_1 + f_2 \leq 0. \tag{5}$$

The singular sourcing $h_\gamma : \mathbb{R}^* \times \mathbb{R} \mapsto \mathbb{R}$ is continuous and must verify $0 \leq h_\gamma(s,r) \leq \frac{s}{r^\gamma}$ with γ being a real parameter such as $0 < \gamma \leq 1$.

We assume that the functions

$$u_{1,0} \text{ and } u_{2,0} \text{ are such that } u_{1,0}, u_{2,0} \geq 0 \text{ and } \quad u_{1,0}, u_{2,0} \in L^\infty(\Omega). \tag{6}$$

Define the following spaces:

$$V_1 = \left\{\varphi \in W^1L_M(\Omega) : \varphi = 0 \text{ on } \Gamma_1\right\} \text{ and } V_2 = \left\{\psi \in W^1L_M(\Omega) : \psi = 0 \text{ on } \Gamma_2\right\},$$

where $W^1L_M(\Omega)$ is the Musielak-Orlicz-Sobolev space defined below.

The operator $A(u_i) = -div\Big(a(x,t,\nabla u_i)\Big)$ is a generalized Leray-Lions operator defined on the inhomogeneous Musielak-Orlicz spaces (see assumptions (7)–(9)).

The investigation of quenching phenomena traces back to 1975, when Kawarada published a paper [10] delving into a dynamical model. This pivotal work catalyzed an extensive exploration of the quenching systems by numerous researchers dedicating their efforts to the aspects such as the existence and characterization of quenching points, and the asymptotic behaviour of solutions, among other topics.

The solution maintains its boundedness in quenching systems, but the first-order time derivative undergoes unbounded growth within a finite time frame. This concept is also elaborated upon in Salin's research [15] and Selçuk's work [16]. For the numerical approach, Beauregard M.A. et al. [3] proposed a fully adaptive approximation for quenching systems over circular domains. Recently, Marion P.T. [13] studied quenching phenomena due to a concentrated nonlinear source in an infinitely long cylinder.

Hence, the aim here is to explore a concept referred to as "quenching", which extends the notion of explosion. Our goal is to investigate systems exhibiting nonstandard growth and inhomogeneous nonlinearities, which have notable relevance in the realm of physics. Furthermore, we consider the inclusion of non-homogeneous Neumann boundary conditions, where we introduce a generalization of systems previously studied in [5] for the $(p,q)-$Laplacian with the Dirichlet boundaries ($M(x,t) = |t|^p$ in Lebesgue spaces).

$$\begin{cases} -\mathrm{div}\big(h_1(x)|\nabla u|^{p-2}\nabla u)\big) = \lambda F_{u_1}(x,u_1,u_2) & \text{in} \quad \Omega, \\ -\mathrm{div}\big(h_2(x)|\nabla u|^{q-2}\nabla u)\big) = \lambda F_{u_2}(x,u_1,u_2) & \text{in} \quad \Omega, \\ u_1(x) = u_2(x) & \text{in} \quad \partial\Omega, \end{cases}$$

where F is a $\mathcal{C}^1-$function with $\nabla F = (F_{u_1}, F_{u_2})$. And $(p(x),q(x))-$Laplacian systems were considered in [9], where the authors explore the system with cross-diffusion in Lebesgue spaces with variable exponents.

$$\begin{cases} \partial_t u - \mathrm{div}\big(a(x,t,\nabla u)\big) = \mathrm{div}(|F|^{p(x,t)-2}F) & \text{in} \quad \Omega, \\ \partial_t v - \mathrm{div}\big(a(x,t,\nabla u)\big) = \delta\Delta u & \text{in} \quad \Omega, \end{cases}$$

where $\delta \geq 0$, and $a(x,t,.)$ is a Leray-Lions vector field, see also [1, 2] and references therein.

To our knowledge, the results obtained in this paper have not been covered in the existing literature. The organization of this contribution is given as follows. We start in Section 2 by presenting some definitions and properties in the framework of Museilack spaces. After that, in Section 3, we give our essential assumptions and set up the definition of the weak solution of the system (**S**). Then we establish the main existence Theorem 4.1.

2 Mathematical Preliminaries

2.1 Musielak-Orlicz function

In addressing the issue presented in system (2)-(3), we employ the framework of Orlicz-Sobolev spaces. This choice arises from the presence of a nonhomogeneous function $M(x,t)$ within the differential operator of system (2)-(3). Consequently, our approach commences by introducing fundamental concepts on Museilack-Orlicz-Sobolev spaces (see [14]). Let $M : \Omega \times \mathbb{R}_+ \to \mathbb{R}$ satisfy

(M_1): $M(x,.)$ is an N-function for all $x \in \Omega$ (i.e., it is convex, non-decreasing, continuous, $M(x,0) = 0$, $M(x,t) > 0$ for $t > 0$, $\lim\limits_{t\to 0}\sup\limits_{x\in\Omega}\dfrac{M(x,t)}{t} = 0$ and $\lim\limits_{t\to\infty}\inf\limits_{x\in\Omega}\dfrac{M(x,t)}{t} = \infty$).

(M_2): $M(.,t)$ is a measurable function for all $t \geq 0$. A function M which satisfies the conditions (M_1) and (M_2) is called a Musielak-Orlicz function.

We say that γ grows essentially less rapidly than M at 0 (resp. near infinity), and we write $\gamma \ll M$. For every positive constant c, we have
$\lim\limits_{t\to 0}\big(\sup\limits_{x\in\Omega}\dfrac{\gamma(x,ct)}{M(x,t)}\big) = 0$ (resp. $\lim\limits_{t\to\infty}\big(\sup\limits_{x\in\Omega}\dfrac{\gamma(x,ct)}{M(x,t)}\big) = 0\big)$.

2.2 Musielak-Orlicz space

For a Musielak-Orlicz function M and a measurable function $u : \Omega \to \mathbb{R}$, we define the functional, $\varrho_{M,\Omega}(u) = \displaystyle\int_\Omega M(x, |u(x)|)dx$.

$$L_M(\Omega) = \{u : \Omega \to \mathbb{R} \quad \text{mesurable}: \quad \varrho_{M,\Omega}(\frac{u}{\lambda}) < \infty, \quad \text{for some} \quad \lambda > 0\}.$$

For any Musielak-Orlicz function M, we put $\bar{M}(x,s) = \sup_{t\geq 0}(st - M(x,s))$, the conjugate of M. We say that a sequence of function $u_n \in L_M(\Omega)$ is modularly convergent to $u \in L_M(\Omega)$ if there exists a constant $\lambda > 0$ such that $\lim_{n\to\infty} \varrho_{M,\Omega}(\frac{u_n - u}{\lambda}) = 0$.

This implies convergence for $\sigma(\Pi L_M, \Pi L_{\bar{M}})$ (see [4]).

In the space $L_M(\Omega)$, we define the following norm:

$$\|u\|_M = \inf\Big\{\lambda > 0 : \int_\Omega M(x, \frac{|u(x)|}{\lambda})dx \leq 1\Big\},$$

which is called the Luxemburg norm. The closure in $L_M(\Omega)$ of the set of bounded measurable functions with compact support in $\overline{\Omega}$ is denoted $E_M(\Omega)$. It is a separable space and $(E_M(\Omega))^* = L_M(\Omega)$. We define

$$W^1L_M(\Omega) = \{u \in L_M(\Omega) : D^\alpha u \in L_M(\Omega), \quad \forall \alpha \leq 1\}$$
$$W^1E_M(\Omega) = \{u \in E_M(\Omega) : D^\alpha u \in E_M(\Omega), \quad \forall \alpha \leq 1\},$$

where $D^\alpha u$ denotes the distributional derivatives. The space $W^1L_M(\Omega)$ is called the Musielak-Orlicz-Sobolev space. Let $\overline{\varrho}_{M,\Omega}(u) = \sum_{|\alpha|\leq 1} \varrho_{M,\Omega}(D^\alpha u)$ and $\|u\|^1_{M,\Omega} = \inf\{\lambda > 0 : \overline{\varrho}_{M,\Omega}(\frac{u}{\lambda}) \leq 1\}$ for $u \in W^1L_M(\Omega)$. The pair $(W^1L_M(\Omega), \|u\|^1_{M,\Omega})$ is a Banach space if M satisfies the following condition (see [14]). There exists a constant $c > 0$ such that $\inf_{x\in\Omega} M(x,1) > c$.

2.3 Inhomogeneous Musielak-Orlicz-Sobolev spaces

For each $\alpha \in \mathbb{N}^N$, denote by D^α_x the distributional derivative on Q of order α with respect to the variable $x \in \mathbb{R}^N$. The inhomogeneous Musielak-Orlicz-Sobolev spaces of order 1 are defined as follows:

$$W^{1,x}L_M(Q) = \big\{u \in L_M(Q) : \forall |\alpha| \leq 1, \quad D^\alpha_x u \in L_M(Q)\big\}$$

$$W^{1,x}E_M(Q) = \big\{u \in E_M(Q) : \forall |\alpha| \leq 1, \quad D^\alpha_x u \in E_M(Q)\big\}.$$

The last is a subspace of the first one, and both are Banach spaces under the norm

$$\|u\| = \sum_{|\alpha|\leq m} \|D^\alpha_x u\|_{M,Q}.$$

$\mathcal{D}(\Omega)$ is dense in $L_M(\Omega)$ with respect to the modular topology, $\mathcal{D}(\Omega)$ is dense in $W^1_0L_M(\Omega)$ for the modular convergence and $\mathcal{D}(\overline{\Omega})$ is dense in $W^1_0L_M(\Omega)$ for the modular convergence.

Lemma 2.1 *(see [8]) Suppose that Ω satisfies the segment property and let $u \in W^1_0L_M(\Omega)$. Then there exists a sequence $u_n \in \mathcal{D}(\Omega)$ such that $u_n \to u$ for modular convergence in $W^1_0L_M(\Omega)$. Furthermore, if $u \in W^1_0L_M(\Omega) \cap L^\infty(\Omega)$, then $\|u_n\|_\infty \leq (N+1)\|u\|_\infty$.*

Lemma 2.2 *(see [8]) If a sequence* $u_n \in L_M(\Omega)$ *converges a.e. to* u *and if* u_n *remains bounded in* $L_M(\Omega)$*, then* $u \in L_M(\Omega)$ *and* $u_n \rightharpoonup u$ *for* $\sigma(L_M(\Omega), E_{\bar{M}}(\Omega))$.

Lemma 2.3 *(see [8]) Let* $u_n, u \in L_M(\Omega)$. *If* $u_n \to u$ *with respect to the modular convergence, then* $u_n \rightharpoonup u$ *for* $\sigma(L_M(\Omega), L_{\bar{M}}(\Omega))$.

3 Essential Assumptions

Suppose Ω satisfies the segment property, and let M and γ be two Musielak-Orlicz functions such that M and its complementary $\bar{M}$ satisfy conditions of Lemma 2.2 from [1], $\gamma \ll M$ and $t^2 \leq M(x,t)$.

$A : D(A) \subset W^{1,x}L_M(\Omega) \to W^{-1,x}L_{\bar{M}}(\Omega)$ defined by $A(u_i) = -\mathrm{div}(a(x,t,\nabla u_i))$, where $a : Q_T \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function such that for a.e. $x \in \Omega$ and for all $s \in \mathbb{R}$, $\xi, \xi^* \in \mathbb{R}^N$ with $\xi \neq \xi^*$.

$$|a(x,t,\xi)| \leq \beta(c(x) + \bar{M}_x^{-1}(M(x,\nu_2|\xi|))),\ \beta > 0,\ c(x) \in E_{\bar{M}}(\Omega), \tag{7}$$

$$(a(x,t,\xi) - a(x,t,\xi^*)(\xi - \xi^*) > 0, \tag{8}$$

$$a(x,t,\xi)\xi \geq \alpha M(x,|\xi|). \tag{9}$$

$f_i : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function.

Lemma 3.1 *The following continuous inclusions are valid:* $L_M(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow L_{\bar{M}}(\Omega)$. *In particular,* $W_0^1 L_M(\Omega) \hookrightarrow H_0^1(\Omega)$, $H^{-1}(\Omega) \hookrightarrow W^{-1}L_{\bar{M}}(\Omega)$, $L^2(0,T,H^{-1}(\Omega)) \hookrightarrow W^{-1,x}L_{\bar{P}}(\Omega)$ *and* $\hookrightarrow W^{-1,x}L_{\bar{M}}(\Omega)$.

Example 3.1 The N-function $M(t) = m(x)t^2 log(e+t)$, where $m \in (L^\infty(\Omega))^+$, verifies the assumptions above.

In all the rest of this paper and for all measurable subset E of Q_T, we designate by $meas(E)$ the Lebesgue measure of E. First, we will define the weak solution of the problem (2)-(4) used here.

Definition 3.1 A weak solution to problem (3)-(4) is a nonnegative couple

$$(u_1,u_2) \in \Big[L^2(0,T;V_1) \cap L^\infty(0,T;L^2(\Omega))\Big] \times \Big[L^2(0,T;V_2) \cap L^\infty(0,T;L^2(\Omega))\Big]$$

with

$$(u_{1,t},u_{2,t}) \in \Big[L^2(0,T;V_1^*) + L^1\big(0,T;L^1_{loc}(\Omega)\big)\Big] \times \Big[L^2(0,T;V_2^*) + L^\infty\big(0,T;L^1_{loc}(\Omega)\big)\Big],$$

$$u_1(0,x) = u_{1,0}(x), u_2(0,x) = u_{2,0}(x), \quad a.e. \quad x \in \Omega.$$

$$\int_\Omega f_1 \frac{u_2}{(u_1)^\gamma}\phi < \infty \quad , \quad \int_\Omega f_2 \frac{u_2}{(u_1)^\gamma}\eta < \infty,$$

and

$$-\int_\Omega u_{1,0}(x)\varphi(0,x) - \int_{Q_T} u_1 \frac{\partial \varphi}{\partial t} + \int_{Q_T} a(t,x,\nabla u_1)\nabla\varphi = \int_{Q_T} F_1\varphi,$$

$$-\int_\Omega u_{2,0}(x)\psi(0,x) - \int_{Q_T} u_2 \frac{\partial \psi}{\partial t} + \int_{Q_T} a(t,x,\nabla u_2)\nabla\psi = \int_{Q_T} F_2\psi$$

for all $\varphi, \psi, \eta, \phi \in C_0^\infty([0,T) \times \Omega)$.

4 Existence Result

We shall establish the following existence theorem.

Theorem 4.1 *Suppose that* (5)–(6) *hold, then system (3)-(4) admits at least a couple of weak solutions in the sense of Definition 3.1.*

The proof is divided into 5 steps.

4.1 Step 1: Regularized system.

For $n > 0$ and $i = 1, 2$,

$$f_{1,n}(t,x) = \min \big\{n, f_1(t,x)\big\}, \tag{10}$$

$$f_{2,n}(t,x) = \text{-}\min \big\{n, -f_2(t,x)\big\}. \tag{11}$$

The functions $f_{1,n}$ and $f_{2,n}$ verify the same properties as f_1 and f_2, moreover, $0 \leq f_{1,n} \leq f_1$, $\lim\limits_{n\to+\infty} f_{1,n} = f_1$, and $f_2 \leq f_{2,n} \leq 0$, $\lim_{n\to+\infty} f_{2,n} = f_2$. Let us now consider the regularized problem

$$(\mathcal{S}_n)\begin{cases} \dfrac{\partial u_{i,n}}{\partial t} - div\big(a(x,t,\nabla u_{i,n})\big) = F_{i,n}(t,x,u_{1,n},u_{2,n}) & \text{in} \quad Q_T,\\ u_{i,n}(0,x) = u_{i,n}^{(0)}(x) & \text{in} \quad \Omega,\\ u_{i,n}(t,x) = 0 & \text{on} \quad \Gamma_i \times (0,T),\\ \dfrac{\partial u_{1,n}}{\partial \nu} = 0 & \text{on} \quad \Gamma_2 \times (0,T),\\ \dfrac{\partial u_{2,n}}{\partial \nu} = 0 & \text{on} \quad \Gamma_1 \times (0,T). \end{cases} \tag{12}$$

$$F_{i,n}(t,x,u_{1,n},u_{2,n}) = \begin{cases} f_{i,n}(t,x)h_\gamma(u_{1,n} + \dfrac{1}{n}, u_{2,n}) & \text{if } u_{1,n} \geq 0 \text{ and } \ u_{2,n} \geq 0,\\ 0 & \text{otherwise.}\end{cases}$$

Without obviating that $u_{1,n}^{(0)}, u_{2,n}^{(0)} \in L_\infty(\Omega) \cap W_0^1 L_M(\Omega)$ is a couple of regularized sequences of the initial data obtained since $W_0^1 L_M(\Omega) \hookrightarrow H_0^1(\Omega)$), by a standard technique (see [6]), we have

$$\lim_{n\to+\infty} \frac{1}{n}||u_{1,n}^{(0)}||_{W_0^1 L_M(\Omega)} = 0 \qquad \text{and} \qquad \lim_{n\to+\infty} \frac{1}{n}||u_{2,n}^{(0)}||_{W_0^1 L_M(\Omega)} = 0. \tag{13}$$

Lemma 4.1 *The problem (12) admits a nonnegative couple of solutions*

$$(u_{1,n}, u_{2,n}) \in [L^2(0,T;V_1) \cap L^\infty(Q_T)] \times [L^2(0,T;V_2) \cap L^\infty(Q_T)]$$

such that

$$-\int_\Omega u_{1,n}^{(0)}(x)\varphi(0,x) - \int_{Q_T} u_{1,n}\frac{\partial \varphi}{\partial t} + \int_{Q_T} a(t,x,\nabla u_{1,n})\nabla\varphi = \int_{Q_T} F_{1,n}(t,x,u_{1,n},u_{2,n})\varphi, \tag{14}$$

$$-\int_\Omega u_{2,n}^{(0)}(x)\psi(0,x) - \int_{Q_T} u_{2,n}\frac{\partial \psi}{\partial t} + \int_{Q_T} a(t,x,\nabla u_{2,n})\nabla\psi = \int_{Q_T} F_{2,n}(t,x,u_{1,n},u_{2,n})\psi \tag{15}$$

for every $\varphi, \psi, \in C_0^\infty([0,T) \times \Omega)$.

Proof. Without losing generality, we assume $u_{1,n}^{(0)} = 0$ and $u_{2,n}^{(0)} = 0$. Then, by using the Stampacchia method [18], we show the positivity of the solution. Take $\varphi = -u_{1,n}^-$ as a test function in the first equation of the problem (12), where $u_{1,n} = u_{1,n}^+ - u_{1,n}^-$, $u_{1,n}^+ = \max\big\{u_{1,n}, 0\big\}, u_{1,n}^- = \max\big\{-u_{1,n}, 0\big\}$.

Since $u_{1,n}^+ = 0$ on supp$u_{1,n}^-$, the right-hand side of (14) is zero because

$$F_n(t,x,u_{1,n},u_{2,n}) = \begin{cases} f_{1,n}(t,x)h_\gamma(u_{1,n} + \dfrac{1}{n}, u_{2,n}) & \text{if } u_{1,n} \geq 0 \text{ and } \ u_{2,n} \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

So we have

$$\int_{Q_T} (u_{1,n})_t(-u_{1,n}^-) + \int_{Q_T} a(t,x,\nabla u_{1,n})\nabla(-u_{1,n}^-) = 0,$$

which gives

$$\int_{Q_T} (u_{1,n}^+ - u_{1,n}^-)_t(-u_{1,n}^-) + \int_{Q_T} a(t,x,\nabla u_{1,n})\nabla(-u_{1,n}^-) = 0.$$

We observe that on the support of $u_{1,n}$, we have $u_{1,n}^+ u_{1,n}^- = 0$ on supp$u_{1,n}$, it comes

$$\int_{Q_T} (u_{1,n}^+ - u_{1,n}^-)_t(-u_{1,n}^-) + \int_{Q_T\cap\{u_{1,n}\geq 0\}} a(t,x,\nabla u_{1,n})\nabla(-u_{1,n}^-)$$

$$+ \int_{Q_T\cap\{u_{1,n}\leq 0\}} a(t,x,\nabla u_{1,n})\nabla(-u_{1,n}^-) = 0$$

$$\int_\Omega \frac{1}{2}(u_{1,n})^2 + \int_{Q_T} a(t,x,\nabla(-u_{1,n}^-))\nabla(-u_{1,n}^-) = 0$$

$$\int_\Omega \frac{1}{2}(u_{1,n})^2 + \int_{Q_T} a\big(t,x,,\nabla(-u_{1,n}^-)\big)\nabla(-u_{1,n}^-) \geq \int_\Omega \frac{1}{2}(u_{1,n}^-)^2 + \alpha \int_{Q_T} M\big(|\nabla(u_{1,n}^-)|\big),$$

$$u_{1,n}^- = 0 \quad \text{a.e in}\, Q_T, \tag{16}$$

i.e., $u_{1,n} \geq 0$ a.e. in Ω and for all $t \in [0,T)$. Analogously, we prove that $u_{2,n} \geq 0$ by the testing function $\psi = -u_{2,n}^-$. Let us denote by T_k the truncation function $T_k(s) = \max\big\{-k, \min\big\{k,s\big\}\big\}$, $k \geq 0, s \in \mathbb{R}$, where $\langle .,. \rangle$ denotes the duality product between V_i^* and V_i for $i = 1,2$.

4.2 Step 2: L^∞ and energy estimates for $(u_{1,n}, u_{2,n})$

Proposition 4.1 *There exist positive constants C_1 and C_2, independent of n, such that*

$$||u_{1,n}||_{L^\infty(Q_T)} \leq C_1, \ \textit{and}\ ||u_{2,n}||_{L^\infty(Q_T)} \leq C_2. \tag{17}$$

Proof. The estimate (17) for $\{u_{1,n}\}$ follows directly by Proposition 2.13 [7] with some abbreviations that go along with our problem, it remains to prove the uniform boundedness of $u_{2,n}$.

For simplicity, we suppose $u_{2,n}^{(0)}(x)=0$, to handle the equation solved by $u_{2,n}$, we choose as a test function $\psi=(u_{2,n}-C_2)^+$, with $C_2>1$ fixed, we obtain with $Q_t=\Omega\times[0,t)$,

$$\int_\Omega (u_{2,n})_t(u_{2,n}-C_2)^+ + \int_{Q_T} a(t,x,\nabla u_{2,n})\nabla(u_{2,n}-C_2)^+$$
$$= \int_\Omega f_{2,n}(t,x)h_\gamma(u_{1,n}+\frac{1}{n},u_{2,n})(u_{2,n}-C_2)^+ \leq 0. \tag{18}$$

$$\begin{aligned}\int_{Q_T} a(t,x,\nabla u_{2,n})\nabla(u_{2,n}-C_2)^+ &= \int_{\{(t,x)\in Q_T/u_{2,n}(t,x)\leq C_2\}} a(t,x,\nabla u_{2,n})\nabla(v_n-C_2)^+\\ &+\int_{\{(t,x)\in Q_T/u_{2,n}(t,x)\geq C_2\}} a(t,x,\nabla u_{2,n})\nabla(u_{2,n}-C_2)^+\\ &=\int_{\{(t,x)\in Q_T/u_{2,n}(t,x)\geq C_2\}} a(t,x,,\nabla u_{2,n})\nabla u_{2,n}.\end{aligned} \tag{19}$$

When neglecting the nonnegative term on the left-hand side of (18), it comes

$$(u_{2,n}-C_2)^+=0, \quad \text{a.e.} \quad \text{in } Q_T,$$

which completes the proof. Choosing as a test function $\varphi=u_{1,n}\in L^2(0,T;V)$ in the first equation of problem (12), by integrating over $\Omega\times[0,t)$, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_{Q_T}(u_{1,n})^2+\int_{Q_T} a(t,x,\nabla u_{1,n})\nabla u_{1,n} = \int_{Q_T} f_{1,n}(t,x)h_\gamma(u_{1,n}+\frac{1}{n},u_{2,n})u_{1,n},$$

in other words,

$$\frac{1}{2}\int_\Omega u_{1,n}^2(t)+\int_{Q_T} a(t,x,\nabla u_{1,n})\nabla u_{1,n} = \frac{1}{2}||u_{0,n}||^2_{L^2(\Omega)}+\int_{Q_T} f_{1,n}(t,x)h_\gamma(u_{1,n}+\frac{1}{n},u_{2,n})u_{1,n}.$$

Due to the fact that $h_\gamma(u_{1,n}+\frac{1}{n},u_{2,n})\leq \frac{u_{1,n}}{(u_{1,n}+\frac{1}{n})^\gamma}\leq u_{1,n}^{1-\gamma}$ and $0<1-\gamma<1$, we deduce

$$\int_{Q_T} f_{1,n}(t,x)h_\gamma(u_{1,n}+\frac{1}{n},u_{2,n})u_{2,n}\leq \int_{Q_T} f_{1,n}(t,x)u_{1,n}^{1-\gamma}u_{2,n}\leq \sup u_{1,n}^{1-\gamma}u_{2,n}\int_{Q_T} f_{1,n}\leq C,$$

again, we have

$$||u_{1,n}||_{L^\infty(0,T;L^2(\Omega))}\leq C.$$

For the second equation of problem (12), we choose as a test function $\psi=u_{2,n}\in L^2(0,T;V_2)$. We obtain

$$\begin{aligned}\frac{1}{2}\int_\Omega u_{2,n}^2(t)+\int_{Q_T} a(t,x,\nabla u_{2,n})\nabla u_{2,n} &= \frac{1}{2}||u_{2,n}^{(0)}||^2_{L^2(\Omega)}+\int_\Omega f_{2,n}(t,x)h_\gamma(u_{1,n}+\frac{1}{n},u_{2,n})u_{2,n}\\ &\leq \frac{1}{2}||u_{2,n}^{(0)}||^2_{L^2(\Omega)},\end{aligned} \tag{20}$$

which gives $||u_{2,n}||_{L^\infty\left(0,T;L^2(\Omega)\right)} \le C$, hence the inequality (15).

Note that $h_\gamma(u_{1,n}+\frac{1}{n},u_{2,n}) \le \dfrac{u_{1,n}}{(u_{1,n}+\frac{1}{n})^\gamma} \le u_{1,n}^{1-\gamma}$ and $0<1-\gamma<1$, using the L^∞-estimate of $u_{1,n}$ and (9), by plugging $\varphi = u_{1,n} \in L^2(0,T;V_1)$ as a test function in (12), we easily get

$$||u_{1,n}||_{L^\infty(0,T;L^2(\Omega))} \le C_3,$$

where C_3 is a positive constant independent of n. Similarly, by plugging $\psi = u_{2,n} \in L^2(0,T;V_2)$ in (12), we get $||u_{2,n}||_{L^\infty\left(0,T;L^2(\Omega)\right)} \le C_4$, hence, there exists a positive constant C_5, independent of n,

$$||u_{1,n}||_{L^2(0,T;V_1)} + ||u_{2,n}||_{L^2(0,T;V_2)} \le C_5. \tag{21}$$

Proposition 4.2 *There exists a positive constant $C>0$, independent of n, such that*

$$\int_{Q_T} f_{1,n}(t,x)h_\gamma(u_{1,n}+\frac{1}{n},u_{2,n})\varphi^2(x) \le C \text{ for all } n\in\mathbb{N} \tag{22}$$

for every $\varphi\in C_0^\infty(Q_T)$. For $\delta>0$, we have

$$\int_{Q_T\cap\{0\le u_{1,n}\le\delta\}} f_{1,n}(t,x)h_\gamma(u_{1,n}+\frac{1}{n},u_{2,n})\varphi^2(x) \le C\delta, \tag{23}$$

$$\int_{Q_T\cap\{0\le u_{2,n}\le\delta\}} f_{1,n}(t,x)h_\gamma(u_{1,n}+\frac{1}{n},u_{2,n})\varphi^2(x) \le \begin{cases} C\delta^{1-\gamma} & \text{if } 0<\gamma<1,\\ C\sqrt{\delta} & \text{if } \gamma=1\end{cases} \tag{24}$$

for every $\varphi\in C_0^\infty(Q_T)$ with $\varphi\ge 0$.

Proof. The proof follows the same lines as Proposition 2.20 in [7], see also [12], with slight modifications since the growth of $a(t,x,\nabla u)$ with embedding in Lemma 3.1, we examine only the estimate (22) for the convenience. Multiplying the first equation of problem (12) by the test function $\varphi^2(x)$, we get

$$\int_0^T \left\langle (u_{1,n})_t, \varphi^2(x)\right\rangle + 2\int_{Q_T} a(t,x,\nabla u_{1,n}).\varphi\nabla\varphi = \int_{Q_T} f_{1,n}(t,x)h_\gamma(u_{1,n}+\frac{1}{n},u_{2,n})\varphi^2(x),$$

which gives

$$\int_{Q_T} f_{1,n}(t,x)h_\gamma(u_{1,n}+\frac{1}{n},u_{2,n})\varphi^2(x) \le 2\int_{Q_T} |a(t,x,\nabla u_{1,n})||\varphi||\nabla\varphi| + C,$$

this gives, by applying Hölder's inequality and the previous proposition,

$$\int_{Q_T} f_{1,n}(t,x)h_\gamma(u_{1,n}+\frac{1}{n},u_{2,n})\varphi^2(x) \le C+2\left(\int_{Q_T}|a(t,x,\nabla u_{1,n})|^2\varphi^2\right)^{\frac{1}{2}}.\left(\int_{Q_T}|\nabla(\varphi)|^2\right)^{\frac{1}{2}}$$
$$\le C_2,$$

which proves the inequality (22).

A similar estimate is in the following Proposition 4.3, by multiplying problem (12) by the test function $\psi^2(x)$.

Proposition 4.3 *For $\delta > 0$, we have*

$$\int_{Q_T} |f_{2,n}(t,x)| h_\gamma(u_{1,n} + \frac{1}{n}, u_{2,n}) \psi^2(x) \leq C \, , \textit{ for all } n \in \mathbb{N}, \tag{25}$$

$$\int_{Q_T \cap \{0 \leq u_{2,n} \leq \delta\}} \big| f_{2,n}(t,x) | h_\gamma(u_{1,n} + \frac{1}{n}, u_{2,n}) \varphi^2(x) \leq C\delta, \tag{26}$$

$$\int_{Q_T \cap \{0 \leq u_{1,n} \leq \delta\}} \big| f_{2,n}(t,x) | h_\gamma(u_{1,n} + \frac{1}{n}, u_{2,n}) \varphi^2(x) \leq \left\{ \begin{array}{ll} C\delta^{1-\gamma} & \textit{if } 0 < \gamma < 1 \, , \\ C\sqrt{\delta} & \textit{if } \gamma = 1. \end{array} \right. \tag{27}$$

4.3 Step 3: Convergence and compactness results

To pass to the limit as $n \to \infty$ in (14) and (15), we need strongly convergent subsequences, which is ensured by the following proposition.

Proposition 4.4 *There exists a couple*

$$(u_1, u_2) \in \Big[L^2(0,T;V_1) \cap L^\infty(Q_T) \Big] \times \Big[L^2(0,T;V_2) \cap L^\infty(Q_T) \Big]$$

such that, as $n \to \infty$, we have, up to a subsequence,

$$u_{n,i} \rightharpoonup u_i \quad \textit{weakly in} \quad L^2(0,T;V_i) \quad \textit{for} \quad i = 1,2, \tag{28}$$

$$u_{n,i} \rightharpoonup u_i \quad \textit{weakly in} \quad L^\infty(Q_T) \quad \textit{for} \quad i = 1,2, \tag{29}$$

$$u_{n,i} \to u_i \quad \textit{strongly in} \quad L^1(Q_T) \quad \textit{for} \quad i = 1,2, \tag{30}$$

$$u_{n,i} \to u_i \quad \textit{a.e.in} \quad Q_T \quad \textit{for} \quad i = 1,2, \tag{31}$$

$$\lim_{n \to +\infty} \int_{Q_T} \big| \nabla(u_{n,i} - u_i) \big|^2 = 0 \quad \textit{for} \quad i = 1,2. \tag{32}$$

Therefore,

$$\nabla u_{n,i} \to \nabla u_i \ \textit{a.e. in} \ Q_T \quad \textit{for} \quad i = 1,2. \tag{33}$$

Proof. Convergences (28) are direct consequences of (21). The same thing applies to convergences (29). To prove (30) , we observe that the estimate (22) leads to

$$f_{1,n}(t,x) h_\gamma(u_{1,n} + \frac{1}{n}, u_{2,n}) \varphi^2(x) \in L^1(Q_T) \quad , \quad \forall \varphi \in C_0^\infty(\Omega). \tag{34}$$

In addition, we have

$$\frac{\partial(u_{1,n}\varphi^2)}{\partial t} \quad \text{is bounded in} \quad L^2(0,T;V_1^*) + L^1(Q_T). \tag{35}$$

By (35), choosing s such that $s > \frac{N}{2} + 1$ and using the same argument as in Lemma 2.3 from [17], we deduce that $\frac{\partial(u_{1,n}\varphi)}{\partial t}$ is bounded in $L^2(0,T;H^{-s})$, consequently, since $s > \frac{N}{2}$, we find that $V_1 \subset L^p(\Omega) \subset H^{-s}(\Omega)$ and the embedding $V_1 \hookrightarrow L^p(\Omega)$ is compact. Applying now Corollary 4 from [17], by (35) and the compactness results, we deduce that $u_{1,n}\varphi$ is relatively compact in $L^2(Q_T)$. Hence, up to a subsequence, convergences (30) and (31) are satisfied.

(32) and (33) can be obtained as a particular case of Proposition 3.14 from [7].

4.4 Step 4: Uniform estimate near the singularity

We consider the set $\{(t,x) \in Q_T : u_1(t,x) = 0\}$. As a consequence of the uniform estimate near the singularity (23), we have the following proposition.

Proposition 4.5 *The couple (u_1, u_2) as a solution to (2) and (3), in the sense of Definition 2.1, satisfies*

$$\int_{Q_T \cap \{u_1=0\}} f_1(t,x) h_\gamma(u_1,u_2)\varphi = 0 \tag{36}$$

$$\int_{Q_T \cap \{u_1=0\}} f_2(t,x) h_\gamma(u_1,u_2)\psi = 0 \tag{37}$$

for all $\varphi, \psi \in C_0^\infty\big(\Omega \times [0,T)\big)$ with $\varphi, \psi \geq 0$. Moreover,

$$\int_{Q_T} f_1(t,x) h_\gamma(u_1,u_2)\varphi = \int_{Q_T \cap \{u>0\}} f_1(t,x) h_\gamma(u_1,u_2)\varphi \tag{38}$$

$$\int_{Q_T} f_2(t,x) h_\gamma(u_1,u_2)\psi = \int_{Q_T \cap \{u>0\}} f_2(t,x) h_\gamma(u_1,u_2)\psi. \tag{39}$$

Proof.

1. Let $\psi \in C_0^\infty\big(\Omega \times [0,T)\big), \psi \geq 0$, with supp$\psi = Y \times [0, T_1]$, $T_1 < T$, $Y \subset\subset E \subset\subset \Omega$ and $\varphi \in C_0^1\big(\Omega\big)$ with $\varphi(x) = 1$ over Y , $\varphi \geq 0$ with supp $\varphi = E$, by the uniform estimate (23), we obtain

$$\begin{aligned}
&\int_{Q_T \cap \{u_{1,n}<\delta\}} f_{1,n}(t,x) h_\gamma\left(u_{1,n} + \frac{1}{n}, u_{2,n}\right)\psi(t,x) \\
&\leq ||\psi||_\infty \int_{[0,T]\times Y} f_{1,n}(t,x) h_\gamma(u_{1,n} + \frac{1}{n}, u_{2,n})\chi_{\{u_{1,n}<\delta\}} \\
&\leq ||\psi||_\infty \int_{Q_T} f_{1,n}(t,x) h_\gamma\left(u_{1,n} + \frac{1}{n}, u_{2,n}\right)\varphi^2(x)\chi_{\{u_{1,n}<\delta\}}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\int_{Q_T} f_{1,n}(t,x) h_\gamma\left(u_{1,n} + \frac{1}{n}, u_{2,n}\right)\chi_{\{u_{1,n}<\delta\}}\psi(t,x) \\
&= \int_{Q_T} f_{1,n}(t,x) h_\gamma\left(u_{1,n} + \frac{1}{n}, u_{2,n}\right)\chi_{\{u_{1,n}<\delta\}}\chi_{\{u_1=\delta\}}\psi(t,x) \\
&+ \int_{Q_T} f_{1,n}(t,x) h_\gamma\left(u_{1,n} + \frac{1}{n}, u_{2,n}\right)\chi_{\{u_{1,n}<\delta\}}\chi_{\{u_1\neq\delta\}}\psi(t,x) \\
&\leq C\delta.
\end{aligned} \tag{40}$$

We observe that there exists at most a countable set D_1 such that meas$\{(t,x) : u_1(t,x) = \delta\} > 0$. We choose δ outside of this set D_1 so that, in (40),

$$\int_{Q_T} f_{1,n}(t,x) h_\gamma\left(u_{1,n} + \frac{1}{n}, u_{2,n}\right)\chi_{\{u_{1,n}<\delta\}}\chi_{\{u_1=\delta\}}\psi(t,x) = 0.$$

So, we have

$$\begin{aligned}&\int_{Q_T} f_{1,n}(t,x)h_\gamma\left(u_{1,n}+\frac{1}{n},u_{2,n}\right)\chi_{\{u_{1,n}<\delta\}}\psi(t,x)\\&=\int_{Q_T} f_{1,n}(t,x)h_\gamma\left(u_{1,n}+\frac{1}{n},u_{2,n}\right)\chi_{\{u_{1,n}<\delta\}}\chi_{\{u_1\neq\delta\}}\psi(t,x)\le C\delta.\end{aligned}\tag{41}$$

By (31), we have $\chi_{\{u_{1,n}<\delta\}}\chi_{\{u_1\neq\delta\}}\to\chi_{\{u_1<\delta\}}$ a.e. in Q_T.
Fix δ, in (41), by Fatou's lemma, we get

$$\int_{Q_T} f_1(t,x)h_\gamma(u_1,u_2)\chi_{\{u<\delta\}}\psi(t,x)\le C\delta.$$

Let δ to 0, Fatou's lemma leads to

$$\int_{Q_T} f_1(t,x)h_\gamma(u_1,u_2)\chi_{\{u=0\}}\psi(t,x)=\int_{Q_T\cap\{u_1=0\}} f_1(t,x)h_\gamma(u_1,u_2)\psi(t,x)=0,\tag{42}$$

this leads to

$$\int_{Q_T} f_1(t,x)h_\gamma(u_1,u_2)\psi(t,x)=\int_{Q_T\cap\{u_1>0\}} f_1(t,x)\psi(t,x).\tag{43}$$

These are the identities we are aiming to establish.

2. Similarly, with some simplifications, we establish the validity of (37) and (39). By plugging a test function $\phi\in C_0^\infty([0,T)\times\Omega),\phi\ge 0$ with supp $\phi=Y\times[0,T_2],T_2<T,\bar{Y}\subset\subset E\subset\subset\Omega$ and $\varphi\in C_0^1(\Omega)$ with $\varphi(x)=1$ over $\bar{Y}$, $\varphi\ge 0$ with supp $\varphi=E$. By the uniform estimate (27), we obtain

$$\begin{aligned}&\int_{Q_T\cap\{u_{1,n}\le\delta\}} |f_{2,n}(t,x)|h_\gamma\left(u_{1,n}+\frac{1}{n},u_{2,n}\right)\phi(t,x)\\&\le\|\phi\|_\infty\int_{[0,T]\times Y}\big|f_{2,n}(t,x)\big|h_\gamma\left(u_{1,n}+\frac{1}{n},u_{2,n}\right)\chi_{\{u_{1,n}<\delta\}}\\&\le\|\phi\|_\infty\int_{[0,T]\times Y}\big|f_{2,n}(t,x)\big|h_\gamma\left(u_{1,n}+\frac{1}{n},u_{2,n}\right)\varphi^2(x)\chi_{\{u_{1,n}<\delta\}},\\&\le\begin{cases}C\delta^{1-\gamma} & \text{if } 0<\gamma<1\\ C\sqrt{\delta} & \text{if } \gamma=1.\end{cases}\end{aligned}$$

On the other hand,

$$\begin{aligned}&\int_{Q_T}\big|f_{2,n}(t,x)\big|h_\gamma\left(u_{1,n}+\frac{1}{n},u_{2,n}\right)\chi_{\{u_{1,n}<\delta\}}\phi(t,x)\\&=\int_{Q_T}\big|f_{2,n}(t,x)\big|h_\gamma\left(u_{1,n}+\frac{1}{n},u_{2,n}\right)\chi_{\{u_{1,n}<\delta\}}\chi_{\{u_1=\delta\}}\phi(t,x)\\&+\int_{Q_T}\big|f_{2,n}(t,x)\big|h_\gamma\left(u_{1,n}+\frac{1}{n},u_{2,n}\right)\chi_{\{u_{1,n}<\delta\}}\chi_{\{u_1\neq\delta\}}\phi(t,x),\\&\le\begin{cases}C\delta^{1-\gamma} & \text{if } 0<\gamma<1\\ C\sqrt{\delta} & \text{if } \gamma=1.\end{cases}\end{aligned}\tag{44}$$

Note that there exists at most a countable set $\bar{D}_2$: meas $\{(t,x) : u_1(t,x) = \delta\} > 0$. Select $\delta \notin \bar{D}_2$, so, in (44), the integral

$$\int_{Q_T} |f_{2,n}(t,x)| h_\gamma \left(u_{1,n} + \frac{1}{n}, u_{2,n}\right) \chi_{\{u_{1,n}<\delta\}} \chi_{\{u_1=\delta\}} \phi(t,x) = 0.$$

So, we have

$$\begin{aligned} &\int_{Q_T} |f_{2,n}(t,x)| h_\gamma \left(u_{1,n} + \frac{1}{n}, u_{2,n}\right) \chi_{\{u_{1,n}<\delta\}} \phi(t,x) \\ &= \int_{Q_T} |f_{2,n}(t,x)| h_\gamma \left(u_{1,n} + \frac{1}{n}, u_{2,n}\right) \chi_{\{u_{1,n}<\delta\}} \chi_{\{u_1\neq\delta\}} \phi(t,x), \\ &\leq \begin{cases} C\delta^{1-\gamma} & \text{if } 0<\gamma<1 \\ C\sqrt{\delta} & \text{if } \gamma = 1. \end{cases} \end{aligned} \tag{45}$$

By (31), we have $\chi_{\{u_{1,n}<\delta\}} \chi_{\{u_1\neq\delta\}} \to \chi_{\{u_1<\delta\}}$ a.e. in Q_T.
For δ fixed, Fatou's lemma in (45) leads to

$$\int_{Q_T} f_2(t,x) h_\gamma(u_1,u_2) \chi_{\{u_1<\delta\}} \phi(t,x) \leq \begin{cases} C\delta^{1-\gamma} & \text{if } 0<\gamma<1 \\ C\sqrt{\delta} & \text{if } \gamma = 1. \end{cases}$$

Again, by Fatou's lemma, we get for $\delta \to 0$,

$$\int_{Q_T} |f_2(t,x)| h_\gamma(u_1,u_2) \chi_{\{u_1=0\}} \phi(t,x) = \int_{Q_T \cap \{u_1=0\}} |f_2(t,x)| h_\gamma(u_1,u_2) \phi(t,x) = 0, \tag{46}$$

this leads to

$$\int_{Q_T} |f_2(t,x)| h_\gamma(u_1,u_2) \chi_{\{u_1=0\}} \phi(t,x) = \int_{Q_T \cap \{u>0\}} |f_2(t,x)| h_\gamma(u_1,u_2) \phi(t,x), \tag{47}$$

which also means

$$\int_{Q_T} f_2(t,x) h_\gamma(u_1,u_2) \chi_{\{u_1=0\}} \phi(t,x) = \int_{Q_T \cap \{u_1>0\}} f_2(t,x) h_\gamma(u_1,u_2) \phi(t,x),$$

this is the desired result.

4.5 Step 5: Proof of main theorem

Now, we present the proof of the primary result in this paper. Since $u_{1,n}, u_{2,n} \geq 0$ a.e. in Q_T , thanks to (29), we obtain $u_1, u_2 \geq 0$. Thanks to (30), we can proceed to take the limit in the sections related to the time derivatives of (14) and (15). By (14), $(\nabla u_{1,n})$ and $(\nabla u_{2,n})$ are equi-integrable. By (31) and (33), Vitali's theorem (see Theorem 1.0.16 in [11]) leads to

$$\nabla u_{1,n} \to \nabla u_1 \quad \text{in} \quad L^2(Q_T), \tag{48}$$

$$\nabla u_{2,n} \to \nabla u_2 \quad \text{in} \quad L^2(Q_T). \tag{49}$$

We deal now with the singular lower-order terms. Let $D = K \times [0, T_1], T_1 < T$ such that $K \subset\subset E \subset\subset \Omega$ and $\psi \in C_0^\infty(\Omega)$ with supp$\psi = D$. Let ϕ be a function such that

$\phi(x) = 1$ on the set K, $0 \leq \phi \leq 1$ and supp $\phi = E$.

Let $\delta > 0$, we have

$$\begin{cases} \displaystyle\int_{Q_T} f_{1,n}(t,x)h_\gamma(u_{1,n} + \frac{1}{n}, u_{2,n})\psi = \int_{Q_T \cap \{0 \leq u_{1,n} < \delta\}} f_{1,n}(t,x)h_\gamma(u_{1,n} + \frac{1}{n}, u_{2,n})\psi \\ \displaystyle + \int_{Q_T \cap \{u_{1,n} \geq \delta\}} f_{1,n}(t,x)h_\gamma(u_{1,n} + \frac{1}{n}, u_{2,n})\psi = \mathit{\Delta}_1 + \mathit{\Delta}_2. \end{cases}$$

Concerning the term $\mathit{\Delta}_1$, we proceed as

$$\begin{aligned} \mathit{\Delta}_1 \quad &\leq ||\psi||_\infty \int_{D \cap \{0 \leq u_{1,n} < \delta\}} f_{1,n}(t,x)h_\gamma\left(u_{1,n} + \frac{1}{n}, u_{2,n}\right)\varphi^2(x) \\ &\leq ||\psi||_\infty \int_{Q_T \cap \{0 \leq u_{1,n} < \delta\}} f_{1,n}(t,x)h_\gamma\left(u_{1,n} + \frac{1}{n}, u_{2,n}\right)\varphi^2(x). \end{aligned}$$

By (23), we get to

$$\mathit{\Delta}_1 \leq C\delta, \tag{50}$$

where C is a constant independent of n. For $\mathit{\Delta}_2$, we have

$$\begin{aligned} \mathit{\Delta}_2 = &\int_{Q_T} f_{1,n}(t,x)h_\gamma\left(u_{1,n} + \frac{1}{n}, u_{2,n}\right)\psi(t,x)\chi_{\{u_{1,n} \geq \delta\}}\chi_{\{u_1 \neq \delta\}} \\ &+ \int_{Q_T} f_{1,n}(t,x)h_\gamma\left(u_{1,n} + \frac{1}{n}, u_{2,n}\right)\psi(t,x)\chi_{\{u_{1,n} \geq \delta\}}\chi_{\{u_1 = \delta\}} \\ &= \mathit{\Delta}_3 + \mathit{\Delta}_4. \end{aligned}$$

Note that there is at most a countable set Θ_1 such that meas$\{(t,x) : u_1(t,x) = \delta\} > 0$. Choose $\delta \notin \Theta_1$, so $\mathit{\Delta}_4 = 0$. Since (31) holds, for $\mathit{\Delta}_3$, we have that $\chi_{\{u_{1,n} \geq \delta\}}\chi_{\{u_1 \neq \delta\}} \to \chi_{\{u_1 > \delta\}}$ a.e. in Q_T. $f_{1,n}(t,x)h_\gamma\left(u_{1,n} + \frac{1}{n}, u_{2,n}\right)\chi_{\{u_{1,n} \geq \delta\}}\chi_{\{u_1 \neq \delta\}}\psi(t,x) \leq f_1(t,x)h_\gamma(\delta, u_{2,n})\psi(t,x) \in L^1(Q_T)$. By (31) and the Lebesgue theorem, we have

$$\begin{aligned} \lim_{n \to +\infty} &\int_{Q_T} f_{1,n}(t,x)h_\gamma\left(u_{1,n} + \frac{1}{n}, u_{2,n}\right)\chi_{\{u_{1,n} \geq \delta\}}\chi_{\{u_1 \neq \delta\}}\psi(t,x) \\ &= \int_{Q_T} f_1(t,x)h_\gamma(u_1, u_2)\chi_{\{u_1 \geq \delta\}}\psi(t,x). \end{aligned}$$

Then

$$\lim_{n \to +\infty} \mathit{\Delta}_2 = \int_{Q_T} f_1(t,x)h_\gamma(\delta, u_1)\chi_{\{u_1 \geq \delta\}}\psi(t,x). \tag{51}$$

By (4.5), (50), (51) and (43), we get

$$\lim_{n \to +\infty} \int_{Q_T} f_{1,n}(t,x)h_\gamma\left(u_{1,n} + \frac{1}{n}, u_{2,n}\right)\psi(t,x)$$

$$= \lim_{n \to +\infty} \int_{Q_T} f_{1,n}(t,x)h_\gamma\left(u_{1,n} + \frac{1}{n}, u_{2,n}\right)\psi(t,x)\chi_{\{u_{1,n} \geq \delta\}} = \int_{Q_T} f_1(t,x)h_\gamma(\delta, u_2)\psi(t,x).$$

Similarly, for any $\delta > 0$,

$$\begin{aligned}\int_{Q_T} f_{2,n}(t,x)h_\gamma(u_{1,n}+\frac{1}{n},u_{2,n})\psi(t,x) &= \int_{Q_T\cap\{0\le u_{1,n}<\delta\}} f_{2,n}(t,x)h_\gamma(u_{1,n}+\frac{1}{n},u_{2,n})\psi(t,x)\\ &+\int_{Q_T\cap\{u_{1,n}\ge\delta\}} f_{2,n}(t,x)h_\gamma(u_{1,n}+\frac{1}{n},u_{2,n})\psi(t,x)\\ &= I_1+I_2.\end{aligned} \tag{52}$$

Concerning the term I_1, we have

$$\big|I_1\big| \le ||\psi||_\infty \int_{Q_T\cap\{0\le u_{1,n}<\delta\}} f_{2,n}(t,x)h_\gamma(u_{1,n}+\frac{1}{n},u_{2,n})\varphi^2(x).$$

By (27), we get to

$$|I_1| \le ||\psi||_\infty \begin{cases} C\delta^{1-\gamma} & \text{if } 0<\gamma<1,\\ C\sqrt{\delta} & \text{if } \gamma=1.\end{cases} \tag{53}$$

Hence, $\lim\limits_{\delta\to 0}\big|I_1\big|=0$. Concerning I_2, we see that

$$\begin{aligned} I_2 = \lim_{n\to+\infty}\int_{Q_T} f_{2,n}(t,x)h_\gamma\left(u_{1,n}+\frac{1}{n},u_{2,n}\right)\chi_{\{u_{1,n}\ge\delta\}}\chi_{\{u_1\ne\delta\}}\psi(t,x)\\ +\int_{Q_T} f_{2,n}(t,x)h_\gamma\left(u_{1,n}+\frac{1}{n},u_{2,n}\right)\chi_{\{u_{1,n}\ge\delta\}}\chi_{\{u_1=\delta\}}\psi(t,x)=J_1+J_2.\end{aligned}$$

We note that there exists, at most, a countable set Θ_2 such that meas$\{(t,x): u_1(t,x)=\delta\}>0$. Choose $\delta\notin\Theta_2$, so $J_1=0$, since (31) holds, for B_1, we have

$$\chi_{\{u_{1,n}\ge\delta\}}\chi_{\{u_1\ne\delta\}} \to \chi_{\{u_1>\delta\}} \quad \text{a.e. in } \; Q_T$$

$$\begin{aligned}&\big|f_{2,n}(t,x)\big|h_\gamma\left(u_{1,n}+\frac{1}{n},u_{1,n}\right)\chi_{\{u_{1,n}\ge\delta\}}\chi_{\{u_1\ne\delta\}}\psi(t,x)\\ &\le \big|f_2(t,x)\big|h_\gamma(\delta,u_{2,n})\psi(t,x)\in L^1(Q_T).\end{aligned}$$

The Lebesgue theorem combined with (31) leads to

$$\lim_{n\to\infty} I_2 = \int_{Q_T} f_2(t,x)h_\gamma(u_1,u_2)\chi_{\{u_1>\delta\}}\psi(t,x). \tag{54}$$

By (52), (53), (54) and (47), we can get

$$\begin{aligned}&\lim_{n\to+\infty}\int_{Q_T} f_{2,n}(t,x)h_\gamma\left(u_{1,n}+\frac{1}{n},u_{2,n}\right)\psi(t,x)\\ &=\lim_{n\to+\infty}\int_{Q_T} f_{2,n}(t,x)h_\gamma\left(u_{1,n}+\frac{1}{n},u_{2,n}\right)\chi_{\{u_{1,n}\ge\delta\}}\psi(t,x)\\ &=\int_{Q_T} f_2(t,x)h_\gamma(u_1,u_2)\psi(t,x).\end{aligned}$$

Applying a similar argument to $u_{1,n}$ to address the case of $u_{1,n}$, but this time utilizing (24) and (26), we conclude the proof of our main result, Theorem 2.2.

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