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Existence and Uniqueness of Solution for Stochastic Nonlocal Random Functional Integral Equation

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Abstract: In this paper, we use Banach's fixed point theorem to establish sufficient conditions which guarantee the existence and uniqueness of the solution for a stochastic nonlocal random functional integral equation. As applications, an example is presented to illustrate our obtained results.

Keywords: *stochastic differential equation; existence; uniqueness; stochastic integral equations; fixed point theorem*

Mathematics Subject Classification (2020): 70K55, 93-10.

1 Introduction

Stochastic differential equations (SDEs) play an important role in characterizing many social, physical, biological, and engineering problems. The theory of SDEs has developed quickly, the investigation of SDEs has attracted considerable attention of researchers, and many qualitative theories of SDEs have been obtained (see [2, 4, 8]).

In the last two decades, the existence and uniqueness of solution for SDEs have been considered in many publications such as [1, 5, 7, 9, 10].

When random fluctuations have great effects on the parameters and evolution in the mathematical model which describes a certain phenomenon, a stochastic differential equation should be the starting point for deriving the suitable model. Recently, nonlocal stochastic models were introduced by many authors to describe the evolution of the

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studied phenomena. Motivated by the papers [3,6], we study the existence and uniqueness solution of the following nonlocal functional stochastic differential equation:

$$\begin{aligned} dx(t) =& f(t, x(t), A(t)x(t))dt + g(t, x(t), C(t)x(t))dB(t), \ t \in [0, T], \\ x(0) +& \sum_{k=1}^{p} c_k x(t_k) = x_0, \end{aligned} \tag{1}$$

which is equivalent to the following stochastic functional integral equation:

$$\begin{aligned} x(t) = x(0) + & \int_0^t f(s, x(s), A(s)x(s))ds \\ + & \int_0^t g(s, x(s), C(s)x(s))dB(s), \end{aligned} \tag{2}$$

where the first integral is a mean square Riemann integral and the second is an Ito integral.

This paper consists of four sections. In Section 2, we review some concepts, introduce some notation and state our main result, which shows the existence and uniqueness of solution for the stochastic nonlocal random functional integral equation (2). In Section 3, we give the proof of the main result. In Section 4, one example is given to illustrate the theoretical result.

2 Preliminaries and Main Result

Throughout this paper, we assume that $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$ satisfying the usual conditions (i.e., it is right continuous and $\mathcal{F}_0$ contains all P-null sets). Let $\mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P})$ stand for the space of all $\mathbb{R}$-valued random variables $\{X(t), \ t \in [0, T]\}$ such that

$$\mathbf{E}|X|^2 = \int_\Omega |X|^2 \, d\mathbb{P} < \infty.$$

For $X \in \mathcal{L}_2(\Omega, \mathcal{F}, \mathbb{P})$, we let

$$\|X\|_2 := \Big(\int_\Omega |X|^2 \, d\mathbb{P} \Big)^{1/2}.$$

Then $\mathcal{L}_2(\Omega, \mathcal{F}, \mathbb{P})$ is a Hilbert space equipped with the norm $\| \cdot \|_2$. We consider the following nonlocal functional stochastic differential equation:

$$\begin{aligned} dx(t) =& f(t, x(t), A(t)x(t))dt + g(t, x(t), C(t)x(t))dB(t), \ t \in [0, T], \\ x(0) +& \sum_{k=1}^{p} c_k x(t_k) = x_0, \end{aligned}$$

where $0 = t_0 < t_1 < ... < t_p \leq T$, c_k are constants $(k = 1, ..., p)$, $p \in \mathbb{N}$, $\{B_t\}$ is a standard Brownian motion defined on the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ adapted to the filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$. $x(0)$ is an $\mathcal{F}_0$-mesurable random variable independent of B with finite second moment.

$A(t), t \in [0,T]$, and $C(t), t \in [0,T]$, are the families of linear bounded operators defined on $Y := \mathcal{C}([0,T], \mathcal{L}_2(\Omega, \mathcal{F}, \mathbb{P}))$, the space of all continuous stochastic processes defined from $[0,T]$ into $\mathcal{L}_2(\Omega, \mathcal{F}, \mathbb{P})$ with values in Y. The measurable real random functions f and g are defined on $[0,T] \times Y \times Y$ with values in the space $\mathcal{L}_2(\Omega, \mathcal{F}, \mathbb{P})$.

$A(t)$ and $C(t)$ are the families of bounded operators. So, there exist a function $\alpha(t) : [0,T] \to \mathbb{R}^+$ and a function $\gamma(t) : [0,T] \to \mathbb{R}^+$ such that $\|A(t)x\|_Y \le \alpha(t)\|x\|_Y$ and $\|C(t)x\|_Y \le \gamma(t)\|x\|_Y$.

For the nonlocal stochastic differential equation (1), we have the following result. It shows that under some sufficient conditions, there exists a unique solution.

Theorem 2.1 *Assume that the following conditions hold:*

(i) For all $x, y \in Y$ and $t \in [0,T]$, there exists a constant $l > 0$ such that

$$|f(t,x,y)| \le l\sqrt{1 + |x|^2 + |y|^2},$$
$$|g(t,x,y)| \le l\sqrt{1 + |x|^2 + |y|^2};$$

(ii) For all $x, y, x', y' \in Y$ and $t \in [0,T]$, there exists a constant $a > 0$ such that

$$|f(t,x,y) - f(t,x',y')| \le a\sqrt{|x - x'|^2 + |y - y'|^2},$$
$$|g(t,x,y) - g(t,x',y')| \le a\sqrt{|x - x'|^2 + |y - y'|^2};$$

(iii) There exists a real continuous monotone nondecreasing mapping F defined on $[0,T]$ such that $s < t$ implies

$$\mathbf{E}[|B(t) - B(s)|^2] = \mathbf{E}[\|B(t) - B(s)\|^2 \backslash \mathcal{F}_s] = F(t) - F(s);$$

(iv) $\displaystyle\sum_{k=1}^{p} c_k \neq -1$*;*

(v) a satisfies $4a^2\left(1 + \dfrac{\sum_{k=1}^{p} c_k^2}{\left(1 + \sum_{k=1}^{p} c_k\right)^2}\right)[(1+\theta^2)(T^2 + F(T) - F(0)] < 1.$

Then nonlocal functional stochastic integral equation (2) has a unique continuous solution $X_t(\omega)$ belonging to $\mathcal{L}_2(\Omega, \mathcal{F}, \mathbb{P})$, and $X_t(\omega)$ is adapted to the filtration $\{\mathcal{F}_t\}_{t \ge 0}$.

3 Proof of the Theorem

Assume that $\displaystyle\sum_{k=1}^{p} c_k \neq -1$. Integrating the equation (1) yields

$$x(t) = x(0) + \int_0^t f(s, x(s), A(s)x(s))ds + \int_0^t g(s, x(s), C(s)x(s))dB(s).$$

So we have

$$\begin{aligned} x(t_k) = x(0) + \int_0^{t_k} f(s, x(s), A(s)x(s))ds \\ + \int_0^{t_k} g(s, x(s), C(s)x(s))dB(s) \quad (k = 1, ..., p). \end{aligned} \tag{3}$$

By (1) and (3),

$$\begin{aligned} x(0) + \sum_{k=1}^{p} c_k \Big[x(0) + \int_0^{t_k} f(s, x(s), A(s)x(s))ds \\ + \int_0^{t_k} g(s, x(s), C(s)x(s))dB(s) \Big] = x_0. \end{aligned} \tag{4}$$

Since $\sum_{k=1}^{p} c_k \neq -1$, then (4) implies

$$\begin{aligned} x(0) = \Big(x_0 - \sum_{k=1}^{p} c_k \Big[\int_0^{t_k} f(s, x(s), A(s)x(s))ds \\ + \int_0^{t_k} g(s, x(s), C(s)x(s))dB(s) \Big] \Big) \Big/ \Big(1 + \sum_{k=1}^{p} c_k \Big). \end{aligned}$$

Then

$$\begin{aligned} x(t) = \frac{x_0 - \sum_{k=1}^{p} c_k \Big[\int_0^{t_k} f(s, x(s), A(s)x(s))ds + \int_0^{t_k} g(s, x(s), C(s)x(s))dB(s) \Big]}{1 + \sum_{k=1}^{p} c_k} \\ + \int_0^{t} f(s, x(s), A(s)x(s))ds + \int_0^{t} g(s, x(s), C(s)x(s))dB(s) \end{aligned}$$

Let us define the integral operator G by

$$\begin{aligned} Gx(t) = \frac{x_0 - \sum_{k=1}^{p} c_k \Big[\int_0^{t_k} f(s, x(s), A(s)x(s))ds + \int_0^{t_k} g(s, x(s), C(s)x(s))dB(s) \Big]}{1 + \sum_{k=1}^{p} c_k} \\ + \int_0^{t} f(s, x(s), A(s)x(s))ds + \int_0^{t} g(s, x(s), C(s)x(s))dB(s). \end{aligned}$$

Lemma 3.1 *The operator G sends the space $\mathcal{C}([0,T], \mathcal{L}_2(\Omega, \mathcal{F}, \mathbb{P}))$ into itself.*

Proof. Let $0 \leq t_1 \leq t_2 \leq T$ and $x \in \mathcal{C}([0,T], \mathcal{L}_2(\Omega, \mathcal{F}, \mathbb{P}))$. By applying the Cauchy-Schwarz inequality and the condition (iii), we obtain

$$\begin{aligned} \mathbf{E}|Gx(t_2) - Gx(t_1)|^2 \leq 2(t_2 - t_1) \int_{t_1}^{t_2} \mathbf{E}|f(s, x(s), A(s)x(s))|^2 ds \\ + 2 \int_{t_1}^{t_2} \mathbf{E}|g(s, x(s), C(s)x(s))|^2 dF(s). \end{aligned}$$

Let $\theta = \max\{\max_{t\in[0,T]} \alpha(t), \max_{t\in[0,T]} \gamma(t)\}$. Applying the growth condition yields

$$\mathbf{E}|Gx(t_2) - Gx(t_1)|^2 \le 2l^2[(t_2 - t_1)^2 + F(t_2) - F(t_1)][1 + (1 + \theta^2)\|x\|_Y^2].$$

$F(t)$ is continuous, then

$$\lim_{t_1 \mapsto t_2} \|F(t_2) - F(t_1)\| = 0.$$

Therefore,

$$\lim_{t_1 \mapsto t_2} \mathbf{E}|Gx(t_2) - Gx(t_1)|^2 = 0.$$

Consequently, Gx is continuous in mean square on $[0, T]$.

But the function Gx is square integrable with respect to measure probability, has a finite second moment, and is adapted to the filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$. This implies that G maps Y into itself.

Now, we will show that G is a contraction on Y.

Let x and y in Y. Applying the Cauchy-Schwarz inequality and the above conditions, we have

$$\begin{aligned}
\mathbf{E}|Gx(t) - Gy(t)|^2 &\le 4\int_0^t \mathbf{E}|f(s, x(s), A(s)x(s)) - f(s, y(s), A(s)y(s))|^2 ds \\
&+ 4\int_0^t \mathbf{E}|g(s, x(s), C(s)x(s)) - g(s, y(s), C(s)y(s))|^2 dF(s) \\
&+ 4\mathbf{E}\left|\frac{\sum_{k=1}^{p} c_k\left[\int_0^{t_k} f(s, x(s), A(s)x(s)) - f(s, y(s), A(s)y(s))ds\right]}{1 + \sum_{k=1}^{p} c_k}\right|^2 \\
&+ 4\mathbf{E}\left|\frac{\sum_{k=1}^{p} c_k\left[\int_0^{t_k} g(s, x(s), C(s)x(s)) - g(s, y(s), C(s)y(s))dB(s)\right]}{1 + \sum_{k=1}^{p} c_k}\right|^2 \\
&\le 4a^2\left(1 + \frac{\sum_{k=1}^{p} c_k^2}{\left(1 + \sum_{k=1}^{p} c_k\right)^2}\right)\left[(1 + \theta^2)(T^2 + F(T) - F(0)\right]\mathbf{E}|x - y|^2.
\end{aligned}$$

4 Illustrative Example

For simplicity, let $X(0) = x_0$. The process $X_t = e^{0.36845t + 0.87B_t}$ starting from $X(0) = 1$ solves the stochastic differential equation (SDE)

$$\begin{aligned}
dX_t &= 1.01X_t dt + 0.87X_t dB_t, \\
x_0 &= 1.
\end{aligned}$$

5 Conclusion

The main goal of this paper is to discuss the existence and uniqueness of solutions for a kind of stochastic integral equations with nonlocal conditions. We obtained a sufficient condition of the existence and uniqueness of solution for stochastic integral equation with nonlocal condition.

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