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Study of the Existence of Extremal Solutions for Differential Inclusions with Nonlocal Integral Boundary Conditions

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Abstract: In the present paper, we have studied some existence results for extremal solutions for differential inclusions with nonlocal integral boundary conditions, under certain monotonicity assumption. The existence of solutions is obtained via some well known fixed point theorems. An illustrative example is given at the end of this paper.

Keywords: *differential inclusions; fixed point theorems; minimal and maximal solutions; nonlocal integral conditions.*

Mathematics Subject Classification (2020): 34A60, 34K09, 70K75, 70K99.

1 Introduction

There are many problems in applied mathematics, control theory, nonlinear dynamics, as well as economical systems, Hamiltonian systems and mechanical problems, in which one needs to study the differential inclusions

$$x'(t) \in F(t, x(t)),$$

where $F(.,.)$ is a multivalued function, see for instance [6, 20].

In recent years, many authors have investigated the existence of absolutely continuous solutions for the initial value problems of multivalued differential equations, under

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lower or upper semicontinuous assumptions, for more detail, see for instance [10], and the references therein. Also, the existence of continuous, and absolutely continuous solutions for differential inclusions with nonlocal conditions, under the upper semicontinuous assumption, has been extensively studied by several authors, see for instance [11, 17] and the references therein.

The study of integral boundary conditions has a great importance due to their various applications in many scientific fields such as population dynamics [9] and cellular systems [1]. Moreover, the existence of continuous, absolutely continuous, and bounded variation solutions of boundary value problems with integral boundary conditions has been studied by some authors such as Arara and Benchohra [4], Benchohra et al. [11], A. Boucherif [2], Infante [18], see also the references therein.

The existence of minimal and maximal solutions for a single-valued case, i.e., differential equations, is discused by many authors, see for instance [7, 22].

In [3], the authors have proved the existence of the minimal and maximal solutions for the following initial value problem:

$$\begin{cases} x^n(t) \in F(t, x(t)), \quad a.e.\ t \in I = [0, a], \\ x^i(0) = x_i \in \Re, \quad i = 1, 2, ..., n-1. \end{cases} \tag{1}$$

And in [14], the author discuses the existence of the minimal and maximal solutions of the following periodic boundary value problem:

$$\begin{cases} x'(t) \in F(t, x(t)), \quad a.e.\ t \in I = [0, T], \\ x(0) = x(T). \end{cases} \tag{2}$$

Also, the author in [13], has proved the existence of the minimal and maximal solutions of the following integral inclusions:

$$x(t) - q(t) \in \int_0^{\sigma(t)} k(t, s) F(s, x(\eta(s))) ds \tag{3}$$

for $t \in [0, 1]$, where $\sigma, \eta : [0, 1] \to [0, 1], q : [0, 1] \to [0, 1]$,
and $k : [0, 1] \times [0, 1] \to \Re$.
For more detail about the extremal solutions of differential inclusions, see [21].
The main purpose of this paper is to demonstrate the existence of extremal absolutely continuous solutions, by using a lattice fixed point theorem [15], without the convexity condition of the following problem:

$$x'(t) \in F(t, x(t)), \quad a.e.\ t \in I = [0, T], \tag{4}$$

with the integral condition

$$x(0) = \int_0^T g(s, x(s)) ds, \tag{5}$$

where $F : I \times \Re \to P(\Re)$ is a multivalued function, when F is isotone increasing in the second variable, $P(\Re)$ is the family of all nonempty subsets of $\Re$ and $g : I \times \Re \to \Re$ is a given function.

The problem (4)-(7) has been studied recently by Boucherif [3], for the existence of a bounded variation solution, under the bounded variation condition.

This work is organized as follows: in addition to the Introduction, we have two main sections. In Section 2, we give some preliminary tools that we will use later. In Section 3, we present our main results obtained. Finally, as an application, we provide an example to demonstrate the validity of our results.

2 Preliminaries

In this section, we present the necessary notations, definitions, and some basic facts that are used in this paper, for more detail, we refer the reader to [3, 5, 8, 12, 14–16, 19].
Let $(X, \|.\|_X)$ be a normed space.
$P(X)$ is the set of all nonempty subsets of X.
$P_{cl}(X) = \{Y \in P(X) :\ Y\ is\ closed\}$.
$P_{cv}(X) = \{Y \in P(X) :\ Y\ is\ convex\}$.
$P_{bd}(X) = \{Y \in P(X) :\ Y\ is\ bounded\}$.
$P_{cl,cv}(X) = \{Y \in P(X) :\ Y\ is\ closed\ and\ convex\}$.
$P_{cp,cv}(X) = \{Y \in P(X) :\ Y\ is\ compact\ and\ convex\}$.

Definition 2.1 A set-valued function $F : X \to P(X)$ is called convex (closed) valued if $F(x)$ is convex (closed) for all $x \in X$.

Definition 2.2 A set-valued function $F : X \to P(X)$ is called bounded valued on the bounded sets B if $F(B) = \bigcup_{x\in B} F(x)$ is bounded in X for all $B \in P_{bd}(X)$ or, equivalently, $\sup_{x\in B}\{sup\{|u| :\ u \in F(x)\}\} < \infty$.

Definition 2.3 A set-valued function $F : X \to P(X)$ is called upper semicontinuous (u.s.c) on X if for each $x_0 \in X$, the set $F(x_0)$ is a nonempty closed subset of X, and if for each open set N of X containing $F(x_0)$, there exists an open neighborhood N_0 of x_0 such that $F(N_0) \subseteq N$. In other words, F is u.s.c if the set $F^{-1}(A) = \{x \in X :\ F(x) \subset A\}$ is open in X for every open set A in X. Or for every closed subset A of X, the set $F^{+}(A) = \{x \in X :\ A \bigcap F(x) \neq \emptyset\}$ is closed in X.

Definition 2.4 A multivalued map $N : I \to P_{cl}(\Re)$ is said to be measurable if for every $y \in \Re$, the function

$$t \mapsto d(y, N(t)) = inf\{|y - z|; \quad z \in N(t)\}$$

is measurable.

Definition 2.5 A multivalued map $F : I \times \Re \to P(\Re)$ is said to be Carathéodory if

(i) $t \mapsto F(t, x)$ is measurable for each $x \in \Re$,

(ii) $u \mapsto F(t, u)$ is upper semicontinuous for almost all $t \in I$.

For each $x \in C(I, \Re)$, we denote by $S^1_{F,x}$ the set of Lebesgue integrable selections of F, i.e.,

$$S^1_{F,x} = \{v \in L^1(I, \Re); \ \ v(t) \in F(t, x(t)) \ \ a.e. \ \ t \in I\},$$

and $AC(I, \Re)$ is the space of all absolutely continuous real-valued functions on I, and it is a Banach space with respect to the norm

$$\|x\| = sup\{|x(t)|, t \in I\},$$

where $|.|$ is the norm on $\Re$.

Theorem 2.1 *Assume that the multivalued function*

$$F : I \times \Re \to P(\Re)$$

satisfies the following assumptions:

(1) F is a Carathéodory multivalued function,

(2) for each $r > 0, \exists h_r \in L^1(I, \Re_+)$ such that

$$|F(t,x)| = \sup\{|v|, v \in F(t,x)\} \leq h_r(t), \forall |x| \leq r, \ and \ for \ a.e. \ \ t \in I.$$

Then the set $S^1_{F,x}$ is nonempty.

Definition 2.6 A partially ordered set $(N, \leq)$ is called a lattice if for any $x, y \in N, \ \ x \wedge y = inf\{x,y\}$ and $x \vee y = sup\{x,y\}$ exist.

Let A be any subset of N, by $\wedge A$ we mean an element $a_* \in N$ such that $x \wedge a_* = a_*$ for all $x \in A$. Similarly, by $\vee A$ we mean an element a^* such that $x \vee a^* = a^*$ for all $x \in A$. The elements a_* and a^* are respectively called the infimum and supremum of A.

Definition 2.7 A lattice $(N, \leq)$ is called a complete lattice if every subset of N has the infimum and supremum in N.

Definition 2.8 A function $f : N \to N$ is called isotone increasing if for any $x, y \in N, \ \ x \leq y$, we have $f(x) \leq f(y)$.

Definition 2.9 *A multivalued function $T : N \to P(N)$ is called isotone increasing if for any $x, y \in N, \ \ x \leq y$ implies $T(x) \leq T(y)$.*

Definition 2.10 A function $\alpha \in AC(I, \Re)$ is called the lower solution of the problem (4)-(7) if for any $v \in S^1_{F,\alpha}$, we have $\alpha^{'}(t) \leq v(t), \ a.e \ t \in I$ and $\alpha(0) \leq \int_0^T g(s, \alpha(s))ds$.

Definition 2.11 A function $\beta \in AC(I, \Re)$ is called the upper solution of the problem (4)-(7) if for any $v \in S^1_{F,\beta}$, we have $\beta^{'}(t) \geq v(t), \ a.e \ t \in I$ and $\beta(0) \geq \int_0^T g(s, \beta(s))ds$.

Definition 2.12 A function $u \in AC(I, \Re)$ is said to be a maximal solution of (4)-(7) if it satisfies (4)-(7) on I, and for any other solution $x \in AC(I, \Re)$ of (4)-(7) on I, we have $x(t) \leq u(t)$ for $t \in I$. Similarly, we can define a minimal solution $y \in AC(I, \Re)$ of (4)-(7).

Theorem 2.2 *Let $F : [0,T] \times \Re \to P(\Re)$ be a multivalued function, assume that the multivalued function F satisfies the following assumptions:*

(1) $F(t,x)$ is nonempty closed for all $(t,x) \in I \times \Re$,

(2) $F(t,.)$ is lower semicontinuous from $\Re$ into $P(\Re)$,

(3) $F(.,.)$ is measurable.

Then there exists a measurable selection of F.

Theorem 2.3 *Let $(N, \leq)$ be a complete lattice, and $T : N \to P(N)$ be a multivalued map. Suppose the following conditions hold:*

(i) T is isotone increasing on N,

(ii) $infTx \in Tx, \ supTx \in Tx$ for each $x \in N$.

Then the set $P = \{u \in N : \ u \in Tu\}$ *is nonempty, and there exists* $u_1, u_2 \in N$ *with* $u_1 \in Tu_1, \ u_2 \in Tu_2$ *with*

$$u_1 \leq u \leq u_2$$

for all $u \in N$ *with* $u \in Tu$.

Theorem 2.4 *Let* X *be a Banach space, and let* $(X, \leq)$ *be a complete lattice. Suppose that* $T : X \to P(X)$ *is a multivalued function such that*

(i) T *is isotone increasing, and*

(ii) $T(x)$ *is closed for each* $x \in X$.

Then the set of fixed points of T *is non-empty, and has the minimal and maximal elements.*

3 Existence Results

In this section, we state and prove our main results.

Theorem 3.1 *Suppose that the following conditions hold:*

(H1) $F(t, x)$ *is closed for each* $(t, x) \in I \times \Re$,

(H2) F *is isotone increasing in* x *almost everywhere for* $t \in I$,

(H3) F *is Caratheodory,*

(H4) $\exists h_F \in L^1(I, \Re_+)$ *such that*

$$|F(t, x)| \leq h_F(t), \quad a.e. \ t \in I,$$

for all $x \in \Re$,

(H5) $t \mapsto g(t, x)$ *is measurable for each* $x \in \Re$ *and* $x \mapsto g(t, x)$ *is continuous for a.e.* $t \in I$,

(H6) $\exists h_g \in L^1(I, \Re_+)$ *such that*

$$|g(t, x)| \leq h_g(t), \quad a.e. \ t \in I,$$

for all $x \in \Re$,

(H7) g *is isotone increasing in* x *almost everywhere for* $t \in I$.

Then the problem (4)-(7) has the minimal and maximal solutions on I.

Proof. We have from the assumption (H3), there exists a measurable selection v of F (i.e., $v(t) \in F(t, x)$), and from the assumption (H4), this selection is integrable, i.e., $v(.) \in L^1(I, \Re)$ (by Theorem 2.1). Now we define the multivalued function T by

$$\begin{aligned} Tx(t) = \{h \in C(I, \Re) \ s.t; h(t) = &\int_0^T g(s, x(s))ds \\ &+ \int_0^t v(s)ds, v(t) \in F(t, x(t)), a.e. \ \ t \in I\}, \end{aligned} \tag{6}$$

let

$$N = \{x \in AC(I, \Re); \|x\| \le M\},$$

where $M = \|h_g\|_1 + \|h_F\|_1$, we have N is a closed and bounded subset of the complete lattice $(AC(I, \Re), \le)$ and so $(N, \le)$ is complete.
Now, we show that T maps N into $P(N)$, to see this, let $x \in N$, and for each $u \in Tx$, $\exists v \in L^1$ such that $v(t) \in F(t, x(t))$ *a.e.* $t \in I$ with

$$u(t) = \int_0^T g(s, x(s))ds + \int_0^t v(s)ds,$$

and therefore

$$|u(t)| \le \int_0^T |g(s, x(s))|ds + \int_0^t |v(s)|ds \le \|h_g\|_1 + \|h_F\|_1,$$

and hence $T : N \to P(N)$.

Next, we show that Tx is a closed subset of N for each $x \in N$, to see this, it is enough to show that the values of the operator Q defined by $Qx = \{v \in L^1(I, \Re); v \in S^1_{F,x}\}$ are closed, let $(v)_n$ be a sequence in $L^1(I, \Re)$ such that $v_n \to v$, then $v_n \to v$ in measure so there exists a subsequence $(v)_{kn}$ (we take $(v)_n$) such that $v_n \to v$ a.e. and from the assumption (H1), we have that the values of Q are closed in $L_1(I, \Re)$, therefore for each $x \in N$, Tx is a closed and bounded subset of N.

Now, we show that T is isotone increasing on N, to see this, let $x, y \in N$ such that $x \le y$, let $u_1 \in Tx$, then $\exists v_1 \in S^1_{F,x}$ such that $u_1(t) = \int_0^T g(s, x(s))ds + \int_0^t v_1(s)ds$, and since $F(t, .)$ and $g(t, .)$ are isotone increasing, we have $\exists v_2 \in S^1_{F,y}$ such that $v_1(t) \le v_2(t), \forall t \in I$, and hence

$$u_1(t) = \int_0^T g(s, x(s))ds + \int_0^t v_1(s)ds \le \int_0^T g(s, y(s))ds + \int_0^t v_2(s)ds = u_2(t)$$

for all $t \in I, u_2 \in Ty$. Similarly, let $u_2 \in Ty$, then $\exists v_2 \in S^1_{F,y}$ such that

$$u_2(t) = \int_0^T g(s, y(s))ds + \int_0^t v_2(s)ds$$

by the fact that F and g are isotone increasing, $\exists v_1 \in S^1_{F,x}$ such that $v_1(t) \le v_2(t)$ for $t \in I$, hence we have

$$u_2(t) = \int_0^T g(s, y(s))ds + \int_0^t v_2(s)ds \ge \int_0^T g(s, x(s))ds + \int_0^t v_1(s)ds = u_1(t)$$

for all $t \in I$, therefore $u_1 \in Tx$, hence

$$Tx \le Ty,$$

i.e., T is isotone increasing on N. Therefore, all conditions of Theorem 2.4 are satisfied, and hence T has a fixed point, and the set of fixed points has the minimal and maximal elements, and therefore the problem (4)-(7) has the minimal and maximal solutions on I.

Now, we give another existence theorem for the minimal and maximal solutions of the problem (4)-(7).

Theorem 3.2 *Assume that the following assumptions hold:*

(A1) $F(t,x)$ *is nonempty and closed for all* $(t,x) \in I \times \Re$,

(A2) $F(t,x)$ *is isotone increasing in* x *for a.e.* $t \in I$,

(A3) g *is measurable in the first variable and continuous in the second variable,*

(A4) $\exists h_g \in L^1(I, \Re_+)$ *such that* $|g(t,x)| \le h_g(t), a.e.\ \ t \in I$,

(A5) $F(t,.)$ *is lower semicontinuous from* $\Re$ *into* $\Re$,

(A6) $F(.,.)$ *is measurable,*

(A7) $\exists q \in L^1(I, \Re_+)$ *and there exists a continuous nondecreasing* $\psi : [0,+\infty) \to [0,+\infty)$ *such that*

$$|F(t,x)| \le q(t)\psi(|x|)\ \ for\ a.e.\ \ t \in I, \forall x \in \Re,$$

and

$$\int_0^a q(t)dt \le \int_{\|h\|_1}^{+\infty} \frac{du}{\psi(u)},$$

(A8) g *is isotone increasing in* x *for a.e.* $t \in I$.

Then the problem (4)-(7) has the minimal and maximal solutions in I.

Proof. We have from the assumptions (A1),(A5), and (A6) that there exists a measurable selection v of F (i.e., $v(t) \in F(t,x)$), and from the assumption (A7), this selection is integrable, i.e., $v(.) \in L^1(I, \Re)$ (by Theorem 2.2). Now we define the multivalued function T by

$$Tx(t) = \{h \in C(I,\Re) s.t; h(t) = \int_0^T g(s,x(s))ds + \int_0^t v(s)ds, v(t) \in F(t,x(t)), a.e\ t \in I.\}$$

Let

$$N = \{x \in AC(I,\Re); \alpha(t) \le x(t) \le \beta(t),\ \ for\ t \in I\},$$

where

$$\alpha(t) = -J^{-1}(\int_0^t q(s)ds)$$

and

$$\beta(t) = J^{-1}(\int_0^t q(s)ds)$$

with

$$J(z) = \int_{h_g}^z \frac{du}{\psi(u)}.$$

Firstly, we show that α is a lower solution of (4)-(7), and β is an upper solution of (4)-(7), we have for each $t \in I$, $\beta'(t) = \frac{1}{J'(\beta(t))} q(t) = \psi(\beta(t)) \times q(t)$ and $-\alpha(t) = q(t)\psi(-\alpha(t))$. Also, we have $\beta(0) = \|h_g\|_1 \ge \int_0^T g(s,\beta(s))ds$ and $\alpha(0) = -\|h_g\|_1 \le \int_0^T g(s,\alpha(s))ds$.

Now, let $v \in S^1_{F,\beta}$, then by (A7), we have $|v(t)| \le q(t)\psi(\beta(t)) = \beta'(t)\ a.e\ t \in I$, as a result we have

$$v(t) \le |v(t)| \le \beta'(t)\ \ a.e\ t \in I,$$

thus β is an upper solution of the problem (4)-(7).

Now, let $v \in S^1_{F,\alpha}$, and from (A7), we have

$$-v(t) \leq |v(t)| \leq q(t)\psi(|\alpha(t)|) = q(t)\psi(-\alpha(t)) = -\alpha^{'}(t) \ a.e \ t \in I,$$

therefore

$$\alpha^{'}(t) \leq v(t) \ a.e \ t \in I.$$

Thus α is a lower solution of the problem (4)-(7).
Now, we prove that T is isotone increasing, to see this, let $x, y \in N$ such that $x \leq y$, let $u_1 \in Tx$, then $\exists v_1 \in S^1_{F,x}$ such that $u_1(t) = \int_0^T g(s, x(s))ds + \int_0^t v_1(s)ds$ and since $F(t,.)$ and $g(t,.)$ are isotone increasing, we have $\exists v_2 \in S^1_{F,y}$ such that $v_1(t) \leq v_2(t), \forall t \in I$, and hence

$$u_1(t) = \int_0^T g(s, x(s))ds + \int_0^t v_1(s)ds \leq \int_0^T g(s, y(s))ds + \int_0^t v_2(s)ds = u_2(t)$$

for all $t \in I, u_2 \in Ty$. Similarly, let $u_2 \in Ty$, then $\exists v_2 \in S^1_{F,y}$ such that

$$u_2(t) = \int_0^T g(s, y(s))ds + \int_0^t v_2(s)ds$$

by the fact that F and g are isotone increasing, $\exists v_1 \in S^1_{F,x}$ such that $v_1(t) \leq v_2(t)$ for $t \in I$, hence we have

$$u_2(t) = \int_0^T g(s, y(s))ds + \int_0^t v_2(s)ds \geq \int_0^T g(s, x(s))ds + \int_0^t v_1(s)ds = u_1(t)$$

for all $t \in I$. Therefore $u_1 \in Tx$ and hence

$$Tx \leq Ty,$$

i.e., T is isotone increasing on N.
Next, we show that $T : N \to P(N)$, to see this let $x \in N$ ($\alpha(t) \leq x(t) \leq \beta(t)$), so for each $u \in T\beta, \exists v \in S^1_{F,\beta}$($i.e.,\ \ v(t) \in F(t, \beta(t))\ t \in I$) with

$$u(t) = \int_0^T g(s, \beta(s))ds + \int^t v(s)ds,$$

and since β is an upper solution of the problem (4)-(7), we have $v(s) \leq \beta^{'}(s)\ a.e\ s \in I$.

As a result, for each $t \in I$, we have

$$\begin{aligned} u(t) &= \int_0^T g(s, \beta(s))ds + \int_0^t v(s)ds \\ &\leq \int_0^T g(s, \beta(s))ds + \int_0^t \beta^{'}(s)ds \\ &= \int_0^T g(s, \beta(s))ds + \beta(t) - \beta(0) \leq \beta(t) \quad (\beta(0) \geq \int_0^T g(s, \beta(s))ds). \end{aligned}$$

Consequently, $T\beta \leq \beta$. A similar argument guarantees that $\alpha \leq T\alpha$, and since T is isotone increasing on N and $\alpha \leq x \leq \beta$, we have $\alpha \leq T\alpha \leq Tx \leq T\beta \leq \beta$, so $T : N \to P(N)$.

As in the proof of Theorem 3.1, we deduce that Tx is closed for each $x \in N$, thus for each $x \in N$, we have Tx is a nonempty closed and bounded subset of N, so as a result we have that $supTx \in Tx$ (also $infTx \in Tx$), and by Theorem 2.3, we have that T has the minimal and maximal elements, therefore the problem (4)-(7) has the minimal and maximal solutions in N.

4 Example

As an example of our result, we consider the following boundary value problem with integral condition:

$$\begin{cases} x'(t) \in \frac{1}{3e^{t+2}}[1, 1 + |x(t)|], \quad t \in I = [0, T], \\ x(0) = \int_0^T \frac{1}{2e^{s+4}}(1 + x(s))ds. \end{cases} \tag{7}$$

Set

$$F(t, x) = \frac{1}{3e^{t+2}}[1, 1 + |x|], \quad t \in I.$$

For each $x \in \Re$, and $t \in I$, we have

$$\|F(t, x)\| = \sup\{|v| : \ v \in F(t, x)\} \leq \frac{1}{3e^{t+2}}(1 + |x|).$$

Hence for each $r > 0$, $\exists h_{F,r} \in L^1(I, \Re_+)$ such that

$$h_{F,r}(t) = \frac{1 + r}{3e^{t+2}} \quad |x| \leq r$$

and

$$g(t, x(t)) = \frac{1}{2e^{t+4}}(1 + x(t)),$$

we have also

$$|g(t, x)| \leq \frac{1 + r}{2e^{t+4}} = h_g(t)$$

for each $|x| \leq r$.

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