Nonlinear Dynamics and Systems Theory, **25** (1) (2025) 78–90

General Stability Result for a Nonlinear Viscoelastic Wave Equation With Boundary Dissipation

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Received: April 13, 2024; Revised: January 17, 2025

Abstract: In this paper, we consider a model of a dynamic viscoelastic wave equation with a nonlinear source and boundary dissipation. Our fundamental goal is to establish the general decay rates of the energy solutions under a class of generality of the relaxation function $g : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying the inequality $g'(t) \leq -H(g(t))$ for all $t \geq 0$, where H is a function satisfying some specific properties. This work extends the previous works with a viscoelastic wave equation and improves earlier results in the literature.

Keywords: *viscoelastic wave equation; relaxation function; general decay; convexity.*

Mathematics Subject Classification (2020): 35B37, 93D15, 93D20, 74D05, 35L55.

1 Introduction

In this paper, we are concerned with the following nonlinear viscoelastic wave equations:

$$\begin{cases} u_{tt} - k_0 \Delta u + \int_0^t g(t-s)\, div(a(x)\nabla u(s))ds + b(x)\, u_t = |u|^{p-2} u & \text{in } \Omega \times \mathbb{R}^+, \\ k_0 \frac{\partial u}{\partial \nu} - \int_0^t g(t-s)\,(a(x)\nabla u(s))\,\nu ds + h(u_t) = 0 & \text{on } \Gamma_1 \times \mathbb{R}^+, \\ u(x,0) = u_0(x),\, u_t(x,0) = u_1(x), & x \in \Omega, \\ u = 0 & \text{on } \Gamma_0 \times \mathbb{R}^+, \end{cases} \quad (1)$$

where $k_0 > 0$, and Ω is a bounded domain in $\mathbb{R}^n$ $(n \geq 1)$ with a smooth boundary $\Gamma = \Gamma_0 \cup \Gamma_1$. Hence Γ_0 and Γ_1 are closed and disjoint with $mes(\Gamma_0) > 0$ and ν is the unit outward normal to Γ. $b : \Omega \to \mathbb{R}^+$ is a function, and

$$2 < p \leq \frac{2n}{(n-2)}, \quad n \geq 3, \quad (2)$$

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$$p > 2, \quad n = 1, 2.$$

We consider the following hypotheses.
$(G1)$ $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a C^1 function satisfying

$$g(0) > 0, \quad k_0 - \int_0^\infty g(s)\, ds = l > 0. \tag{3}$$

$(G2)$ $H : \mathbb{R}^+ \to \mathbb{R}^+$ is a $C^1(\mathbb{R}^+)$ function with $H(0) = 0$, and H is a linear or strictly increasing and strictly convex C^2 function on $(0, r]$, $r < 1$, such that

$$g'(t) \leq -H(g(t)), \quad \forall t \geq 0. \tag{4}$$

$(G3)$ $h : \mathbb{R} \to \mathbb{R}$ is a nondecreasing function with

$$h(s)\, s \geq \alpha |s|^2, \quad \forall s \in \mathbb{R}, \tag{5}$$

$$|h(s)| \leq \gamma |s|, \quad \forall s \in \mathbb{R}, \tag{6}$$

where α, γ are positive constants.
$(G4)$ $a : \Omega \to \mathbb{R}$ is a non negative function and $a \in C^1\left(\overline{\Omega}\right)$ such that

$$a(x) \geq a_0 > 0, \tag{7}$$

$$|\nabla a(x)|^2 \leq a_1^2 |a(x)|, \quad \forall s \in \mathbb{R},$$

for some positive constant a_1.

This type of problems has been considered by many authors and several results concerning existence, nonexistence, and asymptotic behavior have been established. In this regard, Messaoudi [4, 5] considered

$$u_{tt} - \Delta u + \int_0^t g(t-s)\, \Delta u(s)\, ds = b |u|^{p-2} u \tag{8}$$

for $p \geq 2$ and $b = 0$ or 1, and the relaxation function satisfies a relation of the form

$$g'(t) \leq -\xi(t)\, g(t), \tag{9}$$

where ξ is a differentiable nonincreasing positive function. He established a more general decay result, from which the usual exponential and polynomial decay rates are the only special cases. Also, Messaoudi and Mustafa [1] treated the following system:

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g(t-s)\, \Delta u(s)\, ds = 0 & \text{in } \Omega \times \mathbb{R}^+, \\ \frac{\partial u}{\partial \nu} - \int_0^t g(t-s) \frac{\partial u}{\partial \nu}(s)\, \nu ds + h(u_t) = 0 & \text{on } \Gamma_1 \times \mathbb{R}^+, \\ u = 0 & \text{on } \Gamma_0 \times \mathbb{R}^+, \end{cases} \tag{10}$$

where g satisfies (9) and h satisfies weaker conditions than those in [2], and obtained an explicit and general formula for the decay rate of the energy. In [9], Mustafa considered the nonlinear abstract equation subject to a competing effect of viscoelastic and frictional dampings:

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g(t-s)\, Au(s)\, ds + h(u_t) = j(u), & t > 0 \\ u(0) = u_0, u_t(0) = u_1, \end{cases} \tag{11}$$

and studied the simultaneous effect of viscoelastic and frictional dampings on the energy decay rates, with minimal conditions on both h and g, where g satisfies

$$g'(t) \leq -\xi(t) H(g(t)), \tag{12}$$

and H is an increasing and convex function. In this context, we refer to the work [6] by Alabu-Boussouria and Cannarsa, in which they considered the following viscoelastic problem:

$$\begin{cases} u_{tt} - \triangle u + \int_0^t g(t-s) \triangle u(s)\, ds = 0 & \text{in } \Omega \times \mathbb{R}^+, \\ u = 0 & \text{on } \Gamma \times \mathbb{R}^+, \\ u(x,0) = u_0, u_t(x,0) = u_1, & x \in \Omega, \end{cases} \tag{13}$$

and g is a positive function satisfying

$$g'(t) \leq -\chi(g(t)), \tag{14}$$

where χ is a nonnegative function with $\chi(0) = \chi'(0) = 0$, and χ is strictly increasing and strictly convex on $(0, k_0]$ for some $k_0 > 0$. They also required that

$$\int_0^{k_0} \frac{dx}{\chi(x)} = +\infty, \quad \int_0^{k_0} \frac{x}{\chi(x)} dx < 1, \quad \liminf_{s \to 0^+} \frac{\frac{\chi(s)}{s}}{\chi'(s)} > \frac{1}{2},$$

in this case, an explicit rate of decay is given.

In this work, we present an explicit formula for energy decay, which extends the class of functions g beyond that in [6].

2 Preliminaries

We use the standard Lebesgue and Sobolev spaces with their usual scalar products and norms, we set

$$H^1_{\Gamma_0} = \{u \in H^2(\Omega) : u = 0 \text{ on } \Gamma_0\}.$$

We first have the embedding $H^1_{\Gamma_0} \hookrightarrow L^{2(p+1)}(\Omega)$. Let $B > 0$ be the optimal constant of the Sobolev embedding which satisfies the following inequality:

$$\|u\|_{2(p+1)} \leq B \|\nabla u\|_2, \quad \forall u \in H^1_{\Gamma_0}. \tag{15}$$

Use the trace-Sobolev embedding $H^1_{\Gamma_0} \hookrightarrow L^k(\Gamma_1)$, $1 \leq k \leq \frac{2(n-1)}{n-2}$, in this case, the embedding constant is denoted by B_1, that is,

$$\|u\|_{k,\Gamma_1} \leq B_1 \|\nabla u\|_2. \tag{16}$$

Now, we introduce the following functionals:

$$\begin{aligned} J(t) &= \frac{1}{2}\left(k_0 - a(x)\int_0^t g(s)\, ds\right)\|\nabla u\|_2^2 + \frac{1}{2}(g \circ \nabla u)(t) - \frac{1}{p}\|u\|_p^p, \\ E(t) &= \frac{1}{2}\|u_t\|_2^2 + J(u(t)) \quad \text{for } t \in [0, T), \\ I(t) &= I(u(t)) = \left(k_0 - \int_0^t g(s)\, ds\right)\|\nabla u\|_2^2 + (g \circ \nabla u)(t) - \|u\|_p^p, \end{aligned} \tag{17}$$

where

$$(g \circ v)(t) = \int_0^t g(t-s)\,\|v(t) - v(s)\|_2^2\,ds.$$

Lemma 2.1 *Let u be the solution of* (1)*, then, under assumptions* $(G1)$-$(G3)$*,* $E(t)$ *is a nonincreasing function on* $[0,T)$ *and*

$$E'(t) = -\frac{1}{2}\int_\Omega a(x)\,g(t)\,|\nabla u(t)|^2\,dx + \frac{1}{2}(g' \circ \nabla u)(t) - \int_\Omega b(x)\,|u_t(t)|^2\,dx \le 0. \quad (18)$$

Proof. Multiplying the first equation in (1) by u_t and integrating over Ω and using integration by parts and the boundary condition, and hypotheses $(G1)$, $(G2)$, we obtain (18).

By using the Galerkin method and procedure similar to that from [3, 7], we can have the following local existence result for problem (1).

Theorem 2.1 *Assume that* $u_0 \in H^1_{\Gamma_0} \cap H^2(\Omega)$ *and* $u_1 \in H^1_{\Gamma_0}$*. Then there exists a strong solution* u *of* (1) *satisfying*

$$u \in L^\infty\left([0,T); H^1_{\Gamma_0} \cap H^2(\Omega)\right), u_t \in L^\infty\left([0,T); H^1_{\Gamma_0}\right), u_{tt} \in L^\infty\left([0,T); L^2(\Omega)\right)$$

for some $T > 0$.

Lemma 2.2 *Suppose that* $(G1)$*,* $(G3)$ *and* (2) *hold. Assume further that* $(u_0, u_1) \in H^1_{\Gamma_0} \times L^2(\Omega)$ *such that*

$$\beta = \frac{B^p}{l}\left(\frac{2p}{(p-2)\,l}E(0)\right)^{(p-2)/2} < 1 \quad (19)$$

and $I(u_0) > 0$*, then* $I(u(t)) > 0$*,* $\forall t > 0$*, where* B *is the best Poincaré constant, and* $E(0) = E(u_0, u_1)$.

Proof. Since $I(u_0) > 0$, there exists (by continuity) $T_i < T$ such that

$$I(u(t)) \ge 0, \quad \forall t \in [0, T_i],$$

this gives

$$\begin{aligned} J(t) &= \frac{1}{2}\left(k_0 - \int_0^t g(s)\,ds\right)\|\nabla u\|_2^2 + \frac{1}{2}(g \circ \nabla u)(t) - \frac{1}{p}\|u\|_p^p \\ &= \left(\frac{p-2}{2p}\right)\left(\left(k_0 - \int_0^t g(s)\,ds\right)\|\nabla u(t)\|_2^2 + (g \circ \nabla u)(t)\right) + \frac{1}{p}I(t) \\ &\ge \left(\frac{p-2}{2p}\right)\left(\left(k_0 - \int_0^t g(s)\,ds\right)\|\nabla u(t)\|_2^2 + (g \circ \nabla u)(t)\right). \end{aligned} \quad (20)$$

By using $(G1)$, (17), (18) and (20), we easily have

$$l\,\|\nabla u\|_2^2 \le \left(\frac{2p}{p-2}\right)J(t) \le \left(\frac{2p}{p-2}\right)E(0), \quad \forall t \in [0, T_i]. \quad (21)$$

We then exploit $(G1)$, (15), (19) and (21) to obtain

$$\begin{aligned} \|u\|_p^p &\leq B^p \|\nabla u(t)\|_2^p \leq \frac{B^p}{l} \|\nabla u(t)\|_2^{p-2} l \|\nabla u(t)\|_2^2 \leq \beta l \|\nabla u(t)\|_2^2 \\ &< \left(k_0 - \int_0^t g(s)\, ds\right) \|\nabla u(t)\|_2^2, \quad \forall t \in [0, T_i]. \end{aligned} \tag{22}$$

Therefore

$$I(t) = \left(k_0 - \int_0^t g(s)\, ds\right) \|\nabla u\|_2^2 + (g \circ \nabla u)(t) - \|u\|_p^p > 0$$

for all $t \in [0, T_i]$. By repeating this procedure, and using the fact that

$$\lim_{t \to T_i} \frac{B^p}{l} \left(\frac{2p}{(p-2)\, l} E(0) \right)^{(p-2)/2} \leq \beta < 1,$$

T_i is extended to T.

3 Global Existence

In this section, we give some lemmas and the result on the existence of the global solution.

Lemma 3.1 *For any $u \in C^1\left(0, T; H^1(\Omega)\right)$, we have*

$$\begin{aligned} &\int_\Omega \int_0^t g(t-s) \nabla u(s) \nabla u_t(t)\, ds dx \\ &= -\frac{1}{2} \int_\Omega g(t) |\nabla u(t)|^2 dx + \frac{1}{2} (g' \circ \nabla u)(t) \\ &\quad -\frac{1}{2} \frac{d}{dt} \left[(g \circ \nabla u)(t) - \int_\Omega \int_0^t g(s)\, ds\, |\nabla u(t)|^2 dx \right]. \end{aligned} \tag{23}$$

Proof. See [3].

Lemma 3.2 *There exist positive constants d and t_1 such that*

$$g'(t) \leq -d g(t), \quad \forall t \in [0, t_1]. \tag{24}$$

Proof. By $(G1)$ and $(G2)$, we easily deduce that $\lim_{t \to +\infty} g(t) = 0$. Hence, there is $t_1 \geq 0$ large enough such that

$$g(t_1) = r_1$$

and

$$g(t) \leq r_1, \quad \forall t \geq t_1. \tag{25}$$

As g is non increasing, $g(0) > 0$ and $g(t_1) > 0$, then $g(t) > 0$ for any $t \in [0, t_1]$ and

$$0 < g(t_1) \leq g(t) \leq g(0), \quad \forall t \in [0, t_1].$$

Therefore, since H is a positive continuous function, we get

$$a \leq H(g(t)) \leq b, \quad \forall t \in [0, t_1],$$

for some positive constants a and b.
Consequently, for all $t \in [0, t_1]$,

$$g'(t) \leq -H(g(t)) \leq -a = -\frac{a}{g(0)} g(0) \leq -\frac{a}{g(0)} g(t),$$

which gives

$$g'(t) \leq -dg(t), \quad \forall t \in [0, t_1].$$

Remark 3.1 By $(G1)$ and $(G2)$, we easily deduce that $lim_{t\to+\infty} g(t) = 0$ and

$$max\{g(t), -g'(t)\} < min\{r, H(r), H_0(r)\}, \forall t \geq t_1. \tag{26}$$

Theorem 3.1 *Suppose that* $(G1)$, $(G2)$ *and* (2) *hold. If* $(u_0, u_1) \in H^1_{\Gamma_0} \times L^2(\Omega)$ *and satisfies* (19)*, then the solution is global and bounded.*

4 Decay of Solution

In this section, we state and prove the main result of our work. First, we define some functionals. Let

$$\mathcal{L}(t) = E(t) + \varepsilon_1 \Phi(t) + \varepsilon_2 \Psi(t), \tag{27}$$

where

$$\Phi(t) = \int_\Omega u.u_t \, dx, \tag{28}$$

$$\Psi(t) = \int_\Omega a(x) u_t \int_0^t g(t-s)(u(s) - u(t)) \, ds \, dx, \tag{29}$$

and ε_1, ε_2 are some positive constants to be specified later.

Lemma 4.1 *There exist two positive constants* β_1 *and* β_2 *such that the relation*

$$\beta_1 E(t) \leq \mathcal{L}(t) \leq \beta_2 E(t) \tag{30}$$

holds for ε_1, $\varepsilon_2 > 0$ *small enough.*

Lemma 4.2 *Assume that* $(G1)$-$(G4)$ *hold, then the functional*

$$\Phi(t) = \int_\Omega u u_t \, dx$$

satisfies, along the solution of (1),

$$\begin{aligned} \Phi'(t) \leq & -\frac{1}{4}\int_\Omega |\nabla u|^2 \, dx + \int_\Omega |u_t|^2 \, dx + \frac{(k_0 - l)}{2l}(g \circ \nabla u)(t) \\ & + \frac{2\beta^2 B_*^2}{l}\int_{\Gamma_1} u_t^2 d\Gamma + \frac{2B^2 \|b\|_\infty}{l}\int_\Omega b(x) u_t^2 dx + \|u\|_p^p. \end{aligned} \tag{31}$$

Proof. We estimate the derivative of $\Phi(t)$. From (28) and using (1), we have

$$\begin{aligned}\Phi'(t) &= \int_\Omega u_t^2 dx - k_0 \int_\Omega |\nabla u|^2 dx + \int_\Omega \nabla u(t) a(x) \int_0^t g(t-s) \nabla u(s) ds\, dx \\ &\quad - \int_{\Gamma_1} h(u_t) u\, d\Gamma - \int_\Omega b(x) u u_t dx + \int_\Omega |u|^p dx. \qquad (32)\end{aligned}$$

The third, and the fourth, and the fifth terms on the right-hand side of (32) can be estimated as follows. From Hölder's inequality, Young's inequality and (23), for $\eta > 0$, we have

$$\begin{aligned}&\int_\Omega \nabla u(t) a(x) \int_0^t g(t-s) \nabla u(s) dsdx \\ \le\ & \frac{k_0}{2} \int_\Omega |\nabla u|^2 dx + \frac{1}{2k_0} \int_\Omega a(x) \left(\int_0^t g(t-s)(\nabla u(s) - \nabla u(t) + \nabla u(t)) ds \right)^2 dx \\ \le\ & \left[\frac{k_0}{2} + \frac{1}{2k_0}(1+\eta)(k_0-l)^2 \right] \|\nabla u\|_2^2 + \frac{1}{2k_0}\left(1 + \frac{1}{\eta}\right)(k_0 - l)(g \circ \nabla u)(t). \qquad (33)\end{aligned}$$

Employing Hölder's inequality, Young's inequality, $(G1)$ and (15), for $\delta_1, \delta_2 > 0$, we see that

$$\left| \int_{\Gamma_1} h(u_t) u\, d\Gamma \right| \le \delta_1 B_*^2 \|\nabla u\|_2^2 + \frac{\beta^2}{4\delta_1} \int_{\Gamma_1} u_t^2 d\Gamma, \qquad (34)$$

and

$$\int_\Omega b(x) u u_t dx \le \delta_2 B^2 \|\nabla u\|_2^2 + \frac{1}{4\delta_2} \int_\Omega b(x) u_t^2\ dx. \qquad (35)$$

A substitution of (33) - (34) into (32) yields

$$\begin{aligned}\Phi'(t) \le\ & -\left(\frac{k_0}{2} - \frac{1}{2k_0}(1+\eta)(k_0-l)^2 - \delta_1 B_*^2 - B^2 \|b\|_\infty \delta_2 \right) \int_\Omega |\nabla u|^2 dx \\ & + \frac{1}{2k_0}\left(1+\frac{1}{\eta}\right)(k_0-l)(g \circ \nabla u)(t) + \frac{\beta^2}{4\delta_1} \int_{\Gamma_1} u_t^2 d\Gamma \\ & + \int_\Omega u_t^2 dx + \frac{1}{4\delta_2} \int_\Omega b(x) u_t^2 dx + \int_\Omega |u|^p dx.\end{aligned}$$

Letting $\eta = l/(k_0 - l) > 0$, $\delta_1 = l/8B_*^2$ and $\delta_2 = l/8\beta^2 \|b\|_\infty$ in the above inequality, we obtain

$$\begin{aligned}\Phi'(t) \le\ & -\frac{1}{4} \int_\Omega |\nabla u|^2 dx + \int_\Omega u_t^2 dx + \frac{(k_0 - l)}{2l}(g \circ \nabla u)(t) + \frac{2\beta^2 B_*^2}{l} \int_{\Gamma_1} u_t^2 d\Gamma_1 \\ & + \frac{2B^2 \|b\|_\infty}{4\delta_2} \int_\Omega b(x) u_t^2 dx + \int_\Omega |u|^p dx. \qquad (36)\end{aligned}$$

Then (31) is established.

Lemma 4.3 *Assume that* $(G1)$-$(G4)$ *hold. Then the functional*

$$\Psi(t) = -\int_\Omega a(x) u_t \int_0^t g(t-s)(u(t) - u(s)) ds\ dx$$

satisfies, for some positive constants c_5, c_6,

$$\Psi'(t) \leq -\left(a_0 \int_0^t g(s)\,ds - \delta\right) \|u_t\|_2^2 + \delta c_5 \|\nabla u\|_2^2 + c_6 (g \circ \nabla u)(t) + \delta \|b\|_\infty \int_\Omega b(x) u_t^2\,dx$$
$$- \frac{g(0)\|b\|_\infty^2 B^2}{4a_0\delta} (g' \circ \nabla u)(t) + \beta^2 \delta \int_{\Gamma_1} u_t^2 d\Gamma_1. \quad (37)$$

Proof. The proof is similar to the proof of Lemma 4.2.

Theorem 4.1 *Let* $(u_0, u_1) \in H^1_{\Gamma_0} \times L^2(\Omega)$ *be given, satisfying* (19). *Assume that* $(G1)$ *and* $(G2)$ *hold. Then there exist positive constants* c_1, c_2, c_3 *and* ε_0 *such that the following statements hold:*
(**A**) *In the special case,* $H(t) = ct^p$ *with* $1 \leq p < \frac{3}{2}$, *the solution energy of* (1) *satisfies*

$$E(t) \leq c_1 e^{-c_2 t} \qquad \text{if} \quad p = 1, \quad (38)$$
$$E(t) \leq \frac{c_3}{(c_1 t + c_2)^{\frac{1}{2(p-1)}}} \qquad \text{if} \quad 1 < p < \frac{3}{2}. \quad (39)$$

(**B**) *In the general case, the solution energy of* (1) *satisfies*

$$E(t) \leq c_3 H_1^{-1}(c_1 t + c_2), \quad \forall t \geq 0, \quad (40)$$

where

$$H_1(t) = \int_t^1 \frac{1}{sH_0'(\varepsilon_0 s)} ds \quad \text{and} \quad H_0(t) = H(S(t)),$$

provided that S *is a positive* C^1 *function and that* H_0 *is a strictly increasing and strictly convex* C^2 *function on* $(0, r]$ *with* $S(0) = 0$.

$$\int_0^{+\infty} \frac{g(s)}{H_0^{-1}(-g'(s))} ds < +\infty. \quad (41)$$

Proof. By using (18), (27), (31) and (37), we obtain

$$\begin{aligned}
\mathcal{L}'(t) \leq\ & -(\varepsilon_2(a_0 g_0 - \delta) - \varepsilon_1) \|u_t\|_2^2 - \left(\frac{\varepsilon_1 l}{4} - \varepsilon_2 \delta c_5\right) \|\nabla u\|_2^2 \\
& + \left(\varepsilon_2 c_6 + \frac{(k_0 - l)\varepsilon_1}{2l}\right)(g \circ \nabla u)(t) + \varepsilon_1 \int_\Omega |u|^p\,dx \\
& - \left(1 - \frac{2\varepsilon_1 B^2 \|b\|_\infty}{l} - \varepsilon_2 \delta \|b\|_\infty\right) \int_\Omega b(x) u_t^2 dx \\
& - \left(\frac{1}{2} - \varepsilon_2 \frac{g(0)\|a\|_\infty^2 B^2}{4\delta a_0}\right)(-g \circ \nabla u)(t) \\
& - \left(\alpha - \frac{2B_*^2 \varepsilon_1 \beta^2}{l} - \varepsilon_2 \delta \beta^2\right) \int_{\Gamma_1} |u_t|^2 d\Gamma, \quad \forall t \geq t_1.
\end{aligned}$$

We have used the fact that for any $t_1 > 0$,

$$\int_0^t g(s)\,ds \geq \int_0^{t_1} g(s)\,ds = g_0 \quad \forall t \geq t_1. \quad (42)$$

At this point, we choose δ small enough so that

$$\frac{4\delta c_5}{l} < \frac{a_0 g_0}{2} < a_0 g_0 - \delta, \tag{43}$$

where δ is fixed, the choice of any two positive constants ε_1 and ε_2 satisfying

$$\frac{4\delta c_5 \varepsilon_2}{l} < \varepsilon_1 < \frac{a_0 g_0}{2}\varepsilon_2 \tag{44}$$

will make

$$k_1 = \frac{\varepsilon_1 l}{4} - \varepsilon_2 \delta c_5 > 0 \tag{45}$$

and

$$k_2 = \varepsilon_2 \left(a_0 g_0 - \delta\right) - \varepsilon_1 > 0. \tag{46}$$

Then we choose δ, ε_1 and ε_2 small so that (30) and (43) remain valid, further

$$k_3 = 1 - \frac{2\varepsilon_1 B^2 \left\|b\right\|_\infty}{l} - \varepsilon_2 \delta \left\|b\right\|_\infty > 0 \tag{47}$$

$$k_4 = \alpha - \frac{2B_*^2 \varepsilon_1 \beta^2}{l} - \varepsilon_2 \delta \beta^2 > 0 \tag{48}$$

$$k_5 = \frac{1}{2} - \varepsilon_2 \frac{g\left(0\right) \left\|a\right\|_\infty^2 B^2}{4\delta a_0} > 0. \tag{49}$$

Hence, for all $t_1 > 0$, we arrive at

$$\begin{aligned} \mathcal{L}'\left(t\right) \le\ & -k_1 \left\|\nabla u\right\|_2^2 - k_2 \left\|u_t\right\|_2^2 + c_7 \left(g \circ \nabla u\right)\left(t\right) + c_8 \left(g' \circ \nabla u\right)\left(t\right) \\ & -k_3 \int_\Omega b(x) u_t^2 dx - k_4 \int_{\Gamma_1} \left|u_t\right|^2 d\Gamma + \varepsilon_1 \left\|u\right\|_p^p, \end{aligned} \tag{50}$$

which yields that if needed, one can choose ε_1 sufficiently small

$$\mathcal{L}'\left(t\right) \le -mE(t) + C\left(g \circ \nabla u\right)\left(t\right), \tag{51}$$

where $c_i = 7, 8$ m, C are some positive constants.

Now, we use (18) and (24) to conclude that, for any $t \ge t_1$,

$$\begin{aligned} \int_0^{t_1} g\left(s\right) \int_\Omega \left|\nabla u\left(t\right) - \nabla u\left(t-s\right)\right|^2 dx ds \ & \le -\frac{1}{d} \int_0^{t_1} g'\left(s\right) \int_\Omega \left|\nabla u\left(t\right) - \nabla u\left(t-s\right)\right|^2 dx ds \\ & \le -cE'\left(t\right). \end{aligned} \tag{52}$$

Next, we take $\mathcal{F}\left(t\right) = \mathcal{L}\left(t\right) + cE\left(t\right)$, which is clearly equivalent to $E\left(t\right)$. From (51) and (52), we get, for all $t \ge t_1$,

$$\mathcal{F}'\left(t\right) \le -mE\left(t\right) + c\int_0^{t_1} g\left(s\right) \int_\Omega \left|\nabla u\left(t\right) - \nabla u\left(t-s\right)\right|^2 dx ds. \tag{53}$$

(I) $H\left(t\right) = ct^p$ and $1 \le p < \frac{3}{2}$.
Case 1 $p = 1$: Estimate (53) yields

$$\mathcal{F}'\left(t\right) \le -mE\left(t\right) + c\left(g' \circ \nabla u\right)\left(t\right) \le -mE\left(t\right) - cE'(t), \quad \forall t \ge t_1,$$

which gives

$$(\mathcal{F}+cE)'(t) \le -mE(t), \quad \forall t \ge t_1.$$

Hence, using the fact that $\mathcal{F}+cE \sim E$, we easily obtain (38).

Case 2 $1 < p < \frac{3}{2}$: One can easily show that $\int_0^{+\infty} g^{1-\delta_0}(s)\,ds < +\infty$ for any $\delta_0 < 2-p$ (see [7]). Using this fact and (18), and choosing t_1 even larger if needed, we deduce that, for all $t \ge t_1$,

$$\eta(t) := \int_{t_1}^{t} g^{1-\delta_0}(s) \int_\Omega |\nabla u(t) - \nabla u(t-s)|^2 dxds$$

$$\le 2\int_{t_1}^{t} g^{1-\delta_0}(s) \int_0^1 \left(|\nabla u(t)|^2 + |\nabla u(t-s)|^2\right) dxds \le cE(0) \int_{t_1}^{t} g^{1-\delta_0}(s) < 1. \quad (54)$$

Then, Jensen's inequality, (18), hypothesis $(G1)$ and (54) lead to

$$\int_{t_1}^{t} g(s) \int_\Omega |\nabla u(t) - \nabla u(t-s)|^2 dxds = \int_{t_1}^{t} g^{\delta_0}(s)\, g^{1-\delta_0}(s) \int_\Omega |\nabla u(t) - \nabla u(t-s)|^2 dxds$$

$$= \int_{t_1}^{t} g^{(p-1+\delta_0)\left(\frac{\delta_0}{p-1+\delta_0}\right)}(s)\, g^{1-\delta_0}(s) \int_\Omega |\nabla u(t) - \nabla u(t-s)|^2 dxds$$

$$\le c\left[\int_{t_1}^{t} -g'(s) \int_\Omega |\nabla u(t) - \nabla u(t-s)|^2 dxds\right]^{\frac{\delta_0}{p-1+\delta_0}} \le c\left[-E'(t)\right]^{\frac{\delta_0}{p-1+\delta_0}}.$$

Then, particularly for $\delta_0 = \frac{1}{2}$, we find that (53) becomes

$$\mathcal{F}'(t) \le -mE(t) + c\left[-E'(t)\right]^{\frac{1}{2p-1}}.$$

Now, we multiply by $E^\alpha(t)$, with $\alpha = 2p-2$, to get, using (18),

$$(\mathcal{F}E^\alpha)'(t) \le \mathcal{F}'(t)\, E^\alpha(t) \le -mE^{1+\alpha}(t) + cE^\alpha(t)\left[-E'(t)\right]^{\frac{1}{1+\alpha}}.$$

Then Young's inequality, with $q = 1+\alpha$ and $q' = \frac{1+\alpha}{\alpha}$, gives

$$(\mathcal{F}E^\alpha)'(t) \le -mE^{1+\alpha}(t) + \epsilon E^{1+\alpha}(t) + C_\varepsilon\left(-E'(t)\right).$$

Consequently, picking $\varepsilon < m$, we obtain

$$F_0'(t) \le -m'E^{1+\alpha}(t),$$

where $F_0 = \mathcal{F}E^\alpha + C_\varepsilon E \sim E$. Hence we have, for some $a_0 > 0$,

$$F_0'(t) \le -a_0 F_0^{1+\alpha}(t),$$

from which we easily deduce that

$$E(t) \le \frac{c_3}{(c_1 t + c_2)^{\frac{1}{2(p-1)}}}. \quad (55)$$

(II) The general case: We define $\mathcal{I}(t)$ by

$$\mathcal{I}(t) := \int_{t_1}^{t} \frac{g(s)}{H_0^{-1}(-g'(s))} \int_\Omega |\nabla u(t) - \nabla u(t-s)|^2 dxds,$$

where H_0 is such that (41) is satisfied. As in (54), we find that $\mathcal{I}(t)$ satisfies, for all $t \geq t_1$,

$$0 < \mathcal{I}(t) < 1. \tag{56}$$

We also assume, without loss of generality, that $\mathcal{I}(t) \geq \beta > 0$ for all $t \geq t_1$; otherwise (53) yields an exponential decay. In addition, we define $\lambda(t)$ by

$$\lambda(t) := -\int_{t_1}^{t} g'(s) \frac{g(s)}{H_0^{-1}(-g'(s))} \int_{\Omega} |\nabla u(t) - \nabla u(t-s)|^2 dxds$$

and infer from $(G1)$ and the properties of H_0 and S that

$$\frac{g(s)}{H_0^{-1}(-g'(s))} \leq \frac{g(s)}{H_0^{-1}(H(g(s)))} = \frac{g(s)}{S^{-1}(g(s))} \leq \sigma_0$$

for some positive constant σ_0. Then, using (18) and choosing t_1 even larger (if needed), one can easily see that $\lambda(t)$ satisfies, for all $t \geq t_1$,

$$\begin{aligned} \lambda(t) &\leq -k_0 \int_{t_1}^{t} g'(s) \int_{\Omega} |\nabla u(t) - \nabla u(t-s)|^2 dxds \\ &\leq -cE(0) \int_{t_1}^{t} g'(s)\, ds \leq c g(t_1) E(0) < \min\{r, H(r), H_0(r)\}. \end{aligned}$$

Since H_0 is strictly convex on $(0, r]$ and $H_0(0) = 0$, one has $H_0(\theta x) \leq \theta H_0(x)$, provided $0 \leq \theta < 1$ and $x \in (0, r]$. The use of hypothesis $(G1)$, (26), (56), (57) and Jensen's inequality leads to

$$\begin{aligned} \lambda(t) &= \frac{1}{\mathcal{I}(t)} \int_{t_1}^{t} \mathcal{I}(t) H_0[H_0^{-1}(-g'(s))] \frac{g(s)}{H_0^{-1}(-g'(s))} \int_{\Omega} |\nabla u(t) - \nabla u(t-s)|^2 dxds \\ &\geq \frac{1}{\mathcal{I}(t)} \int_{t_1}^{t} H_0[\mathcal{I}(t) H_0^{-1}(-g'(s))] \frac{g(s)}{H_0^{-1}(-g'(s))} \int_{\Omega} |\nabla u(t) - \nabla u(t-s)|^2 dxds \\ &= H_0\left(\int_{t_1}^{t} g(s) \int_{\Omega} |\nabla u(t) - \nabla u(t-s)|^2 dxds \right). \end{aligned}$$

This implies that

$$\int_{t_1}^{t} g(s) \int_{\Omega} |\nabla u(t) - \nabla u(t-s)|^2 dxds \leq H_0^{-1}(\lambda(t)),$$

then (53) becomes

$$\mathcal{F}'(t) \leq -mE(t) + cH_0^{-1}(\lambda(t)), \forall t \geq t_1. \tag{57}$$

Now, for $\varepsilon_0 < r$ and $c_0 > 0$, using (57) and the fact that $E' \leq 0$, $H_0' > 0$, $H_0'' > 0$ on $(0, r]$, we find that the functional F_1 defined by

$$F_1(t) := H_0'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) \mathcal{F}(t) + c_0 E(t)$$

satisfies, for some $\alpha_1, \alpha_2 > 0$,

$$\alpha_1 F_1(t) \leq E(t) \leq \alpha_2 F_1(t) \tag{58}$$

and

$$F_1'(t) = \varepsilon_0 \frac{E'(t)}{E(0)} H_0'' \left(\varepsilon_0 \frac{E(t)}{E(0)} \right) \mathcal{F}(t) + H_0' \left(\varepsilon_0 \frac{E(t)}{E(0)} \right) \mathcal{F}'(t) + c_0 E'(t)$$

$$\leq -mE(t) H_0' \left(\varepsilon_0 \frac{E(t)}{E(0)} \right) + cH_0' \left(\varepsilon_0 \frac{E(t)}{E(0)} \right) H_0^{-1}(\lambda(t)) + c_0 E'(t). \tag{59}$$

Let H_0^* be the convex conjugate of H_0 in the sense of Young (see [8], p.61-64), then

$$H_0^*(s) = s (H_0')^{-1}(s) - H_0 \left[(H_0')^{-1}(s) \right] \quad if\ s \in (0, H_0'(r)] \tag{60}$$

and H_0^* satisfies the following Young's inequality:

$$AB \leq H_0^*(A) + H_0(B) \quad if\ A \in (0, H_0'(r)], B \in (0, r]. \tag{61}$$

With $A = H_0' \left(\varepsilon_0 \frac{E(t)}{E(0)} \right)$ and $B = H_0^{-1}(\lambda(t))$, using (18), (53) and (59)-(61), we arrive at

$$F_1'(t) \leq -mE(t) H_0' \left(\varepsilon_0 \frac{E(t)}{E(0)} \right) + cH_0^* \left(H_0' \left(\varepsilon_0 \frac{E(t)}{E(0)} \right) \right) + c\lambda(t) + c_0 E'(t)$$

$$\leq -mE(t) H_0' \left(\varepsilon_0 \frac{E(t)}{E(0)} \right) + c\varepsilon_0 \left(\frac{E(t)}{E(0)} \right) H_0' \left(\varepsilon_0 \frac{E(t)}{E(0)} \right) - cE'(t) + c_0 E'(t) \tag{62}$$

Consequently, with a suitable choice of ε_0 and c_0, we obtain, for all $t \geq t_1$,

$$F_1'(t) \leq -k \left(\frac{E(t)}{E(0)} \right) H_0' \left(\varepsilon_0 \frac{E(t)}{E(0)} \right) = -kH_2 \left(\varepsilon_0 \frac{E(t)}{E(0)} \right), \tag{63}$$

where $H_2(t) = tH_0'(\varepsilon_0 t)$.

Since $H_2'(t) = H_0'(\varepsilon_0 t) + \varepsilon_0 t H_0''(\varepsilon t)$, using the strict convexity of H_0 on $(0, 1]$, we find that $H_2'(t)$, $H_2(t) > 0$ on $(0, r]$. Thus, with

$$R_0(t) = \alpha_1 \frac{\varepsilon F_1(t)}{E(0)}, \quad 0 < \varepsilon < 1,$$

taking in account (58) and (63), we have

$$R_0(t) \sim E(t) \tag{64}$$

and, for some $k_0' > 0$,

$$R_0'(t) \leq -\varepsilon k_0' H_2(R_0(t)), \quad \forall t \geq t_1.$$

Then a simple integration and a suitable choice of ε yield, for some k_1', $k_2' > 0$,

$$R_0(t) \leq H_1^{-1}(k_1' t + k_2'), \quad \forall t \geq t_1, \tag{65}$$

where $H_1(t) = \int_t^1 \frac{1}{H_2(s)} ds$.

By a combination of (64) and (65), estimate (40) is established.

5 Conclusion

In this paper, we studied the asymptotic behavior of the dynamic viscoelastic wave equation with boundary dissipation and a nonlinear source term. The existence of dissipation through boundary conditions ensures the decay of energy. By using the convexity of the relaxation function g and without imposing any restrictive growth assumption on the damping term, we establish a general decay rate. These results have potential for application in the fields of physics and nonlinear dynamics. A similar study for the models of dynamic viscoelastic wave equations with a logarithmic nonlinear source term and thermal dissipation will be the purpose for future research.

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