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Domination of Hyperbolic Systems with Respect to the Gradient Observation

H. Aichaoui * and S. Benhadid

Department of Mathematics, Faculty of Exact Sciences, University of Constantine 1 Mentouri Brothers, Constantine, Algeria

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Abstract: In this paper, we introduce the notions of domination for a class of controlled and observed hyperbolic systems. We study, with respect to the gradient observation, the possibility to make a comparison of input operators of a controlled system. We give various characterizations and main properties in the general case and then by means of the choice of actuators and sensors. As an application, we examine the case of a one dimension wave equation.

Keywords: *hyperbolic systems; domination; gradient; control; actuators; sensors.*

Mathematics Subject Classification (2020): 35L20, 93B05, 93B07, 93C20.

1 Introduction

Modeling a system consists in representing its dynamic behavior by a mathematical model. The mathematical model obtained is generally in the form of linear or nonlinear differential equations. The methods used in the analysis of linear systems are very powerful because of the existence of available tools. However, these linear analysis methods have several limitations because most systems are not linear, so linear methods are only applicable in a limited domain. These limitations explain the complexity and diversity of nonlinear systems and the analysis methods that apply to them. Therefore, there are no general theories for nonlinear systems, but there are several methods adapted to certain classes of nonlinear systems to overcome these difficulties, a linearization of the system and the output which consists in transforming the dynamics of the nonlinear system into a completely (or partially) linear system, so that the techniques of linear systems can be applied. Therefore, we can extend the concepts presented for linear systems to nonlinear

* Corresponding author: mailto:houda.aichaoui@umc.edu.dz

systems, and among these concepts, there is the concept of domination, which has been discussed in this paper.

This work is an extension of the previous works which concern the analysis of a class of linear systems within certain concepts. These concepts consist of a set of notions such as controllability [13], detectability, observability [10], remediability [3] and domination [5,6] which enable a better knowledge and understanding of the system to be obtained. For some related studies in nonlinear cases, see [8, 12].

The extensions of these concepts that are very important in practical applications are those of gradient controllability [9, 11], gradient detectability [7], gradient observability [15], gradient remediability [14].

This work concerns the notion of domination for a general class of controlled and observed hyperbolic systems used to study the possibility of comparing the input operators with respect to the gradient observation. It is an extension of the previous works on parabolic distributed systems [1, 5]. A more general approach is given in [2–4] for controlled and observed systems in the global, regional and asymptotic cases.

This paper is organized as follows. In Section 2, we present the systems. We define and characterize the concepts of exact and weak domination for controlled hyperbolic systems with respect to the gradient observation in Section 3. In Section 4, we give the main properties and characterization results and the case of sensors and actuators is also examined. Finally, we examine the case of a one dimension wave equation.

2 Considered System

Let Ω be an open and bounded subset of $\mathbb{R}^n$ with a sufficiently regular boundary and $]0,T[$ be the finite time interval. We consider the following system:

$$\begin{cases} \dfrac{\partial^2 y}{\partial t^2}(x,t) = Ay(x,t) + B_1u_1(t) + B_2u_2(t), & \Omega\times]0,T[, \\ y(x,0) = y^0(x),\, \frac{\partial y_1}{\partial t}(x,0) = y^1(x), & \Omega, \\ y(\xi,t) = 0, & \partial\Omega\times]0,T[, \end{cases} \tag{1}$$

where A is a second order elliptic linear operator given by

$$A = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial}{\partial x_j}\right),$$

with the domain $D(A) = H_0^1(\Omega) \cap H^2(\Omega)$ and verified elliptic conditions

$$\begin{cases} a_{ij} = a_{ji} \in L^\infty(\Omega),\ 1 \le i,j \le n, \\ \\ \exists\alpha > 0, \forall\xi = (\xi_1,\xi_2,\ldots,\xi_n) \in \mathbb{R}^n,\ \sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \ge \alpha\sum_{i=1}^n |\xi_i|^2, \quad \text{pp. in } \Omega, \end{cases}$$

$B_1 \in \mathcal{L}(U_1,X)$, $B_2 \in \mathcal{L}(U_2,X)$, $u_1 \in L^2(0,T;U_1)$ and $u_2 \in L^2(0,T;U_2)$, where U_1 and U_2 are two Hilbert spaces (control spaces) and $X = H_0^1(\Omega)$ is the state space.

For $\left(y^0,y^1\right)$ in $H_0^1(\Omega) \times L^2(\Omega)$, the system (1) admits a unique solution y in $C\left(0,T;H_0^1(\Omega)\right) \cap C^1\left(0,T;L^2(\Omega)\right)$. The system (1) is augmented with the output equation

$$z(t) = C\nabla y(t), \tag{2}$$

where $C \in \mathcal{L}\left(\left(L^2(\Omega)\right)^n, \mathcal{O}\right), \mathcal{O}$ is the observation space (Hilbert space). In the case of an observation with q sensors, we take generally $\mathcal{O} = \mathbb{R}^q$.

The gradient operator ∇ is given by the formula

$$\begin{aligned} \nabla : H_0^1(\Omega) &\to \left(L^2(\Omega)\right)^n, \\ y \mapsto \nabla y &= \left(\frac{\partial y}{\partial x_1}, \frac{\partial y}{\partial x_2}, \cdots, \frac{\partial y}{\partial x_n}\right). \end{aligned}$$

We consider the operator $\bar{A}$ defined by

$$\bar{A}\left(y_1, y_2\right) = \left(y_2, A y_1\right), \forall \left(y_1, y_2\right) \in D(\bar{A}),$$

with $D(\bar{A}) = D(A) \times H_0^1(\Omega)$. The operator $\bar{A}$ is linear, closed with a dense domain in the state space $\bar{X} = H_0^1(\Omega) \times H_0^1(\Omega)$, which is a Hilbert space for the inner product

$$\left\langle \left(y_1, y_2\right), \left(z_1, z_2\right)\right\rangle_{H_0^1(\Omega) \times L^2(\Omega)} = \left\langle \sqrt{-A} y_1, \sqrt{-A} z_1 \right\rangle_{L^2(\Omega)} + \left\langle y_2, z_2 \right\rangle_{L^2(\Omega)}.$$

The adjoint operator of $\bar{A}$ is given by $\bar{A}^* = -\bar{A}$.

The operator $\bar{A}$ generates on $\bar{X}$ a strongly continuous semi-group $(S(t))_{t \geq 0}$ defined by

$$S(t)\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} W_1(t)\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \\ W_2(t)\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \end{pmatrix},$$

with

$$W_1(t)\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \sum_{m \geq 1} \sum_{j=1}^{r_m} \left(\left\langle y_1, w_{mj} \right\rangle_{L^2(\Omega)} \cos \sqrt{-\lambda_m} t + \frac{1}{\sqrt{-\lambda_m}} \left\langle y_2, w_{mj} \right\rangle_{L^2(\Omega)} \sin \sqrt{-\lambda_m} t \right) w_{mj}$$

and

$$W_2(t)\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \sum_{m \geq 1} \sum_{j=1}^{r_m} \left(-\sqrt{-\lambda_m} \left\langle y_1, w_{mj} \right\rangle_{L^2(\Omega)} \sin \sqrt{-\lambda_m} t + \left\langle y_2, w_{mj} \right\rangle_{L^2(\Omega)} \cos \sqrt{-\lambda_m} t \right) w_{mj}.$$

Its adjoint is $S^*(t) = S(-t),\ \forall t \geq 0$. On the other hand, we consider the operators

$$\begin{array}{lcl} \bar{B}_1 : U_1 \to \bar{X} & \text{and} & \bar{B}_2 : U_2 \to \bar{X} \\ u_1 \mapsto \bar{B}_1 u_1 = (0, B_1 u_1)^{tr} & & u_2 \mapsto \bar{B}_2 u_2 = (0, B_2 u_2)^{tr}. \end{array}$$

If we put $\bar{y}(t) = \left(y(t), \dfrac{\partial y(t)}{\partial t}\right)^{tr}$, $\bar{y}^0 = \left(y^0, y^1\right)^{tr}$ and $\dfrac{\partial \bar{y}}{\partial t}(t) = \left(\dfrac{\partial y(t)}{\partial t}, \dfrac{\partial^2 y(t)}{\partial t^2}\right)^{tr}$, then the system (1) is equivalent to the following system:

$$\left\{ \begin{array}{l} \dfrac{\partial \bar{y}}{\partial t}(t) = \bar{A}\bar{y}(t) + \bar{B}_1 u_1(t) + \bar{B}_2 u_2(t), \quad 0 < t < T, \\ \bar{y}(0) = \bar{y}^0. \end{array} \right. \tag{3}$$

The unique solution of system (3) is

$$\bar{y}(t) = S(t)\bar{y}^0 + \int_0^t S(t-s)\bar{B}_1 u_1(s) ds + \int_0^t S(t-s)\bar{B}_2 u_2(s) ds.$$

The system (3) is augmented by the output equation

$$\bar{z}(t) = \bar{C}\bar{\nabla}\bar{y}(t), \tag{4}$$

where $\bar{C}$ is defined by

$$\bar{C}\,(y_1, y_2) = Cy_1, \quad \forall (y_1, y_2) \in \left(L^2(\Omega)\right)^n \times \left(L^2(\Omega)\right)^n,$$

and

$$\begin{array}{rl} \bar{\nabla}: & H_0^1(\Omega) \times H_0^1(\Omega) \to \left(L^2(\Omega)\right)^n \times \left(L^2(\Omega)\right)^n \\ & (y_1, y_2) \mapsto \bar{\nabla}\,(y_1, y_2) = (\nabla y_1, \nabla y_2)\,. \end{array}$$

We consider the following operators:

$$\begin{array}{rl} H_1: & L^2(0, T; U_1) \to \bar{X} \\ & u_1 \mapsto H_1 u_1 = \int_0^T S(T-s)\bar{B}_1 u_1(s)ds \end{array}$$

and

$$\begin{array}{rl} H_2: & L^2(0, T; U_2) \to \bar{X} \\ & u_2 \mapsto H_2 u_2 = \int_0^T S(T-s)\bar{B}_2 u_2(s)ds, \end{array}$$

while their adjoints, denoted by H_1^* and H_2^*, are given by $H_1^* = \bar{B_1}^* S(\cdot - T)$ and $H_2^* = \bar{B_2}^* S(\cdot - T)$, respectively. The state of system (3) at time T is given by

$$\bar{y}(T) = S(T)\bar{y}^0 + H_1 u_1 + H_2 u_2.$$

3 Domination with respect to C

Definition 3.1 We say that

1. B_1 dominates B_2 exactly on $[0, T]$ with respect to C if for any $u_2 \in L^2\,(0, T; U_2)$, there exists a control $u_1 \in L^2\,(0, T; U_1)$ such that

$$\bar{C}\bar{\nabla}H_1 u_1 + \bar{C}\bar{\nabla}H_2 u_2 = 0.$$

2. B_1 dominates B_2 weakly on $[0, T]$ with respect to C if for any $\epsilon > 0$ and for any $u_2 \in L^2\,(0, T; U_2)$, there exists a control $u_1 \in L^2\,(0, T; U_1)$ such that

$$\|\bar{C}\bar{\nabla}H_1 u_1 + \bar{C}\bar{\nabla}H_2 u_2\|_{\mathcal{O}} < \epsilon.$$

Lemma 3.1 *Let V, W and Z be reflexive Banach spaces, $P \in \mathcal{L}(V, Z)$ and $Q \in \mathcal{L}(W, Z)$. Then the following properties are equivalent:*

1. $\operatorname{Im} P \subset \operatorname{Im} Q$.

2. $\exists \gamma > 0$ *such that* $\|P^* z^*\|_{V^*} \leq \gamma\, \|Q^* z^*\|_{W^*}\,, \quad \forall z^* \in Z^*$.

Proposition 3.1 *The following properties are equivalent:*

1. B_1 dominates B_2 exactly on $[0, T]$ with respect to C.

2. $\operatorname{Im}\left(\bar{C}\bar{\nabla}H_2\right) \subset \operatorname{Im}\left(\bar{C}\bar{\nabla}H_1\right)$.

3. There exists $\gamma > 0$ *such that for every* $\theta \in \mathcal{O}^*$*, we have*

$$\left\|\bar{B}_2^* S(\cdot - T)\bar{\nabla}^*\bar{C}^*\theta\right\|_{L^2\left(0,T;U_2^*\right)} \leq \gamma \left\|\bar{B}_1^* S(\cdot - T)\bar{\nabla}^*\bar{C}^*\theta\right\|_{L^2\left(0,T;U_1^*\right)}.$$

Proof.
- $1 \Leftrightarrow 2:$ B_1 dominates B_2 exactly on $[0,T]$ with respect to C if and only if

$$\forall u_2 \in L^2(0,T;U_2),\ \exists u_1 \in L^2(0,T;U_1),\ \text{ such that }\ \bar{C}\bar{\nabla}H_1u_1 + \bar{C}\bar{\nabla}H_2u_2 = 0,$$

i.e., if and only if

$$\forall u_2 \in L^2(0,T;U_2),\ \exists u \in L^2(0,T;U_1),\ \text{ such that }\ \bar{C}\bar{\nabla}H_1u_2 = \bar{C}\bar{\nabla}H_2u,$$

where $u = -u_1 \in L^2(0,T;U_1)$, this is equivalent to saying that $\mathrm{Im}\left(\bar{C}\bar{\nabla}H_2\right) \subset \mathrm{Im}\left(\bar{C}\bar{\nabla}H_1\right)$.
- $2 \Leftrightarrow 3:$ In Lemma 3.1, we put

$$P = \bar{C}\bar{\nabla}H_2 \quad \text{and} \quad Q = \bar{C}\bar{\nabla}H_1,$$

where

$$H_1^* = \bar{B_1}^* S(\cdot - T) \quad \text{and} \quad H_2^* = \bar{B_2}^* S(\cdot - T).$$

Hence the result. □

Proposition 3.2 *The following properties are equivalent:*

1. B_1 *dominates* B_2 *weakly on* $[0,T]$ *with respect to* C*.*
2. $\mathrm{Im}\left(\bar{C}\bar{\nabla}H_2\right) \subset \overline{\mathrm{Im}\left(\bar{C}\bar{\nabla}H_1\right)}$.
3. $\ker\left(\bar{B_1}^* S(\cdot - T)\bar{\nabla}^*\bar{C}^*\right) \subset \ker\left(\bar{B_2}^* S(\cdot - T)\bar{\nabla}^*\bar{C}^*\right)$.

Proof.
- $1 \Leftrightarrow 2:$ B_1 dominates B_2 weakly on $[0,T]$ with respect to C if and only if

$$\forall \varepsilon > 0,\ \forall u_2 \in L^2(0,T;U_2),\ \exists u_1 \in L^2(0,T;U_1) \text{ such that } \|\bar{C}\bar{\nabla}H_1u_1 + \bar{C}\bar{\nabla}H_2u_2\|_{\mathcal{O}} < \varepsilon,$$

i.e., if and only if

$$\forall \varepsilon > 0,\ \forall u_2 \in L^2(0,T;U_2),\ \exists u \in L^2(0,T;U_1) \text{ such that } \|\bar{C}\bar{\nabla}H_2u_2 - \bar{C}\bar{\nabla}H_1u\|_{\mathcal{O}} < \varepsilon,$$

where $u = -u_1 \in L^2(0,T;U_1)$, this is equivalent to saying that

$$\mathrm{Im}(\bar{C}\bar{\nabla}H_2u_2) \subset \overline{\mathrm{Im}(\bar{C}\bar{\nabla}H_1)}.$$

- $2 \Rightarrow 3:$ Let $\sigma \in \ker\left(\bar{B_1}^* S(\cdot - T)\bar{\nabla}^*\bar{C}^*\right)$, we have

$$\mathrm{Im}(\bar{C}\bar{\nabla}H_2) \subset \overline{\mathrm{Im}(\bar{C}\bar{\nabla}H_1)},$$

then

$$\mathrm{Im}(\bar{C}\bar{\nabla}H_2) \subset \left[\ker\left(\bar{B_1}^* S(\cdot - T)\bar{\nabla}^*\bar{C}^*\right)\right]^{\perp},$$

hence

$$\langle\bar{C}\bar{\nabla}H_2u_2, \sigma\rangle_{\mathcal{O}\times\mathcal{O}^*} = 0, \quad \forall u_2 \in L^2(0,T;U_2),$$

then $\sigma \in [\mathrm{Im}(\bar{C}\bar{\nabla}H_2)]^{\perp}$, this gives $\sigma \in \ker\left(\bar{B_2}^* S(\cdot - T)\bar{\nabla}^*\bar{C}^*\right)$.
- $3 \Rightarrow 2$: We assume that

$$\ker\left(\bar{B_1}^* S(\cdot - T)\bar{\nabla}^*\bar{C}^*\right) \subset \ker\left(\bar{B_2}^* S(\cdot - T)\bar{\nabla}^*\bar{C}^*\right),$$

let $\sigma \in \ker(\bar{B_1}^* S(\cdot - T)\bar{\nabla}^*\bar{C}^*)$, then $\bar{B_2}^* S(\cdot - T)\bar{\nabla}^*\bar{C}^*\sigma = 0$, and

$$\langle \bar{C}\bar{\nabla}H_2 u_2, \sigma\rangle_{\mathcal{O}\times\mathcal{O}^*} = 0, \ \forall u_2 \in L^2(0,T;U_2),$$

hence

$$\bar{C}\bar{\nabla}H_2 u_2 \in \left[\ker\left(\bar{B_1}^* S(\cdot - T)\bar{\nabla}^*\bar{C}^*\right)\right]^{\perp} = \overline{\mathrm{Im}(\bar{C}\bar{\nabla}H_1)}, \ \forall u_2 \in L^2(0,T;\bar{X}). \quad \square$$

Remark 3.1 Let us give the following properties and remarks:

1. In the case where C is the identity operator, we say that B_1 dominates B_2 exactly on $[0,T]$ (respectively weakly).

2. The exact domination with respect to C implies the weak domination with respect to C but the converse is not true.

3. If the system

$$\begin{cases} \dfrac{\partial^2 y}{\partial t^2}(x,t) = Ay(x,t) + B_1 u_1(t), & \Omega\times]0,T[, \\ y(x,0) = y^0(x), \frac{\partial y_1}{\partial t}(x,0) = y^1(x) & \Omega, \\ y(\xi,t) = 0, & \partial\Omega\times]0,T[, \end{cases}$$

is gradient controllable exactly (respectively weakly), or equivalently to $\mathrm{Im}\left(\bar{\nabla}H_1\right) = \bar{X}$ [9] (respectively $\overline{\mathrm{Im}\left(\bar{C}\bar{\nabla}H_1\right)} = \bar{X}$), then B_1 dominates exactly (respectively weakly) any operator B_2 with respect to any output operator C.

4 Domination with respect to C and Actuators

This section focuses on the notions of actuators and sensors.
In the case where $U_1 = \mathbb{R}^{p_1}$ and $U_2 = \mathbb{R}^{p_2}$, i.e., the system (1) is excited by p_1 zone actuators $(\Omega_i, \mathsf{a}_i)_{1\le i\le p_1}$, where $\mathsf{a}_i \in L^2(\Omega_i), \Omega_i = \mathrm{supp}(\mathsf{a}_i) \subset \Omega$, for $i = 1,2,\dots,p_1$, and by other p_2 zone actuators $\left(\tilde{\Omega}_i, \tilde{\mathsf{a}}_i\right)_{1\le i\le p_2}$, where $\tilde{\mathsf{a}}_i \in L^2\left(\tilde{\Omega}_i\right), \tilde{\Omega}_i = \mathrm{supp}(\tilde{\mathsf{a}}_i) \subset \Omega$, for $i = 1,2,\dots,p_2$, the operators $\bar{B_1}$ and $\bar{B_2}$ are given by

$$\begin{aligned} \bar{B_1} \ : \ & \mathbb{R}^{p_1} \to \bar{X} \\ & u_1(t) = \left(u_1^1(t), u_1^2(t), \dots, u_1^{p_1}(t)\right) \mapsto \bar{B_1}u_1(t) = \begin{pmatrix} 0 & \sum\limits_{i=1}^{p_1} \chi_{\Omega_i}(x)\mathsf{a}_i(x)u_1^i(t)\end{pmatrix}^{tr} \end{aligned}$$

and

$$\begin{aligned} \bar{B_2} \ : \ & \mathbb{R}^{p_2} \to \bar{X} \\ & u_2(t) = \left(u_2^1(t), u_2^2(t), \dots, u_2^{p_2}(t)\right) \mapsto \bar{B_2}u_2(t) = \begin{pmatrix} 0 & \sum\limits_{i=1}^{p_2} \chi_{\tilde{\Omega}_i}(x)\tilde{\mathsf{a}}_i(x)u_2^i(t)\end{pmatrix}^{tr}, \end{aligned}$$

and their adjoints are, respectively,

$$\bar{B_1}^*(y_1, y_2) = \begin{pmatrix} \langle \mathtt{a}_1, y_2\rangle_{\Omega_1} & \langle \mathtt{a}_2, y_2\rangle_{\Omega_2} & \dots & \langle \mathtt{a}_{p_1}, y_2\rangle_{\Omega_{p_1}} \end{pmatrix}^{tr} \in \mathbb{R}^{p_1},$$

$$\bar{B_2}^*(y_1, y_2) = \begin{pmatrix} \langle \tilde{\mathtt{a}}_1, y_2\rangle_{\tilde{\Omega}_1} & \langle \tilde{\mathtt{a}}_2, y_2\rangle_{\tilde{\Omega}_2} & \dots & \langle \tilde{\mathtt{a}}_{p_2}, y_2\rangle_{\tilde{\Omega}_{p_2}} \end{pmatrix}^{tr} \in \mathbb{R}^{p_2}.$$

Corollary 4.1 *$(\Omega_i, \mathtt{a}_i)_{1\le i\le p_1}$ dominates $\left(\tilde{\Omega}_i, \tilde{\mathtt{a}}_i\right)_{1\le i\le p_2}$ exactly on $[0, T]$ with respect to C if and only if there exists $\gamma > 0$ such that for all σ in $\mathcal{O}^*$, we have*

$$\sum_{i=1}^{p_2} \int_0^T \left\langle \tilde{\mathtt{a}}_i, W_2(s-T)\bar{\nabla}^* \bar{C}^* \sigma\right\rangle^2_{\tilde{\Omega}_i} \, \mathrm{d}s \le \gamma \sum_{i=1}^{p_1} \int_0^T \left\langle \mathtt{a}_i, W_2(s-T)\bar{\nabla}^* \bar{C}^* \sigma\right\rangle^2_{\Omega_i} \, \mathrm{d}s.$$

Proof. According to Proposition 3.1, B_1 dominates B_2 exactly on $[0, T]$ if and only if there exists $\gamma > 0$ such that for all σ in $\mathcal{O}^*$, we have

$$\left\|\bar{B_2}^* S(\cdot - T)\bar{\nabla}^* \bar{C}^* \sigma\right\|^2_{L^2(0,T;\mathbb{R}^{p_2})} \le \gamma \left\|\bar{B_1}^* S(\cdot - T)\bar{\nabla}^* \bar{C}^* \sigma\right\|^2_{L^2(0,T;\mathbb{R}^{p_1})}.$$

Firstly, we have

$$S(\cdot - T)\bar{\nabla}^* \bar{C}^* \sigma = \begin{pmatrix} W_1(\cdot - T)\bar{\nabla}^* \bar{C}^* \sigma \\ W_2(\cdot - T)\bar{\nabla}^* \bar{C}^* \sigma \end{pmatrix},$$

and we have then

$$\left\|\bar{B_2}^* S(\cdot - T)\bar{\nabla}^* \bar{C}^* \sigma\right\|^2_{L^2(0,T;\mathbb{R}^{p_2})} = \sum_{i=1}^{p_2} \int_0^T \left\langle \tilde{\mathtt{a}}_i, W_2(s-T)\bar{\nabla}^* \bar{C}^* \sigma\right\rangle^2_{\tilde{\Omega}_i} \, \mathrm{d}s$$

and

$$\left\|\bar{B_1}^* S(\cdot - T)\bar{\nabla}^* \bar{C}^* \sigma\right\|^2_{L^2(0,T;\mathbb{R}^{p_1})} = \sum_{i=1}^{p_1} \int_0^T \left\langle \mathtt{a}_i, W_2(s-T)\bar{\nabla}^* \bar{C}^* \sigma\right\rangle^2_{\Omega_i} \, \mathrm{d}s.$$

Hence the result. $\square$

Corollary 4.2 *$(\Omega_i, \mathtt{a}_i)_{1\le i\le p_1}$ dominates $\left(\tilde{\Omega}_i, \tilde{\mathtt{a}}_i\right)_{1\le i\le p_2}$ exactly on $[0, T]$ with respect to C if and only if there exists $\gamma > 0$ such that for all σ in $\mathcal{O}^*$, we have*

$$\sum_{i=1}^{p_2} \int_0^T \left[\sum_{m\ge 1} \sqrt{-\lambda_m} \sin\left(\sqrt{-\lambda_m}(T-s)\right) \sum_{j=1}^{r_m} \left\langle C^*\sigma, \nabla w_{m_j}\right\rangle_{(L^2(\Omega))^n} \left\langle \tilde{\mathtt{a}}_i, w_{m_j}\right\rangle_{\tilde{\Omega}_i} \right]^2 \mathrm{d}s \le$$
$$\gamma \sum_{i=1}^{p_1} \int_0^T \left[\sum_{m\ge 1} \sqrt{-\lambda_m} \sin\left(\sqrt{-\lambda_m}(T-s)\right) \sum_{j=1}^{r_m} \left\langle C^*\sigma, \nabla w_{m_j}\right\rangle_{(L^2(\Omega))^n} \left\langle \mathtt{a}_i, w_{m_j}\right\rangle_{\Omega_i} \right]^2 \mathrm{d}s.$$

Proof. We have

$$\sum_{i=1}^{p_1}\int_0^T \left\langle \mathsf{a}_i, W_2(s-T)\bar{\nabla}^*\bar{C}^*\sigma\right\rangle_{\Omega_i}^2 \, \mathrm{d}s$$

$$=\sum_{i=1}^{p_1}\int_0^T \left\langle \chi_{\Omega_i}\mathsf{a}_i, \sum_{m\geq 1}\sum_{j=1}^{r_m} -\sqrt{-\lambda_m}\left\langle \nabla^* C^*\sigma, w_{m_j}\right\rangle_\Omega \sin\left(\sqrt{-\lambda_m}(s-T)\right) w_{m_j}\right\rangle_\Omega^2 \mathrm{d}s$$

$$=\sum_{i=1}^{p_1}\int_0^T \left[\sum_{m\geq 1}\sqrt{-\lambda_m}\sin\left(\sqrt{-\lambda_m}(T-s)\right)\sum_{j=1}^{r_m}\left\langle C^*\sigma, \nabla w_{m_j}\right\rangle_{(L^2(\Omega))^n}\left\langle \mathsf{a}_i, w_{m_j}\right\rangle_{\Omega_i}\right]^2 \mathrm{d}s$$

and

$$\sum_{i=1}^{p_2}\int_0^T \left\langle \tilde{\mathsf{a}}_i, W_2(s-T)\bar{\nabla}^*\bar{C}^*\sigma\right\rangle_{\tilde{\Omega}_i}^2 \, \mathrm{d}s$$

$$=\sum_{i=1}^{p_2}\int_0^T \left\langle \chi_{\tilde{\Omega}_i}\tilde{\mathsf{a}}_i, \sum_{m\geq 1}\sum_{j=1}^{r_m} -\sqrt{-\lambda_m}\left\langle \nabla^* C^*\sigma, w_{m_j}\right\rangle_\Omega \sin\left(\sqrt{-\lambda_m}(s-T)\right) w_{m_j}\right\rangle_\Omega^2 \mathrm{d}s$$

$$=\sum_{i=1}^{p_2}\int_0^T \left[\sum_{m\geq 1}\sqrt{-\lambda_m}\sin\left(\sqrt{-\lambda_m}(T-s)\right)\sum_{j=1}^{r_m}\left\langle C^*\sigma, \nabla w_{m_j}\right\rangle_{(L^2(\Omega))^n}\left\langle \tilde{\mathsf{a}}_i, w_{m_j}\right\rangle_{\tilde{\Omega}_i}\right]^2 \mathrm{d}s.$$

Hence, the result follows immediately from Corollary 4.1. □

Corollary 4.3 *$(\Omega_i, \mathsf{a}_i)_{1\leq i\leq p_1}$ dominates $\left(\tilde{\Omega}_i, \tilde{\mathsf{a}}_i\right)_{1\leq i\leq p_2}$ weakly on $[0,T]$ with respect to C if and only if*

$$\sum_{j=1}^{r_m}\left\langle C^*\sigma, \nabla w_{m_j}\right\rangle_{(L^2(\Omega))^n}\left\langle \mathsf{a}_i, w_{m_j}\right\rangle_{\Omega_i} = 0,\ \forall i\in\{1,2,\cdots,p_1\},\ \forall m\geq 1$$
$$\Rightarrow \sum_{j=1}^{r_m}\left\langle C^*\sigma, \nabla w_{m_j}\right\rangle_{(L^2(\Omega))^n}\left\langle \tilde{\mathsf{a}}_i, w_{m_j}\right\rangle_{\tilde{\Omega}_i} = 0,\ \forall i\in\{1,2,\cdots,p_2\},\ \forall m\geq 1.$$

Proof. We assume that $(\Omega_i, \mathsf{a}_i)_{1\leq i\leq p_1}$ dominates $\left(\tilde{\Omega}_i, \tilde{\mathsf{a}}_i\right)_{1\leq i\leq p_2}$ weakly on $[0,T]$ with respect to C and

$$\sum_{j=1}^{r_m}\left\langle C^*\sigma, \nabla w_{m_j}\right\rangle_{(L^2(\Omega))^n}\left\langle \mathsf{a}_i, w_{m_j}\right\rangle_{\Omega_i} = 0,\ \forall i\in\{1,2,\cdots,p_1\},\ \forall m\geq 1.$$

Since $\bar{B_1}^*S(\cdot-T)\bar{\nabla}^*\bar{C}^*\sigma$ is equal to

$$\left(\sum_{m\geq 1}\sqrt{-\lambda_m}\sin\left(\sqrt{-\lambda_m}(T-\cdot)\right)\sum_{j=1}^{r_m}\left\langle C^*\sigma, \nabla w_{m_j}\right\rangle_{(L^2(\Omega))^n}\left\langle \mathsf{a}_i, w_{m_j}\right\rangle_{\Omega_i}\right)_{1\leq i\leq p_1},$$

one has

$$\sigma\in\ker\left(\bar{B_1}^*S(\cdot-T)\bar{\nabla}^*\bar{C}^*\right),$$

hence

$$\sigma \in \ker\left(\bar{B_2}^* S(\cdot - T)\bar{\nabla}^*\bar{C}^*\right),$$

i.e., for all $i \in \{1, \dots, p_2\}$, we have

$$\sum_{m\geq 1} \sqrt{-\lambda_m}\sin\left(\sqrt{-\lambda_m}(T-\cdot)\right)\sum_{j=1}^{r_m}\left\langle C^*\sigma, \nabla w_{m_j}\right\rangle_{(L^2(\Omega))^n}\left\langle \tilde{\mathtt{a}}_i, w_{m_j}\right\rangle_{\tilde{\Omega}_i} = 0.$$

The set $\left(\sin(\sqrt{-\lambda_m}(T-.))\right)_{m\geq 1}$ forms a complete orthogonal set of $L^2(0,T)$, then

$$\sum_{j=1}^{r_m}\left\langle C^*\sigma, \nabla w_{m_j}\right\rangle_{(L^2(\Omega))^n}\left\langle \tilde{\mathtt{a}}_i, w_{m_j}\right\rangle_{\tilde{\Omega}_i} = 0,\ \forall i \in \{1, \dots, p_2\},\ \forall m \geq 1.$$

Conversely, we assume that

$$\begin{aligned}
&\sum_{j=1}^{r_m}\langle C^*\sigma, \nabla w_{m_j}\rangle_{(L^2(\Omega))^n}\langle \mathtt{a}_i, w_{m_j}\rangle_{\Omega_i} = 0,\ \forall i \in \{1, 2, \cdots, p_1\},\ \forall m \geq 1\\
\Rightarrow &\sum_{j=1}^{r_m}\langle C^*\sigma, \nabla w_{m_j}\rangle_{(L^2(\Omega))^n}\langle \tilde{\mathtt{a}}_i, w_{m_j}\rangle = 0,\ \forall i \in \{1, 2, \cdots, p_2\},\ \forall m \geq 1,
\end{aligned}$$

we have

$$\begin{aligned}
&\sigma \in \ker\left(\bar{B_1}^* S(\cdot - T)\bar{\nabla}^*\bar{C}^*\right)\\
\Rightarrow &\sum_{j=1}^{r_m}\langle C^*\sigma, \nabla w_{m_j}\rangle_{(L^2(\Omega))^n}\langle \mathtt{a}_i, w_{m_j}\rangle_{\Omega_i} = 0,\ \forall i \in \{1, 2, \cdots, p_1\},\ \forall m \geq 1\\
\Rightarrow &\sum_{j=1}^{r_m}\langle C^*\sigma, \nabla w_{m_j}\rangle_{(L^2(\Omega))^n}\langle \tilde{\mathtt{a}}_i, w_{m_j}\rangle_{\tilde{\Omega}_i} = 0,\ \forall i \in \{1, 2, \cdots, p_2\},\ \forall m \geq 1\\
\Rightarrow &\sigma \in \ker\left(\bar{B_2}^* S(\cdot - T)\bar{\nabla}^*\bar{C}^*\right),
\end{aligned}$$

then $(\Omega_i, \mathtt{a}_i)_{1\leq i\leq p_1}$ dominates $\left(\tilde{\Omega}_i, \tilde{\mathtt{a}}_i\right)_{1\leq i\leq p_2}$ weakly on $[0,T]$ with respect to C. □
In order to give it a characterization, we use the following definitions: for $m \geq 1$,
- The matrix A_m of order $(p_1 \times r_m)$ is defined by

$$A_m = \left(\left\langle \mathtt{a}_i, w_{m_j}\right\rangle_{\Omega_i}\right)_{ij}, \quad 1 \leq i \leq p_1 \quad \text{and} \quad 1 \leq j \leq r_m.$$

- The matrix $\tilde{A}_m$ of order $(p_2 \times r_m)$ is defined by

$$\tilde{A}_m = \left(\left\langle \tilde{\mathtt{a}}_i, w_{m_j}\right\rangle_{\tilde{\Omega}_i}\right)_{ij}, \quad 1 \leq i \leq p_2 \quad \text{and} \quad 1 \leq j \leq r_m.$$

Corollary 4.4 $(\Omega_i, \mathtt{a}_i)_{1\leq i\leq p_1}$ *dominates* $\left(\tilde{\Omega}_i, \tilde{\mathtt{a}}_i\right)_{1\leq i\leq p_2}$ *weakly on* $[0,T]$ *with respect to* C *if and only if*

$$\bigcap_{m\geq 1}\ker\left(A_m g_m\right) \subset \bigcap_{m\geq 1}\ker\left(\tilde{A}_m g_m\right).$$

Here, for $\sigma \in \mathbb{R}^q$ *and* $m \geq 1$,

$$g_m(\sigma) = \left(\ \langle C^*\sigma, \nabla w_{m_i}\rangle_{(L^2(\Omega))^n}\ \right)_{1\leq i\leq r_m}.$$

Proof. We assume that $(\Omega_i, \mathtt{a}_i)_{1\leq i\leq p_1}$ dominates $\left(\tilde{\Omega}_i, \tilde{\mathtt{a}}_i\right)_{1\leq i\leq p_2}$ weakly on $[0,T]$ with respect to C and $\sigma \in \bigcap\limits_{m\geq 1} \ker\left(A_m g_m\right)$, then $A_m g_m(\sigma) = 0,\ \forall m \geq 1$. Since

$$A_m g_m(\sigma) = \begin{pmatrix} \sum\limits_{j=1}^{r_m} \left\langle C^*\sigma, \nabla w_{m_j}\right\rangle_{(L^2(\Omega))^n} \left\langle \mathtt{a}_1, w_{m_j}\right\rangle_{\Omega_1} \\ \sum\limits_{j=1}^{r_m} \left\langle C^*\sigma, \nabla w_{m_j}\right\rangle_{(L^2(\Omega))^n} \left\langle \mathtt{a}_2, w_{m_j}\right\rangle_{\Omega_2} \\ \vdots \\ \sum\limits_{j=1}^{r_m} \left\langle C^*\sigma, \nabla w_{m_j}\right\rangle_{(L^2(\Omega))^n} \left\langle \mathtt{a}_{p_1}, w_{m_j}\right\rangle_{\Omega_{p_1}} \end{pmatrix}, \quad \forall m \geq 1,$$

one has

$$\sum_{j=1}^{r_m} \left\langle C^*\sigma, \nabla w_{m_j}\right\rangle_{(L^2(\Omega))^n} \left\langle \mathtt{a}_i, w_{m_j}\right\rangle_{\Omega_i} = 0,\ \forall i \in \{1, \ldots, p_1\},\ \forall m \geq 1,$$

hence

$$\sum_{j=1}^{r_m} \left\langle C^*\sigma, \nabla w_{m_j}\right\rangle_{(L^2(\Omega))^n} \left\langle \tilde{\mathtt{a}}_i, w_{m_j}\right\rangle_{\tilde{\Omega}_i} = 0,\ \forall i \in \{1, \ldots, p_2\},\ \forall m \geq 1,$$

i.e.,

$$\tilde{A}_m g_m(\sigma) = 0,\ \forall m \geq 1,$$

and therefore $\sigma \in \bigcap\limits_{m\geq 1} \ker\left(\tilde{A}_m g_m\right)$. $\square$

Now, in the case where $\mathcal{O} = \mathbb{R}^q$, i.e., the output of the system (4) is given by q sensors $(D_i, \mathtt{s}_i)_{1\leq i\leq q}$, where $\mathtt{s}_i \in L^2(D_i)$, $D_i = \operatorname{supp}(\mathtt{s}_i) \subset \Omega$ for $i = 1, 2, \ldots, q$, and $D_i \cap D_j = \emptyset$ for $i \neq j$, the operator $\overline{C} = \begin{pmatrix} C & 0 \end{pmatrix}$ is given by

$$\begin{aligned} C \quad : \quad & \left(L^2(\Omega)\right)^n \to \mathbb{R}^q \\ & y \mapsto Cy = \left(\sum_{i=1}^{n} \langle \mathtt{s}_1, y_i\rangle_{D_1} \quad \sum_{i=1}^{n} \langle \mathtt{s}_2, y_i\rangle_{D_2} \quad \cdots \quad \sum_{i=1}^{n} \langle \mathtt{s}_q, y_i\rangle_{D_q}\right)^{tr}, \end{aligned}$$

and its adjoint is $\overline{C}^* = \begin{pmatrix} C^* & 0 \end{pmatrix}^{tr}$, and for $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_q) \in \mathbb{R}^q$,

$$C^*\sigma = \left(\sum_{i=1}^{q} \chi_{D_i}(x)\sigma_i \mathtt{s}_i(x) \quad \sum_{i=1}^{q} \chi_{D_i}(x)\sigma_i \mathtt{s}_i(x) \quad \ldots \quad \sum_{i=1}^{q} \chi_{D_i}(x)\sigma_i \mathtt{s}_i(x)\right)^{tr}. \tag{5}$$

In this case, the exact and weak dominations with respect to the gradient observation are equivalent. The following result gives a necessary and sufficient condition for domination with respect to the sensors.

Corollary 4.5 *$(\Omega_i, \mathtt{a}_i)_{1\leq i\leq p_1}$ dominates $\left(\tilde{\Omega}_i, \tilde{\mathtt{a}}_i\right)_{1\leq i\leq p_2}$ on $[0,T]$ with respect to the sensors $(D_i, \mathtt{s}_i)_{1\leq i\leq q}$ if and only if there exists $\gamma > 0$ such that for all σ in $\mathbb{R}^q$, we have*

$$\sum_{i=1}^{p_2} \int_0^T \Big[\sum_{m\geq 1} \sqrt{-\lambda_m} \sin(\sqrt{-\lambda_m}(T-s)) \sum_{j=1}^{r_m} \sum_{l=1}^{q} \sum_{k=1}^{n} \sigma_l \langle \mathtt{s}_l, \frac{\partial w_{m_j}}{\partial x_k}\rangle_{D_l} \langle \tilde{\mathtt{a}}_i, w_{m_j}\rangle_{\tilde{\Omega}_i}\Big]^2 \mathrm{d}s$$

$$\leq \gamma \sum_{i=1}^{p_1} \int_0^T \Big[\sum_{m\geq 1} \sqrt{-\lambda_m} \sin(\sqrt{-\lambda_m}(T-s)) \sum_{j=1}^{r_m} \sum_{l=1}^{q} \sum_{k=1}^{n} \sigma_l \langle \mathtt{s}_l, \frac{\partial w_{m_j}}{\partial x_k}\rangle_{D_l} \langle \mathtt{a}_i, w_{m_j}\rangle_{\Omega_i}\Big]^2 \mathrm{d}s.$$

Proof. It suffices to use Corollary 4.2 and the relation (5). □

In order to give it another characterization, we pose the following definition: for $m \geq 1$, -The matrix S_m of order $(q \times r_m)$ is defined by

$$S_m = \left(\sum_{k=1}^{n} \left\langle \mathtt{s}_i, \frac{\partial w_{m_j}}{\partial x_k} \right\rangle_{D_i} \right)_{ij}, \quad 1 \leq i \leq q \quad \text{and} \quad 1 \leq j \leq r_m.$$

Corollary 4.6 $(\Omega_i, \mathtt{a}_i)_{1 \leq i \leq p_1}$ *dominates* $\left(\tilde{\Omega}_i, \tilde{\mathtt{a}}_i\right)_{1 \leq i \leq p_2}$ *on* $[0, T]$ *with respect to the sensors* $(D_i, \mathtt{s}_i)_{1 \leq i \leq q}$ *if and only if*

$$\bigcap_{m \geq 1} \ker \left(A_m S_m^{tr}\right) \subset \bigcap_{m \geq 1} \ker \left(\tilde{A}_m S_m^{tr}\right).$$

Proof. Let $\sigma \in \mathbb{R}^q$, we have

$$\begin{aligned}
&\sigma \in \bigcap_{m \geq 1} \ker \left(A_m S_m^{tr}\right) \Leftrightarrow \ \sigma \in \ker \left(A_m S_m^{tr}\right), \ \forall m \geq 1 \Leftrightarrow \ A_m S_m^{tr} \sigma = 0, \ \forall m \geq 1 \\
\Leftrightarrow \ & \sum_{l=1}^{q} \sum_{j=1}^{r_m} \sum_{k=1}^{n} \sigma_l \langle \mathtt{s}_l, \frac{\partial w_{m_j}}{\partial x_k} \rangle_{D_l} \left\langle \mathtt{a}_i, w_{m_j} \right\rangle_{\Omega_i} = 0, \ \forall i \in \{1, 2, \ldots, p_1\}, \ \forall m \geq 1 \\
\Leftrightarrow \ & \sum_{j=1}^{r_m} \left\langle C^* \sigma, \nabla w_{m_j} \right\rangle_{(L^2(\Omega))^n} \left\langle \mathtt{a}_i, w_{m_j} \right\rangle_{\Omega_i} = 0, \ \forall i \in \{1, 2, \ldots, p_1\}, \ \forall m \geq 1 \\
\Leftrightarrow \ & \left(A_m g_m\right)(\sigma) = 0, \forall m \geq 1 \Leftrightarrow \ \sigma \in \ker \left(A_m g_m\right), \forall m \geq 1 \Leftrightarrow \ \sigma \in \bigcap_{m \geq 1} \ker \left(A_m g_m\right),
\end{aligned}$$

this gives

$$\bigcap_{m \geq 1} \ker \left(A_m S_m^{tr}\right) = \bigcap_{m \geq 1} \ker \left(A_m g_m\right).$$

By the same method, we obtain

$$\bigcap_{m \geq 1} \ker \left(\tilde{A}_m S_m^{tr}\right) = \bigcap_{m \geq 1} \ker \left(\tilde{A}_m g_m\right).$$

From Corollary 4.4, we get the result. □

Corollary 4.7 $(\Omega_i, \mathtt{a}_i)_{1 \leq i \leq p_1}$ *dominates* $\left(\tilde{\Omega}_i, \tilde{\mathtt{a}}_i\right)_{1 \leq i \leq p_2}$ *on* $[0, T]$ *with respect to the sensors* $(D_i, \mathtt{s}_i)_{1 \leq i \leq q}$ *if and only if*

$$\begin{aligned}
&\sum_{j=1}^{r_m} \sum_{k=1}^{n} \langle \mathtt{s}_l, \frac{\partial w_{m_j}}{\partial x_k} \rangle_{D_l} \left\langle \mathtt{a}_i, w_{m_j} \right\rangle_{\Omega_i} = 0, \ \forall l \in \{1, 2, \ldots, q\}, \ \forall i \in \{1, 2, \ldots, p_1\}, \ \forall m \geq 1 \\
\Rightarrow & \sum_{j=1}^{r_m} \sum_{k=1}^{n} \langle \mathtt{s}_l, \frac{\partial w_{m_j}}{\partial x_k} \rangle_{D_l} \left\langle \tilde{\mathtt{a}}_i, w_{m_j} \right\rangle_{\tilde{\Omega}_i} = 0, \ \forall l \in \{1, 2, \ldots, q\}, \ \forall i \in \{1, 2, \ldots, p_2\}, \ \forall m \geq 1.
\end{aligned}$$

Proof. It suffices to use Corollary 4.3. □

Corollary 4.8 *If there exists* $m_0 \geq 1$ *such that* $\operatorname{rank}\left(A_{m_0}S_{m_0}^{tr}\right) = q$*, then* $(\Omega_i, \mathtt{a}_i)_{1\leq i\leq p_1}$ *dominates any zone actuators* $\left(\tilde{\Omega}_i, \tilde{\mathtt{a}}_i\right)_{1\leq i\leq p_2}$ *on* $[0,T]$ *with respect to the sensors* $(D_i, \mathtt{s}_i)_{1\leq i\leq q}$*.*

Proof. If there exists $m_0 \geq 1$ such that $\operatorname{rank}\left(A_{m_0}S_{m_0}^{tr}\right) = q$, and the matrix $\left(A_{m_0}S_{m_0}^{tr}\right)$ is of order $(p \times q)$, then from the rank-nullity theorem, we have

$$\operatorname{rank}\left(A_{m_0}S_{m_0}^{tr}\right) + \dim\left(\ker\left(A_{m_0}^{1}S_{m_0}^{tr}\right)\right) = q,$$

then $\dim\left(\ker\left(A_{m_0}S_{m_0}^{tr}\right)\right) = 0$, which is equivalent to $\ker\left(A_{m_0}S_{m_0}^{tr}\right) = \{0\}$, then

$$\bigcap_{m\geq 1} \ker\left(A_m S_m^{tr}\right) = \{0\}.$$

From Corollary 4.6, the operator $(\Omega_i, \mathtt{a}_i)_{1\leq i\leq p_1}$ dominates any zone actuators $\left(\tilde{\Omega}_i, \tilde{\mathtt{a}}_i\right)_{1\leq i\leq p_2}$ with respect to the sensors $(D_i, \mathtt{s}_i)_{1\leq i\leq q}$. $\square$

Corollary 4.9 *If there exists* $m_0 \geq 1$ *such that*

$$\operatorname{rank}\left(A_{m_0}\right) = r_{m_0} \quad \textit{and} \quad \operatorname{rank}\left(S_{m_0}^{tr}\right) = q,$$

then $(\Omega_i, \mathtt{a}_i)_{1\leq i\leq p_1}$ *dominates any zone actuators* $\left(\tilde{\Omega}_i, \tilde{\mathtt{a}}_i\right)_{1\leq i\leq p_2}$ *on* $[0,T]$ *with respect to the sensors* $(D_i, \mathtt{s}_i)_{1\leq i\leq q}$*.*

Proof. We suppose that

$$\operatorname{rank}\left(S_{m_0}^{tr}\right) = q \text{ and } \operatorname{rank}\left(A_{m_0}\right) = r_{m_0}.$$

The matrix $\left(S_{m_0}^{tr}\right)$ is of order $(r_{m_0} \times q)$, then from the rank-nullity theorem, we have

$$\operatorname{rank}\left(S_{m_0}^{tr}\right) + \dim\left(\ker\left(S_{m_0}^{tr}\right)\right) = q,$$

then

$$\dim\left(\ker\left(S_{m_0}^{tr}\right)\right) = 0,$$

which is equivalent to

$$\ker\left(S_{m_0}^{tr}\right) = \{0\}. \tag{6}$$

Similarly, the matrix $\left(A_{m_0}\right)$ is of order $(p \times r_{m_0})$, then from the rank-nullity theorem, we have

$$\operatorname{rank}\left(A_{m_0}\right) + \dim\left(\ker\left(A_{m_0}\right)\right) = r_{m_0},$$

then

$$\dim\left(\ker\left(A_{m_0}\right)\right) = 0,$$

which is equivalent to

$$\ker\left(A_{m_0}\right) = \{0\}. \tag{7}$$

On the other hand, if $\sigma \in \ker\left(A_{m_0}S_{m_0}^{tr}\right)$, then $\left(A_{m_0}S_{m_0}^{tr}\right)\sigma = 0$, which gives

$$A_{m_0}\left(S_{m_0}^{tr}\sigma\right) = 0.$$

From (6), we obtain $S^{tr}_{m_0}\sigma = 0$, and from (7), we obtain $\sigma = 0$. Then

$$\ker\left(A_{m_0} S^{tr}_{m_0}\right) = \{0\}.$$

From Corollary 4.6, $(\Omega_i, \mathtt{a}_i)_{1\leq i\leq p_1}$ dominates any zone actuators $\left(\tilde{\Omega}_i, \tilde{\mathtt{a}}_i\right)_{1\leq i\leq p_2}$ with respect to the sensors $(D_i, \mathtt{s}_i)_{1\leq i\leq q}$. $\square$

5 Application to the Wave Equation

We consider a hyperbolic system described by the following wave equation:

$$\begin{cases} \dfrac{\partial^2 y}{\partial t^2}(x,t) = \Delta y(x,t) + \sum\limits_{i=1}^{p_1} \chi_{\Omega_i} \mathtt{a}_i(x) u_1^i(t) + \sum\limits_{i=1}^{p_2} \chi_{\tilde{\Omega}_i} \tilde{\mathtt{a}}_i(x) u_2^i(t), & \Omega\times]0,T[, \\ y(x,0) = y^0(x),\ \dfrac{\partial y}{\partial t}(x,0) = y^1(x), & \Omega, \\ y(\xi,t) = 0, & \partial\Omega\times]0,T[, \end{cases} \tag{8}$$

where $\Omega \subset \mathbb{R}^n$ is an open and bounded domain with a sufficiently regular boundary, and we consider the system (8) augmented by the output equation

$$z(t) = \left(\sum_{i=1}^{n} \langle \mathtt{s}_1, \frac{\partial y}{\partial x_i}(\cdot,t)\rangle_{D_1} \quad \sum_{i=1}^{n} \langle \mathtt{s}_2, \frac{\partial y}{\partial x_i}(\cdot,t)\rangle_{D_2} \quad \cdots \quad \sum_{i=1}^{n} \langle \mathtt{s}_q, \frac{\partial y}{\partial x_i}(\cdot,t)\rangle_{D_q} \right)^{tr}.$$

There exists an orthonormal basis of eigenfunctions $\left(w_{m_j}\right)_{\substack{m\geq 1\\ 1\leq j\leq r_m}}$ of Δ associated to eigenvalues $(\lambda_m)_{m\geq 1}$ with multiplicity r_m and given by $\Delta w_{m_j} = \lambda_m w_{m_j}$, $\forall m \geq 1$ and $j = 1, 2, \ldots, r_m$.
For $\Omega =]0,1[$, the eigenfunctions of Δ are

$$w_m(x) = \sqrt{2}\sin(m\pi x),\ \forall m \geq 1,$$

and the simple associated eigenvalues are

$$\lambda_m = -m^2\pi^2,\ \forall m \geq 1.$$

The semigroup generated by Δ is

$$S(t)\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \sum\limits_{m\geq 1} (\langle y_1, w_m\rangle_\Omega \cos(m\pi t) + \dfrac{1}{m\pi}\langle y_2, w_m\rangle_\Omega \sin(m\pi t)) w_m \\ \sum\limits_{m\geq 1} (-m\pi\langle y_1, w_m\rangle_\Omega \sin(m\pi t) + \langle y_2, w_m\rangle_\Omega \cos(m\pi t)) w_m \end{pmatrix}.$$

If $D = \mathrm{supp}(\mathtt{s}) \subset]0,1[$, ($q = 1$ and $\mathcal{O} = \mathbb{R}$), the system is augmented with the following output equation:

$$z(t) = \langle \mathtt{s}, \frac{\partial y}{\partial x}(\cdot,t)\rangle_D,$$

and the system (8) is excited by the zone actuators $(\Omega_1, \mathtt{a}_1)$ and $(\tilde{\Omega}_1, \tilde{\mathtt{a}}_1)$ such that $\Omega_1 = \mathrm{supp}(\mathtt{a}_1) \subset]0,1[$ and $\tilde{\Omega}_1 = \mathrm{supp}(\tilde{\mathtt{a}}_1) \subset]0,1[$.

Using Corollary 4.7, we deduce the following characterization.

Proposition 5.1 *(Ω_1, a) dominates $\left(\tilde{\Omega}_1, \tilde{a}\right)$ on $[0,T]$ with respect to the sensors $(D, \mathtt{s})$ if and only if*

$$\left(\langle \mathtt{s}, w_m' \rangle_D \langle a, w_m \rangle_{\Omega_1} = 0,\ \forall m \geq 1 \right) \Rightarrow \left(\langle \mathtt{s}, w_m' \rangle_D \langle \tilde{a}, w_m \rangle_{\tilde{\Omega}_1} = 0,\ \forall m \geq 1 \right).$$

If there exists m_0 such that $\left\langle \mathtt{s}, w_{m_0}' \right\rangle_D \neq 0$, then, from Corollary 4.9, an actuator (Ω_1, a_1) dominates $\left(\tilde{\Omega}_1, \tilde{a}_1\right)$ if

$$\langle a_1, w_{m_0} \rangle_{\Omega_1} = \int_{\Omega_1} a_1(x) \sin(m_0 \pi x)\, \mathrm{d}x \neq 0.$$

Thus, for example, if $\mathtt{a}_1 = w_{m_0}$, then the actuator $(\Omega_1, \mathtt{a}_1)$ dominates any zone actuator $\left(\tilde{\Omega}_1, \tilde{\mathtt{a}}_1\right)$ weakly on $[0,T]$ with respect to the considered sensor $(D, \mathtt{s})$.

6 Conclusion

In this paper, important results and general properties related to the notion of domination of a general class of controlled and observed hyperbolic systems with respect to gradient observation are obtained. The role of actuators and sensors is also examined. The obtained results are related to the choice of convenient efficient actuators. An application to the case of a one dimension wave equation was conducted, it illustrates the notion proposed and confirms the results obtained. Many questions remain open, namely some cases of linear and nonlinear systems. These questions are still under consideration and the results will appear in separate papers.

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