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Spectral Density Estimation in Time Series Analysis for Dynamical Systems

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Abstract: The study proposed in this paper introduces an innovative approach for estimating spectral density in time series analysis within the framework of dynamical systems, which is considered to be the most powerful tool in the statistical treatment of stochastic processes. Spectral density estimation is a crucial tool for understanding the frequency domain characteristics of time series data, particularly in complex dynamical systems. Our analytical results are validated by numerical simulation of the stochastic model. The Yule-Walker technique is used to show the attainment level of the model parameter estimation and the comparison is also made for this estimation. Our approach improves accuracy in capturing spectral characteristics, addressing the challenges posed by nonlinearities inherent in the data. Through empirical and theoretical validation, we demonstrate its efficacy in unraveling time series complexities.

Keywords: *asymptotic properties; ARMA models; Fourier analysis; dynamical periodogram; spectral density.*

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1 Introduction

In various scenarios, information about a particular medium is often depicted as a series of measurements taken over consecutive time intervals, commonly known as a time series. The key disparity between the analysis of time series and situations typically examined in classical statistics lies in the fact that measurements within a time series tend to exhibit stochastic dependence, whereas classical statistics assumes independence among observations. One potential contributor to the intricate nature of time series is the presence of random elements such as measurement errors, system noise, and so forth. At the highest level of randomness, we may encounter a time series representing a sequence of outcomes from independent random variables without any discernible structure. Conversely, the theory of dynamical systems explores the opposite of pure randomness, where future evolution is uniquely determined by the initial state and governing laws. However, the behavior of a deterministic system is not necessarily straightforward. Indeed, advancements in nonlinear dynamical systems reveal the existence of deterministic time series exhibiting highly erratic behavior, resembling realizations of random processes. Such instances are often referred to as chaotic dynamics.

The primary objective of time series analysis revolves around capturing the relationship between future observations and their preceding ones. Dating back to Yule's introduction of linear autoregression in 1927 to analyze sunspot data, linear models have held sway in time series analysis for roughly fifty years. To accommodate complex behaviors within such a simplistic framework, the presence of external random perturbations is necessary. In conventional models propelled by noise, for example, the AR and ARMA models, a future observation is construed as a combination of a specific number of preceding observations and random disturbances, often Gaussian in nature, referred to as innovations (see [1]). However, there are straightforward examples of time series such as those related to chaotic dynamical systems. This poses new challenges: how to recognize such time series and which methods to employ for their modeling and prediction.

In general, when discussing stationary time series, we have a concept of representing the model X_t, where $t \in \mathbb{Z}$, representing the observations of the dynamical system, from which we can define a set of autocovariance as

$$\gamma(t; s) = E\left[(X_t - \mu)(X_s - \mu)\right].$$

This autocovariance depends only on the distance between t and s, $\gamma(t; s) = \gamma(t + h; s + h)$ for all $h \in \mathbb{Z}$. The idea here is to approximate an analytical function by a weighted sum of sine or cosine functions [12], [13]. The idea is as follows: we seek a model of the form

$$X_t = \sum_i a_i \cos(\omega_i t) + b_i \sin(\omega_i t) + \varepsilon_i = \sum_i \sqrt{a_i^2 + b_i^2}\sin(\omega_i - \theta_i) + \varepsilon_i, \tag{1}$$

where (ε_i) is a sequence of independently and identically distributed random variables. If we define $\rho_i = \sqrt{a_i^2 + b_i^2}$, then ρ_i represents the amplitude of the i^{th} periodic component, indicating the weight of that component within the sum.

When considering a sample $X_0, X_2, ...X_{N-1}$ and using frequencies $\omega_j = \frac{2\pi j}{N}$, the dynamical periodogram is defined as

$$I(\omega_j) = \frac{1}{2\pi N}\left|\sum_{k=0}^{k=N-1} X_k e^{ik\omega_j}\right|^2 .$$

It is then possible to demonstrate that $\frac{I(\omega_j)}{N}$ is a consistent estimator of ρ_j in the sense that this estimator converges in probability as the number of observations increases.

Spectral density estimation is an important problem and there is a rich literature. However, restrictive structural conditions have been imposed in many earlier results. For example, Brillinger [5] assumed that all moments existed and cumulants of all orders were summable. Reuman et al. [8] revealed a spectral analysis of stochasticity on nonlinear population dynamics. Recently, Grytsay and Musatenko [4] gave invariant measurements in studying the dynamics of a metabolic process for spectral analysis. Spectral analysis is commonly used in signal processing with the aim of enhancing our understanding of a signal by exploring its frequency domain. Spectral analysis seeks to extract the energy spectrum of a signal. When assuming stationarity, the spectrum becomes a one-dimensional representation of frequency and fully describes the signal's energy content up to the second order. Since most signals originate from random processes, spectral analysis often relies on the domains of probability and statistics [15], [16]. A spectrum can be estimated through a diverse range of methods that utilize information from the observed signal and possibly a priori signal models, whether they are physical or mathematical models. This leads to algorithmic complexity generated by a set of parameters whose choice influences performance. The choices made, the a priori assumptions, and the statistical performance are crucial elements for interpreting the spectrum [10].

Our work involves the analysis of time series for the system with estimation of the spectral density. We employ a technique to construct a spectral density estimator, this is carried out as an asymptotic study. The paper is organized as follows. Section 2 presents the formulation of the baseline model, including some essential concepts and our problem definition. Section 3 lists the asymptotic properties, and exhibits our results. In Section 4, we present a numerical example with simulations.

2 Baseline Model Formulation

Consider any set of observations $x_1, \ldots, x_n$ that can take complex values. If $u = (u_1, \ldots, u_n)'$ and $v = (v_1, \ldots, v_n)'$ are two vectors in $\mathbb{C}^n$, we can define the inner product of u and v as follows:

$$\langle u, v\rangle = \sum_{i=1}^{i=n} u_i \overline{v_i}. \tag{2}$$

Let F_n be a dynamical system of Fourier frequencies defined as

$$F_n = j \in \mathbb{Z}, -\pi < \omega_j = \frac{2\pi j}{n} \leq \pi = -[\frac{n-1}{2}], \ldots, [\frac{n}{2}],$$

where $[x]$ denotes the floor function of x.

We define the vectors e_j for j in F_n as

$$e_j = n^{-1/2}(e^{i\omega j}, e^{2i\omega j}, \ldots, e^{ni\omega j})'. \tag{3}$$

Proposition 2.1 *The vectors e_j, for $j \in F_n$, defined above form an orthonormal basis for $\mathbb{C}^n$.*

Proof. We have

$$\begin{aligned}\langle e_j, e_k\rangle &= n^{-1}\sum_{r=1}^{r=n} e^{ir(\omega_j-\omega_k)} \\ &= \begin{cases} n^{-1}\sum_{r=1}^{r=n} e^0 = \frac{n}{n} = 1, \text{if } j = k; \\ n^{-1}e^{i(\omega_j-\omega_k)}(\frac{1-e^{in(\omega_j-\omega_k)}}{1-e^{i(\omega_j-\omega_k)}}) = 0, \text{if} j \neq k. \end{cases}\end{aligned}$$

Let us further justify the case when $j \neq k$. The first equality arises because we have a geometric series with a common ratio of $e^{i(\omega_j-\omega_k)}$.

Furthermore, the numerator of the fraction becomes zero because, by the definition of ω_j and ω_k,

$$e^{in(\omega_j-\omega_k)} = e^{i2\pi(j-k)} = 1$$

since $j - k$ is a non-zero integer. □

The value $I(\omega j)$ of the dynamical periodogram of the vector $x = (x1, \dots, x_n)$ at frequency $\omega_j = \frac{2\pi j}{n}$ is given by

$$\begin{aligned} I(\omega_j) &= \frac{1}{n}\left|\sum_{t=1}^{t=n} x_t e^{-it\omega j}\right|^2 \\ &= \frac{1}{n}\left[\left(\sum t = 1^{t=n} x_t \cos\omega jt\right)^2 + \left(\sum t = 1^{t=n} x_t \sin\omega_j t\right)^2\right]. \end{aligned} \tag{4}$$

The dynamical periodogram is a powerful tool for detecting a signal because if X contains a sinusoidal component with frequency ω_0, then, when we are close to this frequency, the factors $X(t)$ and $e^{-i\omega_0 t}$ are in phase and make a significant contribution to the sum in equation (4). For other values of ω, some terms in the sum are positive, while others are negative, thus canceling each other out in the sum, which becomes small. In summary, we can detect the presence of a sinusoidal signal when a large value of $I(.)$ appears for a certain value of ω.

Proposition 2.2 *If ω_j is a non-zero Fourier frequency, then*

$$I(\omega_j) = \sum_{|K|<n} \widehat{\gamma}(k) e^{-ik\omega_j},$$

where

$$\widehat{\gamma}(k) = n^{-1}\sum_{t=1}^{t=n-k}(x_{t+k} - m)(\overline{x_t} - \overline{m})$$

with m being the empirical mean of x_i, $m = n^{-1}\sum_{t=1}^{t=n} x_t$, and $\overline{(.)}$ denoting complex conjugation. Also, $\widehat{\gamma}(-k) = \overline{\widehat{\gamma}(k)}$ when $k < 0$.

We show a strong resemblance between the obtained expression for $I(\omega_j)$ and the expression of the spectral density of a stationary dynamical system given by

$$f(\omega) = \frac{1}{2\pi} \sum_{k=-\infty}^{+\infty} \gamma(k) e^{-ik\omega} \quad \text{when} \quad \sum_{k=-\infty}^{+\infty} |\gamma(k)| < \infty.$$

So, we can take $\frac{I(\omega_j)}{2\pi}$ as an estimator of $f(\omega_j)$.

3 Asymptotic Properties and Main Results

In this section, we will focus on the asymptotic properties of the periodogram of a stationary dynamical system with a mean μ and a covariance function that is absolutely summable, i.e., $\sum_{k=-\infty}^{+\infty} |\gamma(k)| < \infty$. Based on the previous remark, we take the estimator of $f(\omega_j)$ as $I(\omega_j)/(2\pi)$.

We begin by extending the dynamical periodogram to all $\omega \in [-\pi, \pi]$ so that it is no longer limited to the Fourier frequencies.
For any $\omega \in [-\pi, \pi]$, the dynamical periodogram is defined as follows:

$$I_n(\omega) = \begin{cases} I_n(\omega_j) & = n^{-1} |\sum_{t=1}^{t=n} X_t e^{-it\omega_j}|^2 \\ & \text{if } \frac{-\pi}{n} < \omega \leq \frac{\pi}{n} \text{and} \quad \omega \in [0, \pi]; \\ I_n(-\omega) & \text{if } \omega \in [-\pi, 0]. \end{cases}$$

For every $\omega \in [0, \pi]$, let us denote $g(n, \omega)$ as the multiple of $\frac{2\pi}{n}$ closest to ω. For every $\omega \in [-\pi, 0]$, we define $g(n, \omega) = g(n, -\omega)$. Thus

$$I_n(\omega) = I_n(g(n, \omega)).$$

Proposition 3.1 *Let $(X_t)t \in \mathbb{Z}$ be a second-order stationary dynamical system with a mean μ and an absolutely summable autocovariance function $\gamma(.)$ $\left(\sum_{k=-\infty}^{\infty} |\gamma(k)| < \infty \right)$, then*

$$\textit{when} \quad n \to \infty : \begin{cases} (E(I_n(0)) - n\mu^2) \longrightarrow 2\pi f(0), \\ E(I_n(\omega)) \longrightarrow 2\pi f(\omega) \quad \textit{if} \quad \omega \neq 0 \end{cases}$$

with

$$I_n(0) = n|\overline{X}|^2 \quad \textit{and} \quad I(\omega_j) = \sum_{|K|<n} \widehat{\gamma}(k) e^{-ik\omega_j} \quad \textit{if} \quad \omega_j \neq 0.$$

Remark 3.1 If $\mu = 0$, then $E(I_n(\omega))$ converges uniformly to $2\pi f(\omega)$ for all $\omega \in [-\pi, \pi]$.

Proof.

$$E(I_n(0)) = nE(|\overline{X}|^2) = n[Var(\overline{X}) + (E(\overline{X}))^2] = nVar(\overline{X}) + n\mu^2.$$

So

$$E(I_n(0)) - n\mu^2 \quad = \quad nVar(\overline{X}).$$

On the other hand,

$$nVar(\overline{X}) = \frac{1}{n}\sum_{i=1}^{i=n}\sum_{j=1}^{j=n} cov(X_i, X_j) = \sum_{|h|<n}(1 - \frac{|h|}{n})\gamma(h)$$

if $\sum_{k=-\infty}^{\infty}|\gamma(k)| < \infty$. According to the dominated convergence theorem (Theorem 3.3.1) from [6], we have

$$\lim_{n\to\infty} nVar(\overline{X}) = \lim_{n\to\infty}\sum_{|h|<n}(1 - \frac{|h|}{n})\gamma(h) = \sum_{h=-\infty}^{\infty}\gamma(h) = 2\pi f(0),$$

$$E(I_n(\omega)) = \sum_{|k|<n}\frac{1}{n}\sum_{t=1}^{n-|k|} E\left[(X_t-\mu)(X_{t+|k|}-\mu)\right]e^{-ikg(n,\omega)} = \sum_{|k|<n}\big(1-\frac{|k|}{n})\gamma(k)e^{-ikg(n,\omega)}.$$

Since $\sum_{k\in\mathbb{Z}}|\gamma(k)| < \infty$ and according to the dominated convergence theorem (Theorem 3.3.1) from [6],

$$\sum_{|k|<n}\big(1 - \frac{|k|}{n})\gamma(k)e^{-ikg(n,\lambda)} \quad\to\quad 2\pi f(\lambda).$$

On the other hand, we have $g(n,\omega) \to \omega$. So

$$E(I_n(\omega)) \quad\to\quad 2\pi f(\omega).$$

□

Theorem 3.1 *Let $\{X_t\}$ be a time series process defined by*

$$X_t = \sum_{k=-\infty}^{+\infty}\psi_k\varepsilon_{t-k},$$

where ε_t is a strong white noise $IID(0,\sigma^2)$ with $\mathbb{E}(\varepsilon_t^2) < \infty$. We assume that $\sum_{j=-\infty}^{+\infty}|\psi_j||j|^{\frac{1}{2}}$ and $\psi(e^{-i\lambda}) = \sum_{k=-\infty}^{+\infty}\psi_k e^{-ik\lambda} \neq 0$. We know that

$$f_X(\omega) = \frac{\sigma^2}{2\pi}|\psi(e^{-i\omega})|^2.$$

1. *For fixed frequencies $0 < \lambda_1 < \dots\lambda_m < \pi$ as $n \to +\infty$, the random vector $(I_{n,X}(\lambda_1)/f_X(\lambda_1), ..., I_{n,X}(\lambda_m)/f_X(\lambda_m))'$ converges by law to a vector of independent random variables with the same exponential distribution with a mean of 1.*

2. *We have*

$$Var(I_{n,X}(\omega_j)) = \begin{cases} 2f_X^2(\omega_j) & +O(n^{-\frac{1}{2}}) \\ & \text{if } \omega_j = 0 \text{ or } \pi; \\ f_X^2(\omega_j) & +O(n^{-\frac{1}{2}}) \\ & \text{if } 0 < \omega_j < \pi \end{cases}$$

and $\quad Cov(I_{n,X}(\omega_j), I_{n,X}(\omega_k)) = O(n^{-1}) \quad$ *if* $\quad \omega_j \neq \omega_k$.

Thus, we show the previous Theorem 3.1 by using the following intermediate lemmas.

Lemma 3.1 *Let* $\{\varepsilon_t\}_{\{t\in\mathbb{Z}\}}$ *be an i.i.d. white noise process with a zero mean and finite variance* $\sigma^2 < \infty$. *Its spectral distribution has a density function of* $f_\varepsilon(\omega) = \frac{\sigma^2}{2\pi}$ *and let* I_n *be the dynamical periodogram of* $\{\varepsilon_t\}$.

1. *Suppose* $0 < \lambda_1 < \lambda_2 < \ldots \lambda_m < \pi$ *are fixed frequencies. Then, as* $n \to +\infty$, *the random vector* $(I_n(\lambda_1), I_n(\lambda_2), \ldots, I_n(\lambda_m))'$ *converges by law to a vector of independent random variables same as a mean of* $\frac{\sigma^2}{2\pi}$;
2. *If* $E(\varepsilon_t^4) = \eta\sigma^4 < \infty$ *and* $0 \leq \omega_j = \frac{2\pi j}{n} \leq \pi$ *are the Fourier frequencies, then*

$$Var(I_n(\omega_j)) = \begin{cases} 2f_\varepsilon^2(\omega_j) & +\frac{\kappa_4}{4\pi^2 n} \\ & \text{if } \omega_j = 0 \text{ or } \pi; \\ f\varepsilon^2(\omega_j) & +\frac{\kappa_4}{4\pi^2 n} \\ & \text{if } \quad 0 < \omega_j < \pi \end{cases} = \begin{cases} \frac{\sigma^4}{2\pi^2} & +\frac{\kappa_4}{4\pi^2 n} \\ & \text{if } \omega_j = 0 \text{ or } \pi; \\ \frac{\sigma^4}{4\pi^2} & +\frac{\kappa_4}{4\pi^2 n} \\ & \text{if } \quad 0 < \omega_j < \pi \end{cases}$$

   and $\quad Cov(I_n(\omega_j), I_n(\omega_k)) = \frac{\kappa_4}{4\pi^2 n} \quad$ *if* $\quad \omega_j \neq \omega_k$, *where* κ_4 *is the fourth-order cumulant of the variable* ε_t *defined as* $\kappa_4 = \mathbb{E}\{\varepsilon_t^4\} - 3(\mathbb{E}\{\varepsilon_t^2\})^2$;
3. *Let us assume that the random variables* ε_t *are Gaussian. In this case,* $\kappa_4 = 0$ *and for all* n, *the random variables* $I_n(\omega_k)/f_\varepsilon(\omega)$, $k \in \{1, \ldots\lfloor\frac{(n-1)}{2}\rfloor\}$ *are independent and identically distributed according to an exponential distribution with a mean of 1.*

Proof.

1. Let us note

$$\begin{cases} \alpha_n(\omega_j) = \sqrt{2\pi n}\sum_{t=1}^{n} \varepsilon_t cos(\omega_j t), \\ \beta_n(\omega_j) = \sqrt{2\pi n}\sum_{t=1}^{n} \varepsilon_t sin(\omega_j t) \end{cases} \tag{5}$$

are the real and imaginary parts of the discrete Fourier transform of ε_t at the frequency points $\omega_j = \frac{2\pi j}{n}$. For an arbitrary frequency ω, we have

$$I_n(\omega_j) = \frac{1}{2}(\alpha_n^Z(g(n,\omega_j))^2 + \beta_n^Z(g(n,\omega_j))^2)^2.$$

Recall that if a sequence of random vectors Y_n converges by law to a random variable Y and ϕ is a continuous function, then $\phi(Y_n)$ converges by law to $\phi(Y)$. It is sufficient to show that the random vector

$$(\alpha_n(\omega_1), \beta_n(\omega_1), \ldots, \alpha_n(\omega_m), \beta_n(\omega_m)) \tag{6}$$

converges by law to a normal distribution with a zero mean and an asymptotic covariance matrix $(\frac{\sigma^2}{4\pi})I_{2m}$, where I_2m is the identity matrix $(2m \times 2m)$. We will first focus on the case $m = 1$. The proof follows from the following corollary.

Corollary 3.1 *Let $U_{n,t}$, where $t = 1, \dots, n$ and $n = 1, 2, \dots,$ be a triangular sequence of centered random variables with finite variances. For all n, the dynamics variables $\{U_{n,1}, \dots, U_{n,n}\}$ are independent. We define $Y_n = \sum_{t=1}^{n} U_{n,t}$ and $\vartheta_n^2 = \sum_{t=1}^{n} var(U_{n,t})$. Then, if for every $\epsilon > 0$,*

$$\lim_{n\to+\infty} \sum_{t=1}^{n} \frac{1}{\vartheta_n^2} \mathbb{E}\left[U_{n,t}^2 \mathbf{I}(|U_{n,t}| > \epsilon\vartheta_n)\right] = 0,$$

we have

$$Y_n/\vartheta_n \longrightarrow_d \mathcal{N}(0,1).$$

Let u and v be arbitrary fixed real numbers, and $\omega_j \in (0, \pi)$. Consider the variable $Y_n = u\alpha_n(g(n,\omega_j)) + v\beta_n(g(n,\omega_j))$, which can also be written as

$$Y_n = \sum_{t=1}^{n} U_{n,t}, \text{where} \quad U_{n,t} = \frac{1}{\sqrt{2\pi n}}(u\cos(g(n,\omega_j)t) + v\sin(g(n,\omega_j)t))\varepsilon_t.$$

Note that, for a fixed n, the random variables $\{U_{n,t}\}$ are independent. Furthermore, for all $\omega_j \neq 0$, it is easy to verify that

$$\sum_{t=1}^{n} \cos^2(g(n,\omega_j)t) = \sum_{t=1}^{n} \sin^2(g(n,\omega_j)t) = \frac{n}{2}$$

and

$$\sum_{t=1}^{n} \cos(g(n,\omega_j)t)\sin(g(n,\omega_j)t) = 0.$$

As a result, we can write

$$\begin{aligned}
\vartheta_n^2 &= \sum_{t=1}^{n} \mathrm{var}(U_{n,t}) \\
&= \frac{1}{2\pi n} \sum_{t=1}^{n} \Big[u^2\cos^2(g(n,\omega_j)t) + v^2\sin^2(g(n,\omega_j)t) \\
&\quad + 2uv\cos(g(n,\omega_j)t)\sin(g(n,\omega_j)t)\Big] \\
&= \frac{(u^2+v^2)}{4\pi} = \vartheta_1^2.
\end{aligned}$$

Hence, by setting $c_0 = (|u|+|v|)/2\pi\vartheta_1$ and $\epsilon' = \epsilon\sqrt{2\pi}\vartheta_1/(|u|+|v|)$, we have

$$\sum_{t=1}^{n} \frac{1}{\vartheta_n^2} \mathbb{E}\left[U_{n,t}^2 \mathbf{I}(|U_{n,t}| \geqslant \epsilon\vartheta_n)\right] \leq \frac{c_0}{n} \sum_{t=1}^{n} \mathbb{E}\left[\varepsilon_t^2 \mathbf{I}(|\varepsilon_t| \geqslant \epsilon'\sqrt{n})\right] = c_0 \mathbb{E}\left[\varepsilon_t^2 \mathbf{I}(|\varepsilon_t| \geqslant \epsilon'\sqrt{n})\right].$$

The last term tends to 0 as we have $\mathbb{E}\left[\varepsilon_1^2 \mathbf{I}(|\varepsilon_1| \geqslant \epsilon'\sqrt{n})\right] \leq \mathbb{E}\left[|\varepsilon_1|^3\right]/\epsilon'\sqrt{n})$ and $\mathbb{E}\left[|\varepsilon_1|^3\right] < \infty$ since $\mathbb{E}\left[|\varepsilon_1|^4\right] < \infty$.

The proof can be easily extended to a set of frequencies $\omega_1 \dots \omega_m$ using the Cramer-Wold method as we recall the following. Let $\{V_n\}_{n\geqslant 0}$ be a sequence of real random vectors of dimension m. $V_n \longrightarrow_d W$ if and only if, for any sequence $\{\omega_1 \dots \omega_m\} \in \mathbb{R}^m$, the random variable $Y_n = \omega_1 V_{n,1} + \dots + \omega_m V_{n,m} \longrightarrow_d \omega_1 W_1 + \dots + \omega_m W_m$.

2. By definition of $I_n(\omega_j)$, we have, at the first order,

$$\mathbb{E}\left[I_n(\omega_j)\right] = \frac{1}{2\pi n}\sum_{s,t=1}^{n}\mathbb{E}\left[\varepsilon_s\varepsilon_t\right]e^{i\omega_j(t-s)} = \frac{\sigma^2}{2\pi}. \tag{7}$$

At the second order, we have

$$\mathbb{E}\left[I_n(\omega_j)I_n^Z(\omega_k)\right] = \frac{1}{(2\pi n)^2}\times\sum_{s,t,u,v=1}^{n}\mathbb{E}\left[\varepsilon_s\varepsilon_t\varepsilon_u\varepsilon_v\right]e^{i(\omega_j(t-s)+\omega_k(v-u))}. \tag{8}$$

Using the fact that the random variables ε_t are independent, centered, have the same variance σ^2 and finite fourth moments, and setting $\mathbb{E}[\varepsilon_1^4] = \kappa_4 + 3\sigma^4$, we obtain

$$\mathbb{E}\left[\varepsilon_s\varepsilon_t\varepsilon_u\varepsilon_v\right] = \kappa_4\delta_{s,t,u,v} + \sigma^4(\delta_{s,t}\delta_{u,v} + \delta_{s,u}\delta_{t,v} + \delta_{s,v}\delta_{t,u}). \tag{9}$$

Plugging this expression into 8, we get

$$\mathbb{E}\left[I_n(\omega_j)I_n(\omega_k)\right] = \frac{\kappa_4}{(4\pi^2 n)} + \frac{\sigma^4}{(4\pi^2 n^2)}\times\left[n^2 + \left|\sum_{t=1}^{n}e^{i(\omega_j+\omega_k)t}\right|^2 + \left|\sum_{t=1}^{n}e^{i(\omega_k-\omega_j)t}\right|^2\right]$$

and therefore

$$\begin{aligned} cov\left[I_n(\omega_j), I_n(\omega_k)\right] &= \mathbb{E}\left[I_n(\omega_j)I_n(\omega_k)\right] - \mathbb{E}\left[I_n(\omega_j)\right]\mathbb{E}\left[I_n(\omega_k)\right] \\ &= \frac{\kappa_4}{(4\pi^2 n)} + \frac{\sigma^4}{(4\pi^2 n^2)}\times\left[\left|\sum_{t=1}^{n}e^{i(\omega_j+\omega_k)t}\right|^2 + \left|\sum_{t=1}^{n}e^{i(\omega_k-\omega_j)t}\right|^2\right]. \end{aligned}$$

This allows us to conclude.

3. When $\{\varepsilon_t\}$ is a centered Gaussian variable, the vector

$$Q_n = \left[\alpha_n(\omega_1)\ \beta_n(\omega_1)\ \dots\ \alpha_n(\omega_{\tilde{n}})\beta_n(\omega_{\tilde{n}})\right].$$

It is sufficient to calculate the mean vector and its covariance matrix. It is easy to verify that the mean vector is zero, and that for $0 < \omega_k \neq \omega_j < \pi$, we have

$$\begin{aligned} &\mathbb{E}\left[(\alpha_n(\omega_k))^2\right] = \mathbb{E}\left[(\beta_n(\omega_k))^2\right] = \frac{1}{4\pi}, && \mathbb{E}\left[\alpha_n(\omega_k)\beta_n(\omega_k)\right] = 0, \\ &\mathbb{E}\left[\alpha_n(\omega_k)\alpha_n(\omega_j)\right] = \mathbb{E}\left[\beta_n(\omega_k)\beta_n^Z(\omega_j)\right] = 0, && \mathbb{E}\left[\alpha_n(\omega_k)\beta_n(\omega_j)\right] = 0. \end{aligned}$$

The covariance matrix is thus $\frac{\sigma^2 I_{\tilde{n}}}{4\pi}$, where $I_{\tilde{n}}$ is the identity matrix of size $\tilde{n}$. Consequently, the components of Q_n are independent. Recall that

$$I_n(\omega_j) = (\alpha_n(\omega_j))^2 + (\beta_n(\omega_j))^2.$$

From the independence of the components of Q_n, we deduce that the random variables $I_n(\omega_j)$ are themselves independent, and that $\frac{4\pi I_n(\omega_j)}{\sigma^2}$ is the sum of the squares of two independent, centered, identically distributed Gaussian variables, each with a variance of 1, whose probability distribution is the law of χ^2 distribution with two degrees of freedom. This concludes the proof. □

The following lemma shows that there is a similar relationship to the previous one that relates the dynamical periodogram $I_{n,X}(\omega)$ of the time series process $\{X_t\}$ and the dynamical periodogram $I_{n,\varepsilon}(\omega)$ of the strong white noise $\{\varepsilon_t\}$.

Lemma 3.2 *Let $\{X_t\}$ be a strong time series process, $X_t = \sum_{k=-\infty}^{+\infty} \psi_k \varepsilon_{t-k}$. Suppose $\sum_{j=-\infty}^{+\infty} |\psi_j||j|^{\frac{1}{2}} < \infty$ and $\mathbb{E}\{\varepsilon_t^4\} < \infty$. Then we have*

$$I_{n,X}(\omega_k) = |\psi(e^{-i\omega_k})|^2 I_{n,\varepsilon}(\omega_k) + R_n(\omega_k),$$

where $R_n(\omega_k)$ satisfies

$$\max_{k\in\{1,...\lfloor\frac{(n-1)}{2}\rfloor\}} \mathbb{E}\{|R_n(\omega_k)|^2\} = O(\frac{1}{n}) \tag{10}$$

and $\omega_k = \frac{2\pi k}{n}$, where $k \in \{1,...\lfloor\frac{(n-1)}{2}\rfloor\}$ are the Fourier frequencies.

Proof. Let us denote $d_n^X(\omega_k)$ and $d_n^Z(\omega_k)$ as the dynamical system of the discrete Fourier transforms of the sequences $\{X_1, ..., X_n\}$ and $\{Z_1, ..., Z_n\}$ at the frequency point $\frac{2\pi k}{n}$ with $k \in \{1, ..., [\frac{(n-1)}{2}]\}$. We can write successively:

$$\begin{aligned}
d_n^X(\omega_k) &= \frac{1}{2\pi n}\sum_{t=1}^{n} X_t e^{-i\omega_k t} \\
&= \frac{1}{2\pi n}\sum_{j=-\infty}^{+\infty} \psi_j e^{-i\omega_k j}\left(\sum_{t=1}^{n} Z_{t-j} e^{-i\omega_k(t-j)}\right) \\
&= \frac{1}{2\pi n}\sum_{j=-\infty}^{+\infty} \psi_j e^{-i\omega_k j}\left(\sum_{t=1-j}^{n-j} Z_t e^{-i\omega_k t}\right) \\
&= \frac{1}{2\pi n}\sum_{j=-\infty}^{+\infty} \psi_j e^{-i\omega_k j}\left(\sum_{t=1}^{n} Z_t e^{-i\omega_k t} + U_{n,j}(\omega_k)\right) \\
&= \psi(e^{-i\omega_k}) d_n^Z(\omega_k) + Y_n(\omega_k).
\end{aligned}$$

Here, we have defined

$$U_{n,j}(\omega_k) = \sum_{t=1-j}^{n-j} Z_t e^{-i\omega_k t} - \sum_{t=1}^{n} Z_t e^{-i\omega_k t} \tag{11}$$

and

$$Y_n(\omega_k) = \frac{1}{\sqrt{2\pi n}}\sum_{j=-\infty}^{+\infty} \psi_j e^{-i\omega_k j} U_{n,j}(\omega_k). \tag{12}$$

We observe that, for $|j| < n$, $U_{n,j}(\omega_k)$ is a sum of $2|j|$ independent centered variables with variance σ^2, while for $|j| \geq n$, $U_{n,j}(\omega_k)$ is a sum of $2n$ independent centered variables with variance σ^2. Therefore, using (11), we have

$$\mathbb{E}\left[|U_{n,j}(\omega_k)|^2\right] \leq 2\sigma^2 \min(|j|, n) \tag{13}$$

and

$$\mathbb{E}\left[|U_{n,j}(\omega_k)|^4\right] \le C_R\sigma^4(\min(|j|,n))^2, \tag{14}$$

where $C_R < \infty$ is a constant. To establish (14), we only need to set $\mathbb{E}\left[Z_t^4\right] = \eta\sigma^4$ and use the inequality (15) for $p = 4$:

$$\mathbb{E}\left[|\sum_{k=1}^{n} X_k|^p\right] \le C(p)\left(\left(\sum_{k=1}^{n}\mathbb{E}[X_k^2]\right)^{\frac{p}{2}} + \sum_{k=1}^{n}\mathbb{E}[|X_k|^p]\right). \tag{15}$$

Now, use (14) to bound $\mathbb{E}\left[|Y_n(\omega_k)|^4\right]$. By adapting the notation $\|X\|_p = (\mathbb{E}[|X|^p])^{\frac{1}{p}}$ (for $p > 0$), we get, following the triangular inequality (Minkovski inequality) $\|X+Y\|_p \le \|X\|_p + \|Y\|_p$,

$$\sup_{k\in\{1,\dots,[\frac{(n-1)}{2}]\}} \|Y_n(\omega_k)\|_4 \le \sup_{k\in\{1,\dots,[\frac{(n-1)}{2}]\}} \frac{1}{\sqrt{2\pi n}} \sum_{j=-\infty}^{+\infty} |\psi_j|\|U_{(n,j)}(\omega_k)\|_4.$$

From (14), $\|U_{(n,j)}(\omega_k)\|_4 \le C\sigma\min(|j|,n)^{\frac{1}{2}}$. Therefore

$$\sup_{k\in\{1,\dots,[\frac{(n-1)}{2}]\}} \|Y_n(\omega_k)\|_4 \le C\sigma(\frac{1}{\sqrt{2\pi n}}) \sum_{j=-\infty}^{+\infty} |\psi_j|\min(|j|,n)^{\frac{1}{2}}.$$

Now we can write

$$\sum_{j=-\infty}^{+\infty} |\psi_j|\min(|j|,n)^{\frac{1}{2}} \le \sum_{j=-\infty}^{+\infty} |\psi_j||j|^{\frac{1}{2}}.$$

Therefore, $\|Y_n(\omega_k)\|_4$ is of the same order as $O(n^{\frac{-1}{2}})$.
We can now express $R_n(\omega_k) = I_n^X(\omega_k) - |\psi(e^{-i\omega_k})|^2 I_n^Z(\omega_k)$ in terms of $Y_n(\omega_k) = d_n^X(\omega_k) - \psi(e^{-i\omega_k})d_n^Z(\omega_k)$. It follows that

$$\begin{aligned} R_n(\omega_k) &= |\psi(e^{-i\omega_k})d_n^Z(\omega_k) + Y_n(\omega_k)|^2 - |\psi(e^{-i\omega_k})|^2 I_n^Z(\omega_k) \\ &= \psi(e^{-i\omega_k})d_n^Z(\omega_k)Y_n(-\omega_k) + \psi(e^{-i\omega_k})d_n^Z(-\omega_k)Y_n(\omega_k) + |Y_n(\omega_k)|^2. \end{aligned}$$

According to Holder's inequality, $\|XY\|_r \le \|X\|_p\|Y\|_q$ if $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. Taking $p = q = 4$ and $r = 2$, we get

$$(\mathbb{E}\left[|R_n(\omega_k)|^2\right])^{\frac{1}{2}} = \|R_n(\omega_k)\|_2 \le 2\sum_j |\psi_j|\|d_n^Z(\omega_k)\|_4\|Y_n(\omega_k)\|_4 + \|Y_n(\omega_k)\|_4.$$

According to Lemma 3.1, $\|d_n^Z(\omega_k)\|_4$ is of the order of $\frac{\sigma}{\sqrt{2\pi}}$. Therefore, $\|R_n(\omega_k)\|_2$ is of the order of $\frac{1}{\sqrt{n}}$ and $\mathbb{E}[|R_n(\omega_k)|^2] = \|R_n(\omega_k)\|_2^2$ is of the order of $\frac{1}{n}$.

This completes the proof of Theorem 3.1. □

4 Applications

4.1 Numerical example

Let X_t be an AR(p) process defined by the equation

$$X_t + \phi_1 X_{t-1} + \phi_2 X_{t-2} + \cdots + \phi_p X_{t-p} = \varepsilon_t. \tag{16}$$

This equation can be written in the following form. Let X_t be an AR(p) process defined by the equation

$$\sum_{k=0}^{p} \phi_k X_{t-k} = \varepsilon_t \quad \text{with} \quad \phi_0 = 1. \tag{17}$$

By multiplying equation (17) by X_{t-1}, we get

$$\sum_{k=0}^{p} \phi_k E\left\{X_{t-k} X_{t-l}\right\} = E\left\{\varepsilon_t X_{t-l}\right\} \quad , a_0 = 1. \tag{18}$$

We can easily identify the terms of autocorrelation and cross-correlation in the Yule-Walker equation:

$$\sum_{k=0}^{N} \phi_k \rho_{xx}[l-k] = \rho_{\varepsilon x}[l] \quad \text{with} \quad \phi_0 = 1. \tag{19}$$

The next step is to calculate the identified cross-correlation term $\rho_{\varepsilon x}(l)$ and relate it to the autocorrelation term $\rho_{xx}(l-k)$.
The term X_{t-l} can also be obtained from equation (16):

$$X_{t-l} = -\sum_{k=1}^{p} \phi_k X_{t-k-l} + \varepsilon_{t-l}. \tag{20}$$

Note that the data and the noise are uncorrelated, so $(X_{t-l} w_t = 0)$. Also, the autocorrelation of the noise is zero at all lags, except for lag 0, where its value is equal to σ^2 (recall the flat power spectral density of white noise and its autocorrelation). These two properties are used in the following steps. We restrict the lags only to non-negative values and zero,

$$\begin{aligned}
\rho_{\varepsilon X}(l) &= E\left\{\varepsilon_t X_{t-l}\right\} \\
&= E\left\{\varepsilon_t \left(-\sum_{k=1}^{N} \phi_k X_{t-k-l} + \varepsilon_{t-l}\right)\right\} \\
&= E\left\{-\sum_{k=1}^{N} \phi_k X_{t-k-l}\varepsilon_t + \varepsilon_{t-l}\varepsilon_t\right\} \\
&= E\left\{0 + \varepsilon_{t-l}\varepsilon_t\right\} \\
&= E\left\{\varepsilon_{t-l}\varepsilon_t\right\} \\
&= \begin{cases} 0 & , l > 0 \\ \sigma^2 & , l = 0. \end{cases}
\end{aligned} \tag{21}$$

By substituting equation (21) into equation (19), we obtain

$$\sum_{k=0}^{N} \phi_k \rho_{xx}[l-k] = \begin{cases} 0, \ l > 0, \\ \sigma^2, \ l = 0, \end{cases} \qquad a_0 = 1. \tag{22}$$

Here, there are two cases to solve, when $(l > 0)$ and when $(l = 0)$. For the case when $l > 0$, equation (22) becomes

$$\sum_{k=1}^{N} \phi_k \rho_{xx}[l-k] = -\rho_{xx}[l]. \tag{23}$$

Equation (23) can be written in the matrix form

$$\begin{pmatrix} \rho_{xx}(0) & \dots & \rho_{xx}(2-P) & \rho_{xx}(1-P) \\ \rho_{xx}(1) & \dots & \rho_{xx}(3-P) & \rho_{xx}(2-P) \\ \vdots & \ddots & \vdots & \vdots \\ \rho_{xx}(P-1) & \dots & \rho_{xx}(1) & \rho_{xx}(0) \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_P \end{pmatrix} = - \begin{pmatrix} \rho_{xx}(1) \\ \rho_{xx}(2) \\ \vdots \\ \rho_{xx}(P) \end{pmatrix}. \quad (24)$$

This is the Yule-Walker system, which consists of a set of P equations and P unknown parameters. Represent equation (24) in a compact format

$$\bar{\varrho}\bar{\phi} = -\bar{\rho}. \quad (25)$$

The solutions $\bar{a}$ can be obtained by

$$\bar{\phi} = -\bar{\varrho}^{-1}\bar{\rho}. \quad (26)$$

Once we solve for $\bar{\phi}$, which corresponds to the model parameters ϕ_k, the noise variance σ^2 is obtained by applying the estimated ϕ_k in equation (22) with $l = 0$. Matlab's "aryule" efficiently solves the Yule-Walker system using the Levinson Algorithm.

4.2 Simulation

We will generate an AR(3) process and assume that we know nothing about the model parameters (see Figure 1).

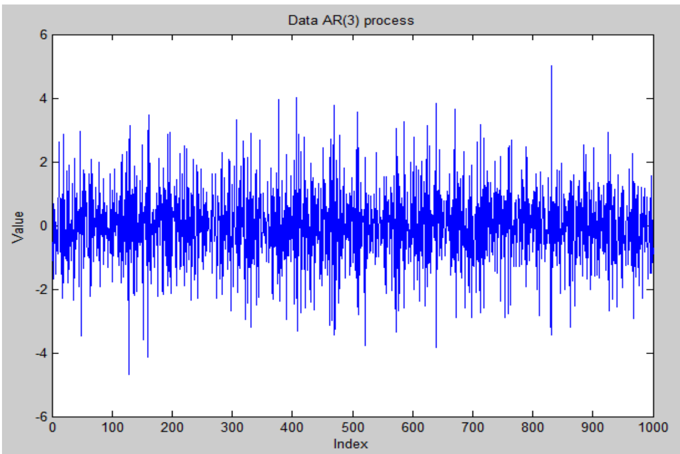

Figure 1: Simulated data for an AR(3) process.

We will take this as an input to the Yule-Walker system and check if it can correctly estimate the model parameters.

$$X_t = -X_{t-1} - 0.8X_{t-2} - 0.4X_{t-3} + \varepsilon_t.$$

Generation of data from the AR(3) process is given above.

To execute the simulation and determine which model order fits best, one should follow the steps below.

Step 1. Plot the dynamical periodogram (Power Spectral Density - PSD) of the simulated data for reference (see Figure 2).

Step 2. Estimate the PSD for three different model orders (e.g., AR(2), AR(3), AR(4)).

Step 3. Compare the estimated PSD for each model order to the reference dynamical periodogram to see which model order best fits the data.

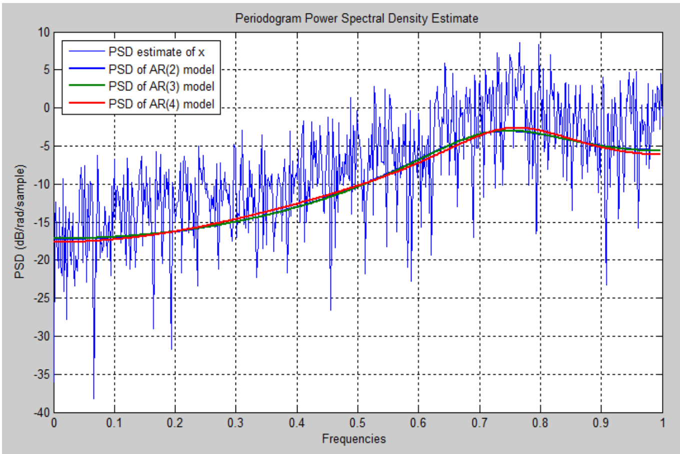

Figure 2: Dynamical periodogram (estimator of the Power Spectral Density "PSD").

The order (P)	The estimated parameters (ϕ_k)	The prediction error
2	[1 0.83 0.43]	1.010
3	[1 0.81 0.39 -0.037]	1.009
4	[1 0.81 0.39 -0.045 -0.009]	1.009

Table 1: The estimated model parameters and the prediction errors.

The estimated model parameters and the noise variances calculated by the Yule-Walker system are provided in Table 1.

It can be established that the estimated parameters are nearly identical to what is expected. See how the error decreases as the model order 'p' increases. The optimal model order in this case is $P = 3$ since the error did not change significantly when increasing the order, and also, the model parameter $\phi 4$ of the AR(4) process is not significantly different from zero.

5 Conclusion

In this paper, the estimation of spectral density provides useful insights into the analysis of time series data. The study emphasizes the importance of understanding the frequency domain of time series by applying techniques of spectral analysis, providing a unique point of view that complements traditional descriptive methods. Incorporating weighted windows, and addressing bias and variance, highlights the possibility for improvements in estimation accuracy. Additionally, one of the most notable achievements of spectral analysis for time series data is its ability to reveal hidden frequencies, allowing for the detection of underlying patterns and behaviors that may not be readily identifiable in the time domain.

It is important to add spectral analysis to the tools for studying time series data, the spectral analysis provides a "frequency" perspective on time series data, the dynamical periodogram is a more sophisticated estimator of the spectrum compared to the autocorrelation function, and the dynamical periodogram is a simple method for estimating the spectral density. This estimator has drawbacks (bias, variance) that can be problematic depending on its use. It is possible to improve this estimator by multiplying it by a weighting window to reduce bias and variance. One of the successes of spectral analysis for time series data is the detection of hidden frequencies.

Our research of spectral density estimation expands our toolbox for analyzing time series data and opens the door to fresh perspectives and potential applications in a variety of disciplines, including signal processing, finance, and economics. The study highlights the usefulness of spectrum analysis in revealing hidden dynamics in time series, making it a potent tool for researchers and analysts in the field.

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