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A Novel Chaotic Jerk System with Multistability and Its Circuit Implementation

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Abstract: In this research paper, a new chaotic jerk system is presented. The specialty of the new system is that it can produce coexisting multiple attractors for different initial conditions. This special behavior of the new system can be used to increase the security of the communication system. The bifurcation diagram, Lyapunov exponents, attractor diagrams, and basin of attraction are the important tools used to validate the multistability of the proposed system. The simulation results indicate that there are multiple coexisting attractors in the new system. Finally, an electronic circuit is designed for the proposed system to realize the coexisting attractors in practice.

Keywords: *jerk system; multistability; bifurcation analysis; circuit simulation.*

Mathematics Subject Classification (2020): 93A30, 70K99.

1 Introduction

In recent years, chaotic jerk systems have been introduced with hidden attractors [1], multi-scroll attractors [2], multi-stability [3], mega-stability [4], hypogenetic system [5], memristor [6], and coexisting attractors [7]. The invention of the chaotic jerk system with coexisting multiple attractors is very important in recent days because of its many engineering applications such as secure communication systems [8], image processing [9], random number generation [10], etc. The coexisting attractors, which means multiple attractors, can be observed in any nonlinear dynamical system for different initial conditions and system parameters. This special behavior of the system increases its complexity

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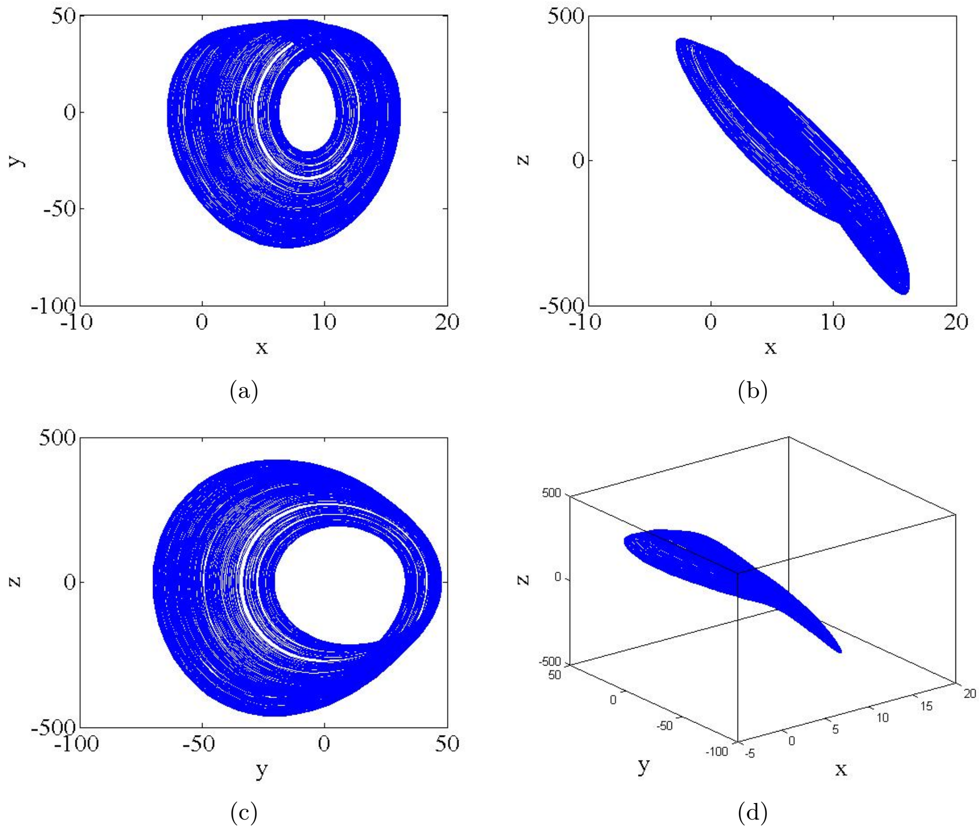


Figure 1: (a-d) The attractors of the new system in 2D and 3D planes.

and can be used to improve the security of the communication system. Recently, many researchers [11–14] introduced chaotic jerk systems with coexisting multiple attractors and analyzed their dynamical properties employing a bifurcation diagram and Lyapunov exponents. It was understood through dynamic analysis that the chaotic system can have coexisting chaotic attractors, stable node attractors, and limit cycle attractors for a certain range of system parameters. When the system produces an infinite number of coexisting attractors for the variation of initial conditions, the phenomenon is called extreme multistability.

The proposed chaotic jerk system has only one cubic nonlinearity and exhibits coexisting attractors in both periodic and chaotic states when initial conditions are changed. The chaotic and coexisting attractor behavior in the new system is verified through the bifurcation diagram, Lyapunov exponents spectrum, and attractor diagram. The dissipativity, equilibrium points, and stability analysis were also conducted to verify the chaotic nature of the proposed system. Furthermore, an electronic circuit is designed for the proposed system using basic electronic components such as resistors, capacitors, and OPAMP and simulated in Multisim software. As a result of the simulation, the chaotic attractors are obtained for the practical implementation of the proposed system.

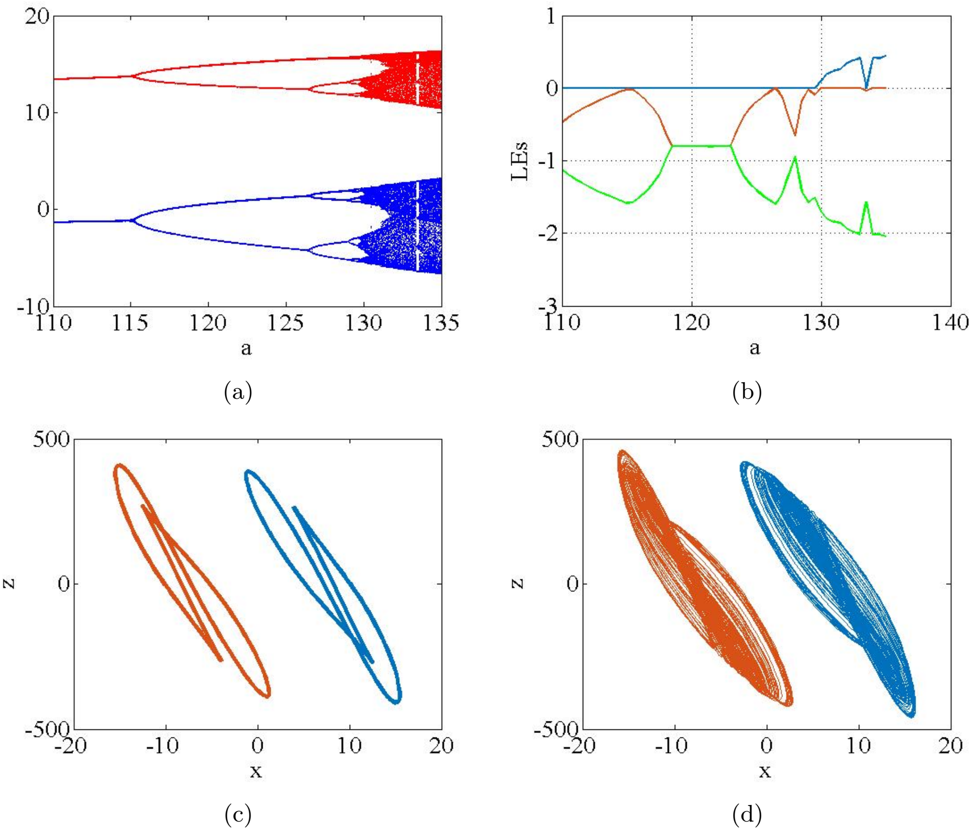


Figure 2: (a) Bifurcation diagram with $X_1(0)$ (Blue) and $X_2(0)$ (Red); (b) Lyapunov exponent plots of the system (1) under the parameter a; (c-d) Coexisting attractors in the xz-plane at $a = 125$ and $a = 133$, respectively.

2 Theoretical Model of Novel Chaotic Jerk System

In this section, a new jerk system is introduced and its dynamical behaviors, including dissipativity, equilibrium points, stability, Lyapunov exponents, and Lyapunov dimension, are analyzed in detail. The new system is of the form

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= z, \\ \dot{z} &= ax - x^3 - by - cz. \end{aligned} \tag{1}$$

Here, x,y,z are the state variables and $a = 133$, $b = 51$, $c = 1.6$ are the parameters of the system (1).

2.1 Lyapunov exponents and Lyapunov dimension

The numerical values of Lyapunov exponents for the new system (1) can be calculated as $LE_1 = 0.404527, LE_2 = 0, LE_3 = -2.028421$. The Lyapunov dimension (D_L) can be

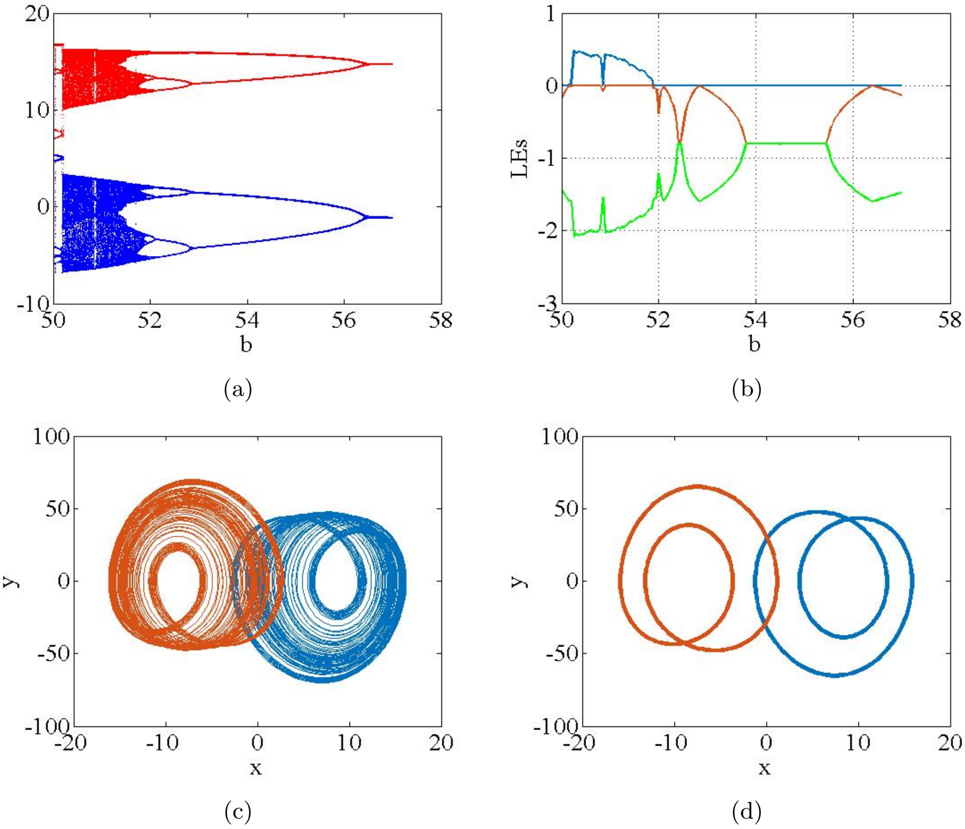


Figure 3: (a) Bifurcation diagram with $X_1(0)$ (Blue) and $X_2(0)$ (Red); (b) Lyapunov exponent plots of the system (1) under the parameter b; (c-d) Coexisting attractors in the xy-plane at $b = 51$ and $b = 54$, respectively.

obtained as $D_L = 2 + \frac{LE_1+LE_2}{|LE_3|} = 2.199$, which indicates the fractional dimension of the proposed system (1).

2.2 Equilibrium points and stability

The equilibrium points can be calculated by letting $\dot{x} = 0, \dot{y} = 0$ and $\dot{z} = 0$ in (1) as given by

$$\begin{aligned} &y = 0, \\ &z = 0, \\ &ax - x^3 - by - cz = 0. \end{aligned} \tag{2}$$

The solution of (2) can be obtained as $x = \pm\sqrt{a}$ and thus the equilibrium points are

$$E_1 = \left[\begin{array}{ccc} 0, & 0, & 0 \end{array}\right], \quad E_{2,3} = \left[\begin{array}{ccc} \pm\sqrt{a}, & 0, & 0 \end{array}\right].$$

Now, the Jacobian matrix (J) of the new system (1) can be written as

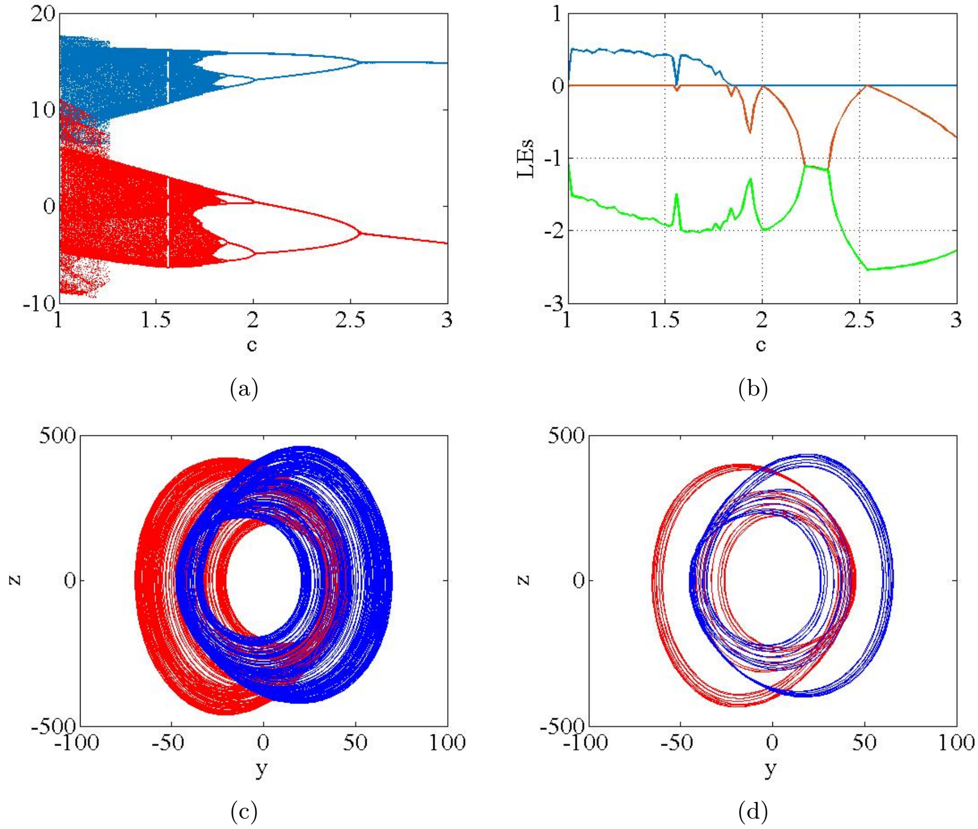


Figure 4: (a) Bifurcation diagram with $X_1(0)$ (Blue) and $X_2(0)$ (Red); (b) Lyapunov exponent plots of the system (1) under the parameter c; (c-d) Coexisting attractors in the yz-plane at $c = 16$ and $c = 18$, respectively.

$$J = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a - 3x^2 & -b & -c \end{pmatrix}. \tag{3}$$

The Jacobian matrix at the equilibrium point E_1 can be written as

$$J(E_1) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a & -b & -c \end{pmatrix}. \tag{4}$$

The corresponding eigenvalues are

$$\lambda_1 = 2.233, \lambda_2 = -1.917 - j7.476, \lambda_3 = -1.917 + j7.476.$$

Since λ_1 is positive, the new system (1) is unstable at the equilibrium point E_1. The Jacobian matrix at the equilibrium points E_2 and E_3 can be written as

$$J(E_2) = J(E_3) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2a & -b & -c \end{pmatrix} \tag{5}$$

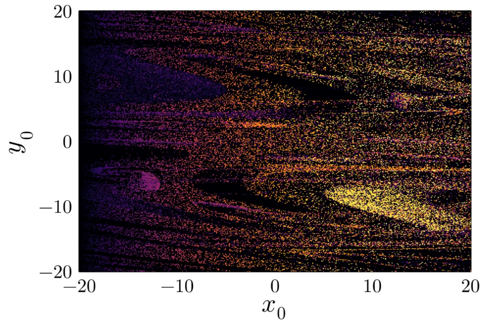
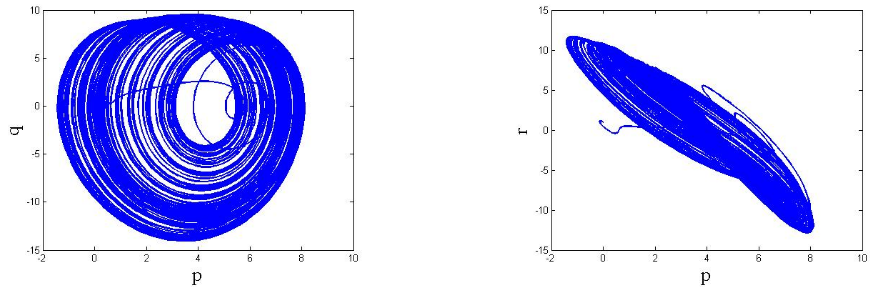


Figure 5: Basin of attraction in the $x_0 - y_0$ plane.



(a) (b)

Figure 6: (a-b) Chaotic attractors of the scaled system (6).

The corresponding eigen values are

$$\lambda_1 = 1.333 + j7.784,\ \lambda_2 = 1.333 - j7.784,\ \lambda_3 = -4.265.$$

Since the real parts of eigenvalues λ_1 and λ_2 are positive, the new system (1) is unstable at the equilibrium points E_2 and E_3. Hence, we can conclude that the proposed system (1) is unstable at all the equilibrium points. The attractors of the system (1) are given in Figure 1 with the initial conditions $X_1(0) = (0, 1, -1)$.

3 Dynamic Analysis

In this section, the multistability in the proposed system (1) is shown with the help of bifurcation diagrams, Lyapunov exponents spectrum, and attractor diagrams. When the initial conditions are changed, the system (1) presents multistability with the same parameter values. The bifurcation diagram and Lyapunov exponents spectrum diagram can be obtained by varying the particular parameter by keeping other parameters constant. All the simulation results of the attractor diagram for the multistability are obtained with the initial conditions $X_1(0) = (0, 1, -1)$ (blue color) and $X_2(0) = (0, -1, 1)$ (red color). The new system (1) presents periodic and chaotic states when we vary the parameter values and coexisting attractors are evolved in both chaotic and periodic states in the new system (1). The simulation results indicate that there is a wealth of chaotic dynamics and the existence of coexisting attractors in the proposed system (1).

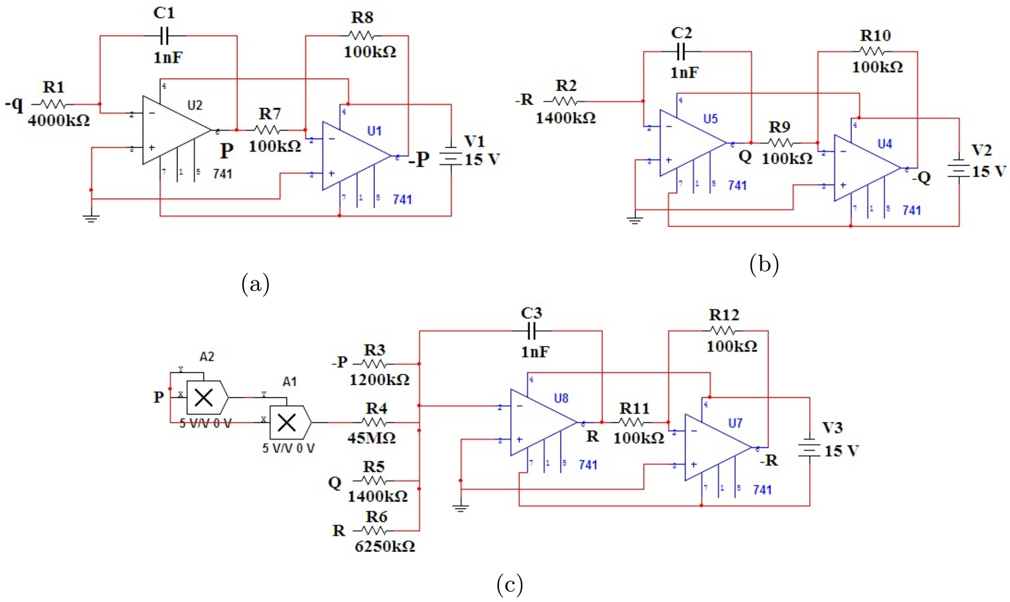


Figure 7: (a-c) Electronic system design for the scaled system (6).

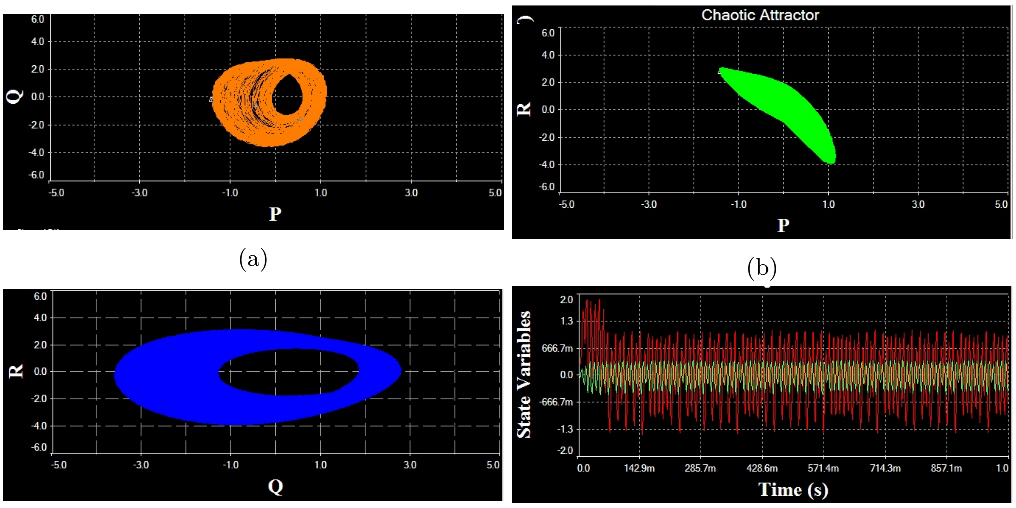
The dynamical behavior of the system (1)) is analysed under the parameters $a\epsilon[110-135]$. Figure 2a shows the bifurcation diagram for the parameter a with the initial conditions $(0,1,-1)$ (Blue) and $(0,-1,1)$ (Red). It is found that the system (1) has a periodic state in $a\epsilon[110-130]$ and a chaotic state beyond $a=130$. The non-overlapping regions in Figure 2a indicate that there is the presence of coexisting multiple attractors in the system (1). Figure 2b represents the variation of Lyapunov exponent values under the parameter a and has at least one positive value beyond $a=130$. Figures 2c and 2d show the coexisting attractors of the system (1) under chaotic and periodic regions.

Next, the dynamic property of the system (1) is analyzed under the parameter $b\epsilon[49-57]$. The corresponding bifurcation diagram and Lyapunov exponents spectrum are shown in Figures 3a and 3b, which demonstrate that there is an inverse periodic doubling nature in the system, i.e., the system (1) is in the chaotic state within $b\epsilon[50.4-51.5]$ and then it enters into the periodic state.

It is also observed from Figure 3a that the states of the system are not modified by the different initial conditions but the bifurcation values are changed. Thus, we can conclude that the system (1) produces coexisting multiple attractors for the different values of initial conditions. The coexisting attractors of the system (1) under $b=51$ and $b=54$ are given in Figures 3c and 3d, respectively.

The dynamic property of the system (1) is analyzed under the parameter $c\in[1-2]$. The corresponding bifurcation diagram and Lyapunov exponents spectrum are shown in Figures 4a and 4b, which demonstrate that the system (1) is in the chaotic state within $c\epsilon[1-1.7]$ and then it produces period - 4, period - 2 and limit cycle attractors.

It is also observed from Figure 4a that the states of the system are not modified by the different initial conditions but the bifurcation values are changed. Thus, we can conclude that the system (1) produces coexisting multiple attractors for the different values of initial conditions. The coexisting attractors of the system (1) under $c=16$ and



(a) (b) (c) (d)

Figure 8: (a-d) Electronic simulation results with $a = 133, b = 51, c = 1.6$.

$c = 18$ are given in Figures 4c and 4d, respectively.

Finally, the presence of coexisting attractors in the proposed system is verified by plotting the riddled basin of attraction relative to the x_0 - y_0 plane as shown in Figure 5, which indicates two types of attractors in the proposed system.

4 Electronic Circuit Implementation

In this section, an electronic circuit is designed for the proposed coexisting attractor jerk system (1) to obtain its attractors for practical applications [15–17]. It is observed from Figure(1) that the range of state signals x,y and z is $(-4, 17)$, $(-50, 70)$ and $(-500, 400)$, respectively. Since the practical circuit realization uses the voltage range $(-15, 15)Volt$, the range of state signals also must be $(-15, 15)Volt$. This can be achieved by scaling the proposed system (1). Now, assume that $p = \frac{x}{2}$, $q = \frac{y}{5}$ and $r = \frac{z}{36}$, where p, q and r are the scaled state variables of the system (1). Thus, the scaled system can be written as in (6), and its attractors are given in Figure 6 with $a = 133, b = 51, c = 1.6$. Note that the phase diagrams of the scaled system are identical to those in Figure 1 but the range of state variables is reduced within $(-15, +15)$.

$$
\begin{aligned}
\dot{p} &= 2.5q, \\
\dot{q} &= 7.2r, \\
\dot{r} &= \tfrac{2a}{36}p - \tfrac{8}{36}p^3 - \tfrac{5b}{36}q - cr.
\end{aligned} \tag{6}
$$

The equations for the proposed system in terms of electrical parameters are given by

$$
\begin{aligned}
\dot{p} &= \tfrac{-1}{R_1C_1}(-q), \\
\dot{q} &= \tfrac{-1}{R_2C_2}(-r), \\
\dot{r} &= \tfrac{-1}{R_3C_3}(-p) + \tfrac{-1}{R_4C_3}(p^3) + \tfrac{-1}{R_5C_3}(q) + \tfrac{-1}{R_6C_3}(r).
\end{aligned} \tag{7}
$$

By comparing (6) and (7), the electrical equations for the parameter values can be obtained as follows:

$$\begin{aligned} a &= \frac{18}{R_3C_3}, \\ b &= \frac{36}{5R_5C_3}, \\ c &= \frac{1}{R_6C_3}. \end{aligned} \tag{8}$$

The electronic circuit design for (7) is displayed in Figure 7, it uses basic components such as resistors, capacitors, and opamps. The resistance values can be obtained with the parameters $a = 133, b = 51, c = 1.6$ and the capacitance $C = C_1 = C_2 = C_3 = 1nF$ as

$$R_1 = 4000K\Omega, R_2 = 1400K\Omega, R_3 = 1200K\Omega, R_4 = 45M\Omega, R_5 = 1400K\Omega, R_6 = 6250K\Omega.$$

The Multisim simulation results for chaotic attractors are given in Figure 8, which is almost identical to Figure 6.

5 Conclusions

In this paper, a new 3-dimensional coexisting multiple attractors chaotic jerk system is presented and its dynamical properties are analyzed in detail. The parameters of the proposed system are analyzed under chaotic and periodic states using a bifurcation diagram with different initial conditions. The dynamic analysis shows that the proposed system can have multiple coexisting attractors under the variation of initial conditions. The riddle basin of attraction is plotted, which confirms the presence of different attractors in the proposed system. The proposed electronic system design for the new system is verified using Multisim software and can be used to solve many issues in practice. The attractors obtained by the Multisim simulation are identical to those obtained by numerical simulation. The complex dynamics and circuit implementation of the proposed nonlinear dynamical system can be used in many practical applications such as communication system. It can be concluded that the proposed system has chaotic and unpredictable behavior.

References

[1] B. Li, B. Sang, M. Liu, X. Hu, X. Zhang and N. Wang. Some jerk systems with hidden chaotic dynamics. *International Journal of Bifurcation and Chaos* **33** (06) (2023) 2350069.

[2] F. Li and J. Zeng. Multi-scroll attractor and multi-stable dynamics of a three-dimensional Jerk system. *Energies* **16** (5) (2023) 2494.

[3] S. Vaidyanathan, I.M. Moroz, A. A. A. El-Latif, B. Abd-El-Atty and A. Sambas. A new multistable jerk system with Hopf bifurcations, its electronic circuit simulation and an application to image encryption. *International Journal of Computer Applications in Technology* **67** (1) (2021) 29–46.

[4] M. Karami, R. Ramamoorthy, A. M. A. Ali and V. T. Pham. Jagged-shape chaotic attractors of a megastable oscillator with spatially square-wave damping. *The European Physical Journal Special Topics* **231** (11) (2022) 2445–2454.

[5] C. Li, J. C. Sprott and H. Xing. Hypogenetic chaotic jerk flows. *Physics Letters A* **380** (11-12) (2016) 1172–1177.

[6] X. Wu, S. He, W. Tan and H. Wang. From memristor-modeled jerk system to the nonlinear systems with memristor. *Symmetry* **14** (4) (2016) 659.

[7] Q. Wang, Z. Tian, X. Wu and W. Tan. Coexistence of multiple attractors in a novel simple jerk chaotic circuit with CFOAs implementation. *Frontiers in Physics* **10** (2022) 835188.

[8] A. Sambas, W. S. Sanjaya, M. Mamat and R. Putra Prastio. Mathematical modelling of chaotic jerk circuit and its application in secure communication system. *Advances in chaos theory and intelligent control* (2016) 133–153.

[9] R. Premkumar, M. Mahdal and M. Elangovan. An efficient chaos-based image encryption technique using bitplane decay and genetic operators. *Sensors* **22** (10) (2022) 8044.

[10] R. Ramar, S. Vaidyanathan, S. Akgul and B. Emin. A new chaotic jerk system with cubic and hyperbolic sine nonlinearities and its application to random number generation and biomedical image encryption. *Scientia Iranica* (2024), Accepted Manuscripts.

[11] H. Chen, S. He, A. D. Pano Azucena, A. Yousefpour, H. Jahanshahi, M. A. Lopez and R. Alcaraz. A multistable chaotic jerk system with coexisting and hidden attractors: Dynamical and complexity analysis, FPGA-based realization, and chaos stabilization using a robust controller. *Symmetry* **12** (4) (2020) 569.

[12] D. Veeman, M. Mehrabbeik, H. Natig, K. Rajagopal, S. Jafari and I. Hussain. A new chaotic system with coexisting attractors. *International Journal of Bifurcation and Chaos* **32** (03) (2022) 2230007.

[13] P. P. Singh and B. K. Roy. Pliers shaped coexisting bifurcation behaviors in a simple jerk chaotic system in comparison with 21 reported systems. *IFAC-PapersOnLine* **55** (1) (2022) 920–926.

[14] Q. Lai, P. D. Kamdem Kuate, H. Pei and H. Fotsin. Infinitely many coexisting attractors in no-equilibrium chaotic system. *Complexity* **2020** (2020) Article ID 8175639.

[15] T. Bonny, S. Vaidyanathan, A. Sambas, K. Benkovide, W. Al Nassan and O. Naqaweh. Multistability and bifurcation analysis of a novel 3d jerk system: Electronic circuit design, fpga implementation, and image cryptography scheme. *IEEE Access* **11** (2023) 78584–78600.

[16] A. Sambas, S. Vaidyanathan, S. F. Al-Azzawi, M. K. M. Nawawi, M. A. Mohamed, Z. A. Zakaria, S. S. Abas and M. Mamat. Modelling and MultiSim Simulation of a New Hyperchaos System with No Equilibrium Point. *Nonlinear Dynamics and Systems Theory* **23** (4) (2023) 422–433.

[17] S. Rouar and O. Zehrour. A new fractional-order three-dimensional chaotic flows with identical eigenvalues. *Nonlinear Dynamics and Systems Theory* **22** (4) (2022) 447–456.