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Variational Analysis and Error Estimate of Contact Problem for Thermo-Viscoelastic Bodies with Long-Term Memory

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Abstract: This paper investigates a contact problem involving the quasistatic interaction between two bodies characterized by thermo-viscoelasticity with long-term memory. The mechanical, thermal contact is captured through the sub-differential condition, which represents the frictional interaction. We establish a variational formulation for the model and we prove the existence of a unique weak solution. Subsequently, a numerical investigation is conducted, employing both finite element and finite difference methods. This computational approach allows for the derivation of a discrete approximation of the error associated with the analyzed model.

Keywords: *fixed point; frictional contact; finite element method; thermo-piezoelectric; weak solution.*

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1 Introduction

Mathematics has always been the beating heart of the science, and the interaction between them has been crucial for centuries. Mathematical theories and fundamental concepts have enabled the description of aspects of the natural world, including motion in mechanics, electricity, gravity, and general relativity.

The connection between mathematics and mechanics is profound as mechanics elucidates the motion of objects influenced by specific forces within the realm of physics. The mathematical exploration of mechanics commences with the definition of physical quantities and geometric representations, often leading to the formulation of graphs and diagrams. Contact mechanics, a subfield of mechanics, delves into the deformation of materials in contact with one another. Mathematical modeling and analysis play pivotal roles in comprehension [11], with nonlinear partial differential equations and variational inequalities being among the primary mathematical tools employed, along with hemivariational inequalities.

Thermal phenomena are closely linked to the mechanics of contact. For example, certain crystals such as quartz, tourmaline, and Rochelle salt exhibit mechanical stresses due to thermal expansion when exposed to heat [5]. Research has extensively explored the laws that govern these thermo-mechanical interactions. The models of thermo-elastic bodies are elaborated in [1]. Additionally, studies examining changes in piezoelectric materials in relation to thermal effects are presented in [4] and [12]. Frictional contact between bodies has taken a considerable place in research, see [7], dealing with a contact problem between materials and physical phenomena (friction, damage and wear), while for a dynamic problem of frictional contact in mechanics in other studies on mathematical numerical solutions of variational inequalities, one can refer to [9].

Furthermore, there are results and research focusing on abstract hemivariational inequalities and numerical simulation outcomes providing numerical evidence regarding the theoretically predicted optimal convergence order, as referenced in [6, 8, 10]. Additionally, in [1], the study delves into the hemivariational inequality and the frictional contact problem with damage, furthermore, there are other studies focusing on numerical aspects, which can be found in [2]. The paper is structured as follows. Notations and preliminaries are detailed in Section 2, while the model, a list of assumptions, and a variational formulation of the problem will be discussed in Section 3. Subsequently, in Section 4, we will cite the results concerning existence and uniqueness as presented in Theorem 4.1. The proof of this theorem relies on variational and hemivariational inequalities as well as results related to the existence and uniqueness of Banach fixed points. Finally, in Section 5, we will present the numerical study, employing the finite element method and finite differences to achieve a precise numerical approach to the solution.

2 Notations and Preliminaries

We present the notations and recall some preliminary concepts.

Let us consider $\Omega^l \subset \mathbb{R}^d$ as a bounded domain with an outer Lipschitz boundary denoted by Γ^l, and let ν represent the unit outer normal on $\partial\Omega^l = \Gamma^l$. We define the spaces

$$H^l = L^2(\Omega^l)^d = \left\{\mathbf{v^l} = \left(v_i^l\right) : v_i^l \in L^2(\Omega^l)\right\}, \quad \mathcal{H}^l = \left\{\boldsymbol{\tau^l} = \left(\tau_{ij}^l\right) \tau_{ij}^l = \tau_{ji}^l \in L^2(\Omega^l)\right\},$$

$$H_1^l(\Omega^l)^d = \{\mathbf{v}^l = \left(v_i^l\right) \in H^1 : \varepsilon(\mathbf{v}^l) \in \mathcal{H}^l\}, \quad \mathcal{H}_1^l = \{\boldsymbol{\tau^l} \in \mathcal{H}^l : Div\boldsymbol{\tau^l} \in H^1\},$$

H^l, $\mathcal{H}^l$, $H_1^l(\Omega^l)^d$ and $\mathcal{H}_1^l$ are real Hilbert spaces equiped with the usual inner products and the associated norms, we also introduce the closed subspaces of $H_1^l(\Omega^l)$ defined by

$$\begin{aligned} V^l &= \left\{\mathbf{v^l} \in H_1^l(\Omega^l)^d : \mathbf{v} = 0 \text{ on } \Gamma_1^l\right\}, \\ Q^l &= \left\{\theta^l \in H_1^l(\Omega^l) : \theta^l = 0 \text{ in } \Gamma_1^l\right\}. \end{aligned}$$

Given that $\mu(\Gamma_a^l) > 0$ and $\mu(\Gamma_1^l) > 0$, the Korn and Friedrichs-Poincaré inequalities are satisfied,

$$\exists C_0 > 0 \quad \|\varepsilon(\mathbf{v}^l)\|_{\mathcal{H}^l} \geq C_0\|\mathbf{v}^l\|_{H_1^l(\Omega^l)^d}, \quad \forall \mathbf{v}^l \in V^l, \tag{1}$$

$$\exists C_2 > 0 \quad \|\nabla \mathbf{w}^l\|_{H^l} \geq C_2\|\mathbf{w}^l\|_{H_1^l(\Omega^l)}, \forall\, \mathbf{w}^l \in Q^l. \tag{2}$$

Moreover, by the Sobolev trace theorem the positive constants C_0 and C_2 exist so that

$$\|\mathbf{v}^l\|_{L^2(\Gamma_3^l)^d} \leq C_0\|\mathbf{v}^l\|_{V^l}, \forall \mathbf{v}^l \in V^l, \tag{3}$$

$$\|\mathbf{z}^l\|_{L^2(\Gamma_3^l)} \leq C_2\|\mathbf{z}^l\|_{Q^l}, \forall z^l \in Q^l. \tag{4}$$

We denote v_ν and $\mathbf{v}_\tau^l$ as the normal and tangential components of $\mathbf{v}^l$ on Γ^l, where v_ν is the perpendicular component and $\mathbf{v}_\tau^l$ is the parallel component, as described in Green's formulas in [4].
For a simpler notation, we use the following spaces:

$$\mathbb{V} = V_1 \times V_2, \quad \mathbb{H} = H^1 \times H^2, \quad \mathbb{H}_1 = H_1^1 \times H_1^2, \quad \mathbb{Q} = Q^1 \times Q^2.$$

2.1 Subdifferential boundary conditions

In the mechanical problem $(\mathcal{P})$, we will use contact laws expressed in terms of the subdifferential $\kappa_\nu \in \partial j(u_\nu)$, in which κ_ν represents an interface force, u_ν signifies the normal displacement and $\partial j(u_\nu)$ represents the subdifferential in the sense of Clarke such that $j : \mathbb{R} \longrightarrow \mathbb{R}$ is a locally Lipschitz function. The generalized (Clarke) directional derivative of j at $x \in \mathbb{R}$ in the direction $v \in \mathbb{R}$ is defined by

$$j^0(x; v) = \limsup_{y\to x, \lambda\downarrow 0} \frac{j(y + \lambda v) - j(y)}{\lambda}.$$

The generalized subdifferential of j at x is a subset of $\mathbb{R}$ expressed as

$$\partial j(x) = \{\zeta \in \mathbb{R} \quad | \quad j^0(x; v) \geq \zeta v \quad \forall v \in \mathbb{R}\}.$$

Some properties of the subdifferential for locally Lipschitz functions can be found in [11].

3 The Model and Assumptions on the Data

Let $\Omega^l, l = 1, 2$, be a bounded domain in $\mathbb{R}^d$ $(d = 2, 3)$ with the outer Lipschitz surface Γ^l, we define two thermo-viscoelastic bodies occupying Ω^l, their boundary is divided into three open disjoint parts Γ_1^l, Γ_2^l and Γ_3^l on one hand, and a partition of $\Gamma_1^l \cup \Gamma_2^l$ into two open parts $\Gamma_a^l$and Γ_b^l on the other hand. We assume that $\mu(\Gamma_1^l) > 0$. Let $T > 0$ and $[0, T]$ be the time interval of interest. The two bodies are subjected to the effect of body forces with specific density $\mathbf{f}_0$, a heat source of constant strength q_{th}^l.

The two bodies are clamped on $\Gamma_1^l \times (0,T)$, so the displacement field vanishes there. A surface traction of density $\mathbf{f}_2^l$ acts on $\Gamma_2^l \times (0,T)$. Also, we suppose that the temperature vanishes on $\left(\Gamma_1^l \cup \Gamma_2^l\right) \times (0,T)$. Moreover, we suppose that the body forces and tractions vary slowly in time, and therefore, the accelerations in the system may be neglected. Neglecting the inertial terms in the equation of motion leads to a quasistatic approach to the process. In the reference configuration, the two bodies can enter in contact along the common part $\Gamma_3^1 = \Gamma_3^2 = \Gamma_3$. The contact model is characterized by the sub-differential of locally Lipschitz functions and the non linear boundary condition of thermal conductivity modeling electric potential exchange between the bodies.

Problem 3.1 For $l = 1, 2$, find the displacement field $u^l : \Omega^l \times [0,T] \to \mathbb{R}^d$ and the temperature $\theta^l : \Omega^l \times [0,T] \to \mathbb{R}$ such that

$$\boldsymbol{\sigma}^{\boldsymbol{l}}(t) = \mathcal{A}^l(\varepsilon(\dot{u}^l(t))) + \mathcal{B}^l\varepsilon(\mathbf{u}^l(t)) + \int_0^T \mathcal{G}^l(t-s)u^l(s)\mathrm{d}s - \mathcal{C}^l\theta^l(t) \quad \text{in } \Omega^l \times (0,T) \tag{5}$$

$$\dot{\theta}^l(t) - div\mathcal{K}^l\left(\nabla\theta^l(t)\right) = \mathcal{M}^l(\varepsilon(\dot{u}^l(t))) + h_0^l \quad \text{in } \Omega^l \times (0,T) \tag{6}$$

$$Div\boldsymbol{\sigma}^l(t) + f_0^l(t) = 0 \quad \text{in } \Omega^l \times (0,T), \tag{7}$$

$$u^l(t) = 0 \quad \text{on } \Gamma_1^l \times (0,T), \tag{8}$$

$$\sigma^l\nu^l = f_2^l \quad \text{on } \Gamma_2^l \times (0,T), \tag{9}$$

$$\begin{cases} \sigma_\upsilon^1(t) = \sigma_\upsilon^2(t) = \sigma_\upsilon(t) \quad -\sigma_\upsilon(t) \in \partial j_\upsilon(\dot{u}_\upsilon(t)) \quad \text{on } \Gamma_3 \times (0,T), \\ \sigma_\tau^1(t) = \sigma_\tau^2(t) = \sigma_\tau(t) \quad -\sigma_\tau(t) \in \partial j_\tau(\dot{u}_\tau(t)) \quad \text{on } \Gamma_3 \times (0,T), \end{cases} \tag{10}$$

$$u_\upsilon^1(t) + u_\tau^2(t) = 0 \quad \text{on } \Gamma_3 \times (0,T) \tag{11}$$

$$-\mathcal{K}\left(\nabla\theta(t)\right)\upsilon \in \partial j_\theta(\theta(t)) \quad \text{on } \Gamma_3 \times (0,T), \tag{12}$$

$$\theta^l = 0 \quad \text{on } (\Gamma_1 \cup \Gamma_2) \times (0,T), \tag{13}$$

$$q(t)^l \cdot \nu^l = h_n \quad \text{on } \Gamma_2^l \times (0,T), \tag{14}$$

$$u^l(0) = u_0^l, \quad \theta^l(0) = \theta_0^l \quad \text{in } \Omega^l. \tag{15}$$

Now, progress to the mechanical presentation of (5)-(15) and provide explanation of the equations and the boundary conditions.

Equations (5) and (6) represent the thermo-viscoelastic with long-term memory constitutive laws between two bodies, where $\mathcal{A}^l$ is a given nonlinear operator, $\mathcal{G}^l$ is the relaxation operator, $\mathcal{B}^l$ represents the elasticity operator and $\mathcal{C}^l$ is the thermal operator, the thermo-viscoelastic constitutive law includes temperature effects described by the parabolic equation given by (6), where $\mathcal{M}^l$ is the thermal expansion tensor and $\mathcal{K}^l$ is the thermal conductivity tensor, equation (7) is the equilibrium equation for the stress, where Div denotes the divergence operator for tensors, then (8), (9), (13) and (14) are the mechanical and thermal boundary conditions and (11) indicates that there is no space between the two bodies, the equations (10) represent the normal stres and normal velocity satisfying the non-monotone damped response condition and the friction law, in which j_ν, j_τ are locally Lipschitz functions and $\partial j_\nu, \partial j_\tau$ denotes the generilized Clarke gradient of the functions j_ν and j_τ, the relation (12) represents the heat exchange between two body, finally, (15) denotes the initial displacement and the temperature conditions.

3.1 Assumptions on the data

We will now enumerate the assumptions regarding the problem's data.
The *viscosity operator* $\mathcal{A}^l : \Omega^l \times \mathbb{S}^d \to \mathbb{S}^d$ satisfies

$$\begin{cases} (a) \quad \mathcal{A}^l(.,\varepsilon) \text{ is Lebesgue measurable on } \Omega^l \text{ for any } \varepsilon \in \mathbb{S}^d, \\ (b) \quad \text{There exists } L_{\mathcal{A}^l} > 0 \text{ such that} \\ \|\mathcal{A}^l(\mathbf{x},\varepsilon_1) - \mathcal{A}^l(\mathbf{x},\varepsilon_2)\| \le L_{\mathcal{A}^l}\|\varepsilon_1 - \varepsilon_2\| \text{ for all } \varepsilon_1,\varepsilon_2 \in \mathbb{S}^d, \\ (c) \quad \text{There exists } m_{\mathcal{A}^l} > 0 \text{ such that for all } \varepsilon_1,\varepsilon_2 \in \mathbb{S}^d, \\ (\mathcal{A}^l(\mathbf{x},\varepsilon_1) - \mathcal{A}^l(\mathbf{x},\varepsilon_2)) \cdot (\varepsilon_1 - \varepsilon_2) \ge m_{\mathcal{A}^l}\|\varepsilon_1 - \varepsilon_2\|^2 \quad \text{(a.e)}\mathbf{x} \in \Omega^l, \\ (d) \quad \mathcal{A}^l(x,0) = 0 \text{ for all } x \in \Omega^l. \end{cases} \tag{16}$$

The elasticity operator $\mathcal{B}^l : \Omega^l \times \mathbb{S}^d \to \mathbb{S}^d$ satisfies

$$\begin{cases} (a) \quad \mathcal{B}^l(x,\varepsilon) \text{ is Lebesgue measurable on } \omega \text{ for all } \varepsilon \in \mathbb{S}^d, \\ (b) \quad \text{There exists } L_{\mathcal{B}^l} > 0 \text{ such that for all } \varepsilon_1,\varepsilon_2 \in \mathbb{S}^d, \\ \|\mathcal{B}^l(\mathbf{x},\varepsilon_1) - \mathcal{B}^l(\mathbf{x},\varepsilon_2)\| \le L_{\mathcal{B}^l}\|\varepsilon_1 - \varepsilon_2\| \quad \text{(a.e)}\mathbf{x} \in \Omega^l, \\ (c) \quad \mathcal{B}^l(x,0) = 0 \text{ for all } x \in \Omega^l. \end{cases} \tag{17}$$

The *relaxation function* $\mathcal{G}^l : \Omega^l \to \mathbb{R}^d$ satisfies

$$\begin{cases} (a) \quad \mathcal{G}^l(\mathbf{x}) \text{ is Lebesgue measurable on } \Omega^l \text{ for any } \mathbf{x} \in \mathbb{R}^d, \\ (b) \quad \text{There exists } L_{\mathcal{G}^l} > 0 \text{ such that} \\ \qquad \|\mathcal{G}^l(\mathbf{x}_1) - \mathcal{G}^l(\mathbf{x}_2)\| \le L_{\mathcal{G}^l}\|\mathbf{x}_1 - \mathbf{x}_2\| \text{ for all } \mathbf{x}_1,\mathbf{x}_2 \in \mathbb{R}^d. \end{cases} \tag{18}$$

The *function* $j_\nu : \Gamma_3^l \times \mathbb{R} \to \mathbb{R}$ satisfies

$$\begin{cases} (a) \quad j_{\nu(.,r)} \text{ is Lebesgue measurable on } \Gamma_3^l \text{ for all } x \in \mathbb{R}, \\ (b) \quad j_{\nu(x,.)} \text{ is locally Lipschitz on } \mathbb{R} \text{ for all } x \in \Gamma_3^l, \\ (c) \quad \text{there exist } c_{0\nu}, c_{1\nu} \geqslant 0 \text{ such that for all } r \in \mathbb{R} \text{ and } x \in \Gamma_3^l \text{ we have} \\ \qquad |\partial j_\nu(x,r)| \le c_{0\nu} + c_{1\nu}|r|, \\ (d) \quad \text{there exists } \alpha_{j\nu} \ge 0 \text{ such that for all } r_1,r_2 \in \mathbb{R} \text{ and } x \in \Gamma_3^l, \text{ we have} \\ \qquad j_\nu^0(x,r_1;r_2 - r_1) + j_\nu^0(x,r_2;r_1 - r_2) \le \alpha_{j\nu}|r_1 - r_2|^2. \end{cases} \tag{19}$$

The *function* $j_\tau : \Gamma_3^l \times \mathbb{R}^d \to \mathbb{R}$ satisfies

$$\begin{cases} (a) \quad j_\tau(.,\xi) \text{ is measurable on } \Gamma_3^l \text{ for all } x \in \mathbb{R}, \\ (b) \quad j_\tau(x,.) \text{ is locally Lipschitz on } \mathbb{R}^d \text{ for all } x \in \Gamma_3^l, \\ (c) \quad \text{there exist } c_{0\tau}, c_{1\tau} \ge 0 \text{ such that for all } r \in \mathbb{R}\text{and } x \in \Gamma_3^l, \text{ we have} \\ \qquad \|\, \partial j_\tau(x,r) \,\| \le c_{0\tau} + c_{1\tau} \,\|\, \xi \,\|_{\mathbb{R}}, \\ (d) \quad \text{there exists } \alpha_{j\tau} \ge 0 \text{ such that for all } \xi_1,\xi_2 \in \mathbb{R} \text{ and } x \in \Gamma_3^l, \text{ we have} \\ \qquad j_\tau^0(x,\xi_1;\xi_2 - \xi_1) + j_\tau^0(x,\xi_2;\xi_1 - \xi_2) \le \alpha_{j\tau} \,\|\, \xi_1 - \xi_2 \,\|^2 . \end{cases} \tag{20}$$

The *function* $j_\theta : \Gamma_3 \times \mathbb{R} \to \mathbb{R}$ satisfies

$$\begin{cases} (a) \quad j_\theta(.,r) \text{ is Lebesgue measurable on } \Gamma_3^l \text{ for all } x \in \mathbb{R}, \\ (b) \quad j_\theta(x,.) \text{ is locally Lipschitz on } \mathbb{R}^d \text{ for all } x \in \Gamma_3^l, \\ (c) \quad \text{there exist } c_{0\theta}, c_{1\theta} \ge 0 \text{ such that for all } r \in \mathbb{R} \text{ and } x \in \Gamma_3^l, \text{ we have} \\ \qquad |\partial j_\theta(x,r)| \le c_{0\theta} + c_{1\theta}|r|, \\ (d) \quad \text{there exists } \alpha_{j\theta} \ge 0 \text{ such that for all } r_1,r_2 \in \mathbb{R} \text{ and } x \in \Gamma_3, \text{ we have} \\ \qquad j_\theta^0(x,r_1;r_2 - r_1) + j_\theta^0(x,r_2;r_1 - r_2) \le \alpha_{j\theta}|r_1 - r_2|^2. \end{cases} \tag{21}$$

On the other hand, we need conditions for the thermal operator $\mathcal{C}^l$, the function $\mathcal{M}^l$ and the thermal conductivity operator$\mathcal{K}^l$, see [6].

Now, we define the forces, tractions, volume and surface charges, as well as the initial functions as follows:

$$f_0^l \in L^2(\Omega^l)^d \quad f_2^l \in L^2(\Gamma_2^l)^d \quad h_0^l \in L^2(\Omega^l)^d.$$

$$h_n^l \in L^2(\Gamma_b^l)^d \quad k \geq 0 \quad u_0 \in V \quad \theta_0 \in Q.$$

By utilizing Riesz's representation theorem, we examine the elements $f^l \in V^l$ and $h \in Q^l$ defined by

$$\langle F, v\rangle_V = \sum_{l=1}^{2} \int_{\Omega^l} f_0^l(t) v^l \mathrm{d}x + \sum_{l=1}^{2} \int_{\Gamma_2^l} f_2^l(t) v^l \mathrm{d}x \quad \text{for all } v \in V$$

$$\langle h, \xi\rangle_Q = \sum_{l=1}^{2} \int_{\Omega^l} h_0^l(t) \xi^l \mathrm{d}x + \sum_{l=1}^{2} \int_{\Gamma_b^l} h_n^l(t) \xi^l \mathrm{d}x \quad \text{for all } \xi \in Q.$$

With the notations mentioned earlier and Green's formulas, we can derive the variational formulation of the mechanical problem $(\mathcal{P})$ for all functions $v^l \in V^l$, $w^l \in Q^l$ and a.e $t \in (0, T)$ given as follows.

3.2 Problem $\mathcal{P}_V$

Find the displacement field $\mathbf{u} = (\mathbf{u}^1, \mathbf{u}^2) : [0, T] \to \mathbb{V}$ and the temperature $\theta = (\theta^1, \theta^2) : [0, T] \to \mathbb{Q}$ such that

$$\begin{aligned} \sum_{l=1}^{2} \left(\sigma^l(t), \varepsilon(v^l - \dot{u}^l(t))\right)_{\mathcal{H}^l} &+ \int_{\Gamma_3} \left(j_\nu^0(\dot{u}_\nu(t); v_\nu - \dot{u}_\nu(t)) + j_\tau^0(\dot{u}_\tau(t); v_\tau - \dot{u}_\tau(t))\right) \mathrm{d}a, \\ &\geq \langle F(t), v - \dot{u}(t)\rangle_{\mathbb{V}}. \end{aligned} \tag{22}$$

$$\begin{aligned} \sum_{l=1}^{2} \Big(\dot{\theta}^l(t), \lambda^l - \theta^l(t)\Big)_{\mathcal{H}^l} &+ \left(\mathcal{K}^l \nabla\theta^l(t), \nabla(\lambda^l - \theta^l(t))\right)_{\mathcal{H}^l} - \left(\mathcal{M}^l \varepsilon(u^l(t)), \lambda^l - \theta^l(t)\right)_{\mathcal{H}^l}, \\ &+ \int_{\Gamma_3} j_\theta^0(\theta(t); \lambda^l - \theta^l(t))\, \mathrm{d}a \geq \langle h(t), \lambda - \theta(t)\rangle_{\mathbb{Q}}, \end{aligned} \tag{23}$$

$$\mathbf{u}(0) = \mathbf{u}_0, \qquad \theta(0) = \theta_0.$$

4 Existence and Uniqueness of a Solution

Let us consider that the following smallness conditions are satisfied:

$$\begin{array}{rcl} \alpha_{\mathcal{A}}^l & \geq & c_0^2(\alpha_{j_\nu} + \alpha_{j_\tau})\sqrt{\mu(\Gamma_3)}, \\ \alpha_{\mathcal{K}}^l & \geq & c_0^2 \alpha_{j_\theta} \sqrt{\mu(\Gamma_3)}, \\ \alpha_{\mathcal{K}}^l - c_0^2 \alpha_{j_\theta} \sqrt{\mu(\Gamma_3)} & \geq & L_{\mathcal{M}}^l T/2. \end{array} \tag{24}$$

Now, we present our result on existence and uniqueness.

Theorem 4.1 *Assume hypotheses (3.16)-(3.29) and (24) are satisfied, then Problem ($\mathcal{P}_V$) has a unique solution* $(\mathbf{u}, \theta)$ *such that*

$$\mathbf{u} \in L^2(0, T, \mathbb{V}), \quad \theta \in L^2(0, T, \mathbb{Q}).$$

In the proof of Theorem (4.1), we follow several steps, based on the results of hemi-variational inequalities and fixed point arguments.

To prove the theorem, we consider the following the auxiliary problems for given $\eta \in L^2(0, T, \mathcal{H}), z \in L^2(0, T, Q)$.

4.1 Problem $\mathcal{PV}_\eta$

Find a displacement field $u_\eta = (u_\eta^1, u_\eta^2) : [0, T] \to \mathbb{V}$ such that for all $t \in [0, T]$, we have

$$\begin{aligned}
&\sum_{l=1}^{2} (\mathcal{A}^l \varepsilon(\dot{u}_\eta^l(t)), \varepsilon(v^l - \dot{u}_\eta^l(t)))_{\mathcal{H}^l} + \int_{\Gamma_3} (j_\nu^0(\dot{u}_{\eta\nu}(t); v_\nu - \dot{u}_{\eta\nu}(t)) \\
&+ (j_\tau^0(\dot{u}_{\eta\tau}(t); v_\tau - \dot{u}_{\eta\tau}(t))\mathrm{d}a + (\eta(t), \varepsilon(v - \dot{u}(t)))_{\mathbb{V}} \geq \langle F(t), v - \dot{u}_\eta(t) \rangle_{\mathbb{V}} \qquad (25) \\
&\qquad u_\eta(0) = u_0.
\end{aligned}$$

4.2 Problem $\mathcal{PV}_\theta$

Find the temperature $\theta_{\eta z} = (\theta_{\eta z}^1, \theta_{\eta z}^2) : [0, T] \to \mathbb{Q}$ such that for all $t \in [0, T]$ and all $\lambda \in \mathbb{Q}$, we have

$$\begin{aligned}
&\sum_{l=1}^{2} (\dot{\theta}_{\eta z}^l(t), \lambda^l - \theta_{\eta z}^l(t))_{\mathcal{H}} + (\mathcal{K}^l \nabla \theta_{\eta z}^l(t), \nabla(\lambda^l - \theta_{\eta z}^l(t)))_{\mathcal{H}} - (\mathcal{M}^l \varepsilon(u_\eta^l(t)), \lambda - \theta_{\eta z}^l(t)) \\
&+ \int_{\Gamma_3} j_\theta^0(\theta_{\eta z}(t); \lambda^l - \theta_{\eta z}(t))\mathrm{d}a \geq \langle h(t), \lambda - \theta_{\eta z}(t) \rangle_{\mathbb{Q}} \qquad (26) \\
&\qquad \theta_{\eta z}(0) = \theta_0.
\end{aligned}$$

Lemma 4.1 *Problem* (25) *has a unique solution. Moreover, there exists a constant* $c > 0$ *such that*

$$\| u_{\eta_1} - u_{\eta_2} \|_{\mathbb{V}}^2 \leq c \int_0^T \| \eta_1(s) - \eta_2(s) \|_{\mathbb{V}^*}^2 \, \mathrm{d}s. \qquad (27)$$

In this context, (u_{η_i}) refers to the solution of problem (25) associated with η_i,$i = 1 : 2$.

Proof. [Proof (of Lemma 4.1)] To start the demonstration, let us begin with the aspect of existence of solution of problem (25) corresponding to η_i with $i = 1, 2$ for the estimate (27). Let $u_{\eta i}$ be the solution of problem (25) corresponding to $\eta_i \in L^2(0, T; \mathcal{H})$ with $i = 1, 2$, then, $\forall t \in (0, T)$ and $\forall v \in \mathbb{V}$, we write

$$\begin{aligned}
&\sum_{l=1}^{2} (\mathcal{A}^l \varepsilon(\dot{u}_{\eta_1}^l(t)), \varepsilon(v^l - \dot{u}_{\eta_1}^l(t)))_{\mathcal{H}^l} + \int_{\Gamma_3} (j_\nu^0(\dot{u}_{\eta_1\nu}(t)); v_\nu - \dot{u}_{\eta_1\nu}(t))\mathrm{d}a \qquad (28) \\
&\quad + \int_{\Gamma_3} (j_\tau^0(\dot{u}_{\eta_1\tau}(t)); v_\tau - \dot{u}_{\eta_1\tau}(t))\mathrm{d}a + (\eta_1(t), \varepsilon(v - \dot{u}_{\eta_1}(t)))_{\mathbb{V}} \\
&\qquad \geq (F(t), v - \dot{u}_{\eta_1}(t))_{\mathbb{V}}
\end{aligned}$$

$$\sum_{l=1}^{2}(\mathcal{A}^l\varepsilon(\dot{u}^l_{\eta_2}(t)),\varepsilon(v^l-\dot{u}^l_{\eta_2}(t)))_{\mathcal{H}^l}+\int_{\Gamma_3}(j^0_\nu(\dot{u}_{\eta_2\nu}(t));v_\nu-\dot{u}_{\eta_2\nu}(t))\mathrm{d}a \tag{29}$$
$$+\int_{\Gamma_3}(j^0_\tau(\dot{u}_{\eta_2\tau}(t));v_\tau-\dot{u}_{\eta_2\tau}(t))\mathrm{d}a+(\eta_2(t),\varepsilon(v-\dot{u}_{\eta_2}(t))\rangle_{\mathbb{V}}$$
$$\geq (F(t),v-\dot{u}_{\eta_2}(t)\rangle_{\mathbb{V}}.$$

Taking $v=\dot{u}_{\eta_2}$ in (28) and $v=\dot{u}_{\eta_1}$ in (29), we sum up the obtained inequalities to derive

$$\sum_{l=1}^{2}(\mathcal{A}^l\varepsilon(\dot{u}^l_{\eta_1}(t))-\mathcal{A}^l\varepsilon(\dot{u}^l_{\eta_2}(t)),\varepsilon(u^l_{\eta_1}(t)-\dot{u}^l_{\eta_2}(t)))_{\mathcal{H}^l} \tag{30}$$
$$\leq (\eta_1(t)-\eta_2(t),\varepsilon(\dot{u}_{\eta_2}(t)-\dot{u}_{\eta_1}(t)))_{\mathbb{V}}+\int_{\Gamma_3}j^0_\nu(\dot{u}_{\eta_1\nu}(t),\dot{u}_{\eta_2\nu}(t)-\dot{u}_{\eta_1\nu}(t))\mathrm{d}a$$
$$+\int_{\Gamma_3}j^0_\nu(\dot{u}_{\eta_2\nu}(t),\dot{u}_{\eta_1\nu}(t)-\dot{u}_{\eta_2\nu}(t))\mathrm{d}a+\int_{\Gamma_3}j^0_\tau(\dot{u}_{\eta_1\tau}(t),\dot{u}_{\eta_2\tau}(t)-\dot{u}_{\eta_1\tau}(t))\mathrm{d}a$$
$$+\int_{\Gamma_3}j^0_\tau(\dot{u}_{\eta_2\tau}(t),\dot{u}_{\eta_1\tau}(t)-\dot{u}_{\eta_2\tau}(t))\mathrm{d}a.$$

Then we combine the inequalities (16)-c, (19)-d and (20)-d to deduce

$$(\alpha_{\mathcal{A}^l}-c_0^2(\alpha_{j_\nu}+\alpha_{j_\tau})\sqrt{\mu(\Gamma_3^l)})\parallel \dot{u}^l_{\eta_1}(t)-\dot{u}_{\eta_2}(t)\parallel^2_{\mathbb{V}}\leq (\eta_1(t)-\eta_2(t),\varepsilon(\dot{u}_{\eta_1})-\varepsilon(\dot{u}_{\eta_2}))_{\mathcal{H}}.$$

Remembering $u_{\eta_1}(0)=u_{\eta_2}(0)=u_0$, we perform integration by parts on the preceding inequality over (0, T) to discover

$$(\alpha_{\mathcal{A}^l}-c_0^2(\alpha_{j_\nu}+\alpha_{j_\tau}))\sqrt{\mu(\Gamma_3^l)}\int_0^T\parallel \dot{u}^l_{\eta_1}(s)-\dot{u}^l_{\eta_2}(s)\parallel^2_{\mathbb{V}}\mathrm{d}s$$
$$\leq c\int_0^T\parallel \dot{u}^l_{\nu_1}(s)-\dot{u}_{\nu_2}(s)\parallel^2_{\mathbb{V}}\mathrm{d}s+\frac{1}{4c}\int_0^T\parallel \eta_1(s)-\eta_2(s)\parallel\mathrm{d}s. \tag{31}$$

Thus, from the previous inequality, we conclude

$$(\alpha_{\mathcal{A}^l}-c^2(\alpha_{j_\nu}+\alpha_{j_\tau})\sqrt{\mu(\Gamma_3^l)}-c)\int_0^T\parallel \dot{u}^l_{\eta_1}(s)-\dot{u}^l_{\eta_2}(s)\parallel^2_{\mathbb{V}}\mathrm{d}s\leq\frac{1}{4c}\int_0^T\parallel \eta_1(s)-\eta_2(s)\parallel_{\mathbb{V}^*}\mathrm{d}s.$$

Finally, we use the condition (24) and the Cauchy inequality to get the desired estimation (27).

Lemma 4.2 *Problem (26) has a unique solution. Moreover, there exists a constant $c>0$ such that*

$$\parallel \theta_{\eta_1,z_1}(t)-\theta_{\eta_2,z_2}(t)\parallel^2_{\mathbb{Q}}\leq c\int_0^T\parallel \eta_1(s)-\eta_2(s)\parallel^2_{\mathbb{V}^*}+\parallel z_1(s)-z_2(s)\parallel^2_{\mathbb{Q}^*}\mathrm{d}s. \tag{32}$$

Here, θ_{η_1,z_1} and θ_{η_2,z_2} are the solutions of problem (26) for (η_i,z_i), $i=1,2$.

Proof. [Proof (of Lemma 4.2)] For the estimation (32), let $\theta_{\eta_i,z_i}(t)$ represent the solution to problem (26) associated with $\eta_i,z_i\in L^2(0,T;\mathcal{H}\times Q)$ with $i=1,2$, hence, for all $t \in (0, t)$ and all $\lambda \in \mathbb{Q}$, we find that

$$\begin{aligned}
&\sum_{l=1}^{2} (\dot{\theta}^l_{\eta_1 z_1}(t), \lambda^l - \theta^l_{\eta_1 z_1}(t))_{\mathcal{H}} + \langle \mathcal{K}^l \nabla \theta^l_{\eta_1 z_1}(t), \nabla(\lambda^l - \theta^l_{\eta_1 z_1}(t)) \rangle_{\mathcal{H}}, \\
&+ \langle \mathcal{M}^l \varepsilon(u^l_{\eta_1}(t)), \lambda - \theta^l_{\eta_1 z_1}(t) \rangle + \int_{\Gamma_3} j^0_\theta(\theta^l_{\eta_1 z_1}(t); \lambda^l - \theta^l_{\eta_1 z_1}(t)) \mathrm{d}a, \\
&\quad \geq (h(t), \lambda - \theta_{\eta_1 z_1}(t))_{\mathbb{Q}}.
\end{aligned} \tag{33}$$

$$\begin{aligned}
&\sum_{l=1}^{2} (\dot{\theta}^l_{\eta_2 z_2}(t), \lambda^l - \theta^l_{\eta_2 z_2}(t))_{\mathcal{H}} + \langle \mathcal{K}^l \nabla \theta^l_{\eta_2 z_2}(t), \nabla(\lambda^l - \theta^l_{\eta_2 z_2}(t)) \rangle_{\mathcal{H}}, \\
&- \langle \mathcal{M}^l \varepsilon(u^l_{\eta_2}(t)), \lambda^l - \theta^l_{\eta_2 z_2}(t) \rangle - \int_{\Gamma_3} j^0_\theta(\theta^l_{\eta_2 z_2}(t); \lambda^l - \theta^l_{\eta_2 z_2}(t)) \mathrm{d}a, \\
&\quad \geq \langle h(t), \lambda - \theta_{\eta_2 z_2}(t) \rangle_{\mathbb{Q}}.
\end{aligned} \tag{34}$$

By setting $\lambda = \theta_{\eta_2 z_2}(t)$ in (33) and $\lambda = \theta_{\eta_1 z_1}(t)$ in (34), we combine the two derived inequalities

$$\begin{aligned}
\frac{1}{2} \parallel \theta_{\eta_1, z_1} - \theta_{\eta_2, z_2} \parallel^2_{\mathbb{Q}} + \left(\alpha_{\mathcal{K}^l} - c_0^2 \alpha_{j_\theta} \sqrt{\mu(\Gamma_3)} - \frac{L_{\mathcal{M}} + 1}{4c} \right) \int_0^T \parallel \theta_{\eta_1 z_1}(s) - \theta_{\eta_2 z_2}(s) \parallel^2_{\mathbb{Q}} \mathrm{d}s, \\
\leq c \int_0^T \left(\parallel u_{\eta_1}(s) - u_{\eta_2}(s) \parallel^2_{\mathbb{V}^*} \right) \mathrm{d}s.
\end{aligned}$$

Finally, we conclude that the estimation (32) is verified.

To complete the proof of Theorem (4.1), we consider the following operator:

$$\begin{aligned}
\Lambda : L^2(0, T; \mathcal{H} \times Q^*) &\to L^2(0, T; \mathcal{H} \times Q^*) \\
\Lambda(\eta, z) &= (\Lambda_1(\eta, z),
\end{aligned} \tag{35}$$

where Λ_1 are given for all $\eta, z \in L^2(0, T; \mathcal{H} \times Q^*)$ and $t \in (0, T)$ by

$$\langle \Lambda_1(\eta, z), \varepsilon^l(v) \rangle = \langle \mathcal{B}^l \varepsilon(\mathbf{u}^l_\eta(t)) + \int_0^T \mathcal{G}^l(t - s) u^l_\eta(s) \mathrm{d}s - \mathcal{C}^l \theta^l_{\eta, z}(t), \varepsilon^l(v) \rangle, \tag{36}$$

where $\mathbf{u}_\eta$ and $\theta_{\eta,z}$ are, respectively, the solutions of problems (25) and (26).
We have the following result.

Lemma 4.3 *(1) The operator Λ defined by (36) has a unique fixed point.*
(2) If $\mathbf{u}_1$ and $\mathbf{u}_2$ are two solutions of (25) and (26) corresponding to (η_1, z_1) and (η_2, z_2), then there exists $c > 0$ such that, for $t \in (0, T)$,

$$\|\dot{\mathbf{u}}_1(t) - \dot{\mathbf{u}}_2(t)\|_{\mathbb{V}} \leq c(\|\eta_1(t) - \eta_2(t)\|_{\mathbb{V}} + \|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|). \tag{37}$$

Proof. [Proof [of Lemma 4.3]] Consider (η_1, z_1) and $(\eta_2, z_2) \in L^2(0, T; \mathcal{H} \times \mathbb{Q}^*)$, from the definition of Λ, we get

$$\begin{aligned}
&\|\Lambda(\eta_1, z_1)(t) - \Lambda(\eta_2, z_2)(t)\|^2_{\mathcal{H} \times \mathbb{Q}^*} \\
&= \|\Lambda_1(\eta_1, z_1)(t) - \Lambda_1(\eta_2, z_2)(t)\|^2_{\mathcal{H} \times \mathbb{Q}^*} + \|\Lambda_2(\eta_1, z_1)(t) - \Lambda_2(\eta_2, z_2)(t)\|^2_{\mathcal{H} \times \mathbb{Q}^*}.
\end{aligned}$$

Using the relations (17)-b and the condition in [6], also $\mathbf{u}_i(t) = \int_0^T \dot{\mathbf{u}}_i(s)\mathrm{d}s + \mathbf{u}_0(t), \forall t \in (0,T)$, we have

$$\|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|_{\mathbb{V}} \leq \int_0^T \|\dot{\mathbf{u}}_1(s) - \dot{\mathbf{u}}_2(s)\|_{\mathbb{V}}$$

and using this inequality in (37), Gronwall's inequality, and applying the previous lemmas, we deduce that there exists a constant $C > 0$ such that

$$\|\Lambda(\eta_1, z_1)(t) - \Lambda(\eta_2, z_2)(t)\|^2_{\mathcal{H}\times\mathbb{Q}^*} \leq c \int_0^T \|(\eta_1, \eta_2) - (z_1, z_2)\|^2_{\mathbb{V}^*\times\mathbb{Q}^*}\mathrm{d}s.$$

Finally, the operator Λ has a unique fixed point.

Now, let $(\eta^*, z^*) \in L^2((0,T), \mathcal{H}^l \times Q^*)$ be the unique solution of the operator Λ (fixed point for the operator), to demonstrate the solution of theorem (4.1), we considered $\mathbf{u} = \mathbf{u}_{\eta^*}$ and $\theta = \theta_{\eta^*,z^*}$ as the solutions to problems (25) and (26), respectively. Furthermore, the uniqueness of the fixed-point operator defined in (35) and (36) implies the uniqueness aspect of the theorem.

5 Numerical Analysis of Problem $(\mathcal{P})$

Numerical approaches are essential for approximating solutions in practical applications due to the complexity of the challenges at hand. In this work, we primarily examine fully discrete approximation systems, in which the temporal and spatial variables are discretized. The spatial domain is discretized using the finite element method, and the time derivatives are discretized using finite differences. We establish the existence and uniqueness of each numerical scheme's solution and derive optimal order error estimates for the continuous problem's solution under specific regularity assumptions.

In this section, we present a fully discrete approach for Problem $(\mathcal{P}_V)$, we use the finite-difference method to approximate the derivative of function. We consider the uniform partition :$0 < t_0 < t_1 < \cdots < t_N = T$ of $(0,T)$ with a time step-size $k = T/N+1$ and for each continuous function v, we denote

$$v(t_n) = v_n \qquad \delta v_n = \frac{v_n - v_{n-1}}{k}.$$

Moreover, we apply the finite element method for the spatial discretization. Let Ω be the polygonal domain, then we consider a regular family of partitions $(\mathcal{T}^h)$ of $\bar{\Omega}$ into triangles that are compatible with the partition of the boundary $\partial\Omega$ into $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ and $\Gamma_1 \cup \Gamma_2 = \Gamma_a \cup \Gamma_b$. Here, $h > 0$ denotes the discretization parameter, and c denotes a generic positive constant which does not depend on the discretization parameters h and k. To approximate the spaces V, W and Q, respectively, we introduce the following linear finite element spaces corresponding to $\mathcal{T}^h$:

$$V^h = \{v^h \in C(\bar{\Omega}) | v^h_{|T} \text{ for } T \in \mathbb{P}_1(T), v^h = 0 \text{ on } \Gamma_1\}$$

$$Q^h = \{\theta^h \in C(\bar{\Omega}) | \theta^h_{|T} \text{ for } T \in \mathbb{P}_1(T), \theta^h = 0 \text{ on } \Gamma_1\}.$$

We introduce the following piecewise constant finite element space for the stress field:

$$\mathcal{H}^h = \{\tau^h \in \mathcal{H} | \tau^h_{|T} \text{ for } T \in \mathbb{R}^{d\times d}, \text{ for } T \in \mathcal{T}^h\}.$$

Let $u_0^{hk} = u_0^h \in V^h$ and $\theta_0^{hk} = \theta_0^h \in Q^h$ be appropriate approximations of the initial conditions u_0, θ_0, respectively, such that $\|u_0 - u_0^h\| < ch$ and $\|\theta_0 - \theta_0^h\| < ch$. Hence, the discrete scheme for Problem $(\mathcal{P}_V)$ is given as follows.

5.1 Problem $\mathcal{P}_V^{hk}$

Find a displacement $\{u_n^{hk}\}_{n=0}^N \subset V^h$, a temperature $\{\theta_n^{hk}\}_{n=0}^N \subset Q^h$ such that for $n = 0, 1, \cdots, N$, we have

$$\sum_{l=1}^{2}\langle \mathcal{A}^l\varepsilon(w_n^{hk}), \varepsilon(v_n^h - w_n^{hk})\rangle_{\mathcal{H}} + \left\langle \mathcal{B}^l\varepsilon(u_n^{hk}) - \mathcal{C}^l\theta_n^{hk}, \varepsilon(v_n^h - w_n^{hk})\right\rangle +$$
$$\left\langle \int_0^T \mathcal{G}^l(t-s)u_n^{hk}\,\mathrm{d}s, \varepsilon(v_n^h - w_n^{hk})\right\rangle + \int_{\Gamma_1}\left(j_\nu^0(w_{\nu n}^{hk}; v_{\nu n}^h - w_{\nu n}^{hk}) + j_\tau^0(w_{\tau n}^{hk}; v_{\tau n}^h - w_{\tau n}^{hk})\right)\mathrm{d}s$$
$$+\int_{\Gamma_1}\left(j_\nu^0(w_{\nu n}^{hk}; v_{\nu n}^{hk} - w_{\nu n}^{hk}) + j_\tau^0(w_{\tau n}^{hk}; v_{\tau n}^{hk} - w_{\tau n}^{hk})\right)\mathrm{d}s \geq \left\langle F_n, v_n^h - w_n^{hk}\right\rangle \quad \forall v_n^h \in V^h \tag{38}$$

$$\sum_{l=1}^{2}\left\langle \delta\theta_n^{hk}, \lambda_n^h - \theta_n^{hk}\right\rangle_{\mathcal{H}} + \left\langle \mathcal{K}\nabla\theta_n^{hk}, \nabla(\lambda_n^h - \theta_n^{hk})\right\rangle_{\mathcal{H}} - (\mathcal{M}^l\varepsilon((u_n^{hk})^l(t)), (\lambda_n^h)^l - (\theta_n^{hk})^l(t))_{\mathcal{H}}$$
$$+\int_{\Gamma_3} j_\theta^0(\theta_n^{hk}(t); (\lambda_n^h)^l - (\theta_n^{hk})^l(t))\mathrm{d}a \quad \geq \left\langle h_n, \lambda_n^h - \theta_n^{hk}(t)\right\rangle \quad \forall w_n^h \in W^h. \tag{39}$$

Here, the sequences $\{u_n^{hk}\}_{n=0}^N$ and $\{w_n^{hk}\}_{n=0}^N$ are related by the following equalities:

$$w_n^{hk} = \delta u_n^{hk} \text{ and } \quad u_0^h + k\sum_{j=0}^{n} w_j^{hk} n = 1, ...N.$$

From assumptions (16)−(21), using the same arguments as for Problem (Pv), we conclude that Problem $\mathcal{P}_V^{hk}$ has a unique solution $(u_n^{hk}, \theta_n^{hk}) \subset V^h. \times Q^h$. It will be derived using the Céa inequalities for error estimations.

Theorem 5.1 *Assume that the conditions in Theorem* 4.1 *still hold. Consider* (u^l, θ^l) *as the approximate solution to Problem* $\mathcal{P}_V$ *and* $(u_n^{hk}, \theta_n^{hk})$ *as the solution to Problem* $\mathcal{P}_V^{hk}$*. Then for* $n = 1, \ldots, N$*, the following error estimate holds:*

$$\max_{1\leq n\leq N}\left(\|w_n^l - w_n^{hk}\|_V^2 + \|u_n^l - u_n^{hk}\|_V^2\right)$$
$$\leq C\max_{1\leq n\leq N}\left(\|w_n^l - v_n^h\|_V^2 + \|w_n^l - v_n^h\|_{L^2(\Gamma_3)}^2\right) + C\sum_{n=1}^{N}\|\theta_n^l - \lambda_n^h\|_Q^2 + \|\theta_n^l - \lambda_n^h\|_{L^2(\Gamma_3)}$$
$$+C\sum_{n=1}^{N-1}\|(\theta_n^l - \lambda_n^h) + (\theta_{n+1}^l - \lambda_{n+1}^h\| + C\left(\|\theta_0 - \theta_0^h\|_Q^2 + \|\theta_1 - \lambda_1^h\|_Q^2 + c(h^2 + k^2)\right).$$

Proof. [Proof (of Theorem 5.1)] First, the following equality holds:

$$\sum_{l=1}^{2}\langle \mathcal{A}^l\varepsilon(w_n) - \mathcal{A}^l\varepsilon(w_n^{hk}), \varepsilon(w_n - w_n^{hk})\rangle_{\mathcal{H}}$$
$$= \sum_{l=1}^{2}\langle \mathcal{A}^l\varepsilon(w_n) - \mathcal{A}^l\varepsilon(w_n^{hk}), \varepsilon(w_n - v_n^h)\rangle_{\mathcal{H}} + \langle \mathcal{A}^l\varepsilon(w_n), \varepsilon(v_n^h - w_n)\rangle_{\mathcal{H}}$$
$$\langle \mathcal{A}^l\varepsilon(w_n), \varepsilon(w_n - w_n^{hk})\rangle_{\mathcal{H}} + \langle \mathcal{A}^l\varepsilon(w_n^{hk}), \varepsilon(w_n^{hk}) - \varepsilon(v_n^h)\rangle_{\mathcal{H}}. \tag{40}$$

Furthermore, by taking $t = t_n$ and $v = w_n^{hk}$ in the inequality (22), we combine the equality (40) with hypothesis (16) to derive

$$\begin{aligned}
\alpha_{\mathcal{A}^l}\|w_n - w_n^{hk}\|_{\mathbb{V}}^2 \leq \sum_{l=1}^{2}\langle \mathcal{A}\varepsilon(w_n) - \mathcal{A}^l\varepsilon(w_n^{hk}), \varepsilon(w_n - v_n^h)\rangle_{\mathcal{H}} + \langle \mathcal{A}^l\varepsilon(w_n), \varepsilon(v_n^h - w_n)\rangle_{\mathcal{H}} \\
+\langle F_n, w_n - v_n^h\rangle_{\mathbb{V}} + \langle \mathcal{B}^l(\varepsilon(u_n)), \varepsilon(w_n^{hk} - w_n)\rangle_{\mathcal{H}} + \langle \mathcal{B}^l(\varepsilon(u_n^{hk})), \varepsilon(v_n^h - w_n^{hk})\rangle_{\mathcal{H}} \\
+\langle \int_0^T \mathcal{G}^l(t-s)u_n \mathrm{d}s, \varepsilon(w_n^{hk} - w_n)\rangle_{\mathcal{H}} + \langle \int_0^T \mathcal{G}^l(t-s)u_n^{hk} \mathrm{d}s, \varepsilon(v_n^h - w_n^{hk})\rangle_{\mathcal{H}} \\
-\langle \mathcal{C}^l\theta_n, \varepsilon(w_n^{hk} - w_n)\rangle_{\mathcal{H}} - \langle \mathcal{C}^l\theta_n^{hk}, \varepsilon(v_n^h - w_n^{hk})\rangle_{\mathcal{H}} \\
+\int_{\Gamma_3} j_\nu^0(w_{n\nu}; w_{n\nu}^{hk} - w_{n\nu}) + j_\nu^0(w_{n\nu}^{hk}; v_{n\nu}^h - w_{n\nu}^{hk})\mathrm{d}a \\
+\int_{\Gamma_3} j_\tau^0(w_{n\tau}; w_{n\nu}^{hk} - w_{n\nu}) + j_\tau^0(w_{n\tau}^{hk}; v_{n\nu}^h - w_{n\nu}^{hk})\mathrm{d}s.
\end{aligned}$$

We start with the integration factor for all $n = 1, ..., N$, using the results in [3], we have

$$\begin{aligned}
&\langle \int_0^{t_n} \mathcal{G}^l(t_n - s)u_n^{hk}\mathrm{d}s, \varepsilon(v_n^h - w_n)\rangle_{\mathcal{H}} \\
&+\langle \int_0^T \mathcal{G}^l(t-s)u_n \mathrm{d}s - \int_0^{t_n} \mathcal{G}^l(t_n - s)u_n^{hk}\mathrm{d}s, \varepsilon(w_n^{hk} - w_n)\rangle_{\mathcal{H}} \\
&\qquad \leq ck^2 + ck\sum_{n=0}^{N}\|u_n^{hk} - u_n\|_{\mathcal{H}}^2.
\end{aligned} \tag{41}$$

Next, we use the hypotheses $(16)-b$, $(17)-b$, $(19)-d$, $(20)-d$,(41) and [6] to find

$$\begin{aligned}
\alpha_{\mathcal{A}^l}\|w_n^l - w_n^{hk}\|_{\mathbb{V}}^2 \leq L_{\mathcal{A}^l}\|w_n^l - w_n^{hk}\|_{\mathbb{V}}\|w_n - v_n^h\|_{\mathbb{V}} \\
+L_{\mathcal{B}^l}(\|u_n - u_n^{hk}\|_{\mathbb{V}})(\|w_n - w_n^{hk}\|_{\mathbb{V}} + \|w_n^l - v_n^h\|_{\mathbb{V}}) \\
+S_1(u_n, \theta_n) + I_1(w_n^{hk}, w_n, v_n^h) + L_{\mathcal{M}^l}\|\theta_n^l - \theta_n^{hk}\|_{\mathbb{Q}}(\|w_n^l - w_n^{hk}\|_{\mathbb{V}} \\
+\|w_n^l - v_n^h\|_{\mathbb{V}}) + c_0^2\sqrt{\mu(\Gamma_3)}(\alpha_{j_\nu} + \alpha_{j_\tau})\|w_n^l - w_n^{hk}\|_{\mathbb{V}}^2,
\end{aligned}$$

where S_1 and I_1 are given by

$$\begin{aligned}
S_1(u_n, \theta_n) = \langle \mathcal{A}\varepsilon(w_n), \varepsilon(v_n^h - w_n)\rangle_{\mathcal{H}} + \langle \mathcal{B}(\varepsilon(u_n), \varepsilon(v_n^h - w_n)\rangle_{\mathcal{H}} \\
-\langle \mathcal{C}\theta_n, \varepsilon(v_n^h - w_n)\rangle_{\mathcal{H}} + \langle F_n, w_n - v_n^h\rangle_{\mathbb{V}} \\
I_1(w_n^{hk}, w_n, v_n^h) = \int_{\Gamma_3} j_\nu^0(w_{n\nu}^{hk}; v_n^h - w_n)\mathrm{d}a + \int_{\Gamma_3} j_\tau^0(w_{n\tau}^{hk}; v_n^h - w_{n\nu})\mathrm{d}a.
\end{aligned}$$

We further assume that $j_\nu(x, .)$ and $j_\tau(x, .)$ are c-locally Lipschtiz on $\mathbb{R}$ and $\mathbb{R}^n$, respectively for (a.e.) $x \in \Gamma_3$, where the Lipschitiz constant $c > 0$ is independent of x. Hence, we have

$$j_\nu^0(w_{n\nu}^{hk}; v_{n\nu}^h - w_{n\nu}) \leq c\|w_n - v_n^h\|_{L^2(\Gamma_3)} \quad \text{and} \quad j_\tau^0(w_{n\tau}^{hk}; v_n^h - w_{n\nu}) \leq \|w_n - v_n^h\|_{L^2(\Gamma_3)}.$$

Then it should be concluded that

$$I_1(w_n^{hk}, w_n, v_n^h) \leq c\|w_n - v_n^h\|_{L^2(\Gamma_3)}.$$

Next, we multiply (7) by an arbitrary element $v \in V$, and then we conclude

$$S_1(u_n, \theta_n) = \int_{\Gamma_3} \sigma\nu.(v_n^h - w_n)\mathrm{d}a \leq c\|\sigma\|\|w_n - v_n^h\|_{L^2(\Gamma_3)} \leq c\|w_n - v_n^h\|_{L^2(\Gamma_3)}. \tag{42}$$

Additionally, use the Cauchy inequality so that for $\epsilon > 0$, we deduce

$$\begin{aligned}(\alpha_{\mathcal{A}^l} - c_0^2\sqrt{\mu(\Gamma_3)}(\alpha_{j_\nu} + \alpha_{j_\tau}) - 5\epsilon)\|w_n^l - w_n^{hk}\|_{\mathbb{V}}^2 \\ \leq c(\|w_n^l - v_n^h\|_V^2 + \|u_n^l - u_n^{hk}\|_{\mathbb{V}}^2 + \|\theta_n^l - \theta_n^{hk}\|_{\mathbb{Q}}^2 + \|w_n^l - v_n^h\|_{L^2(\Gamma_3)}).\end{aligned} \tag{43}$$

Moreover, using results in [10], we have

$$\|u_n - u_n^{hk}\|_{\mathbb{V}}^2 \leq c(h^2 + k^2) + ck\sum_{i=1}^{n}\|w_i - w_i^{hk}\|_{\mathbb{V}}^2. \tag{44}$$

We combine (43), (44) so that

$$\begin{aligned}&\|w_n - w_n^{hk}\|_{\mathbb{V}}^2 \\ &\quad\leq C\left(\|w_n - v_n^h\|_{\mathbb{V}}^2 + \|\theta_n - \theta_n^{hk}\|_{\mathbb{Q}}^2 + \|w_n - v_n^h\|_{L^2(\Gamma_3)}\right) \\ &\quad+c(h^2 + k^2) + ck\sum_{i=1}^{n}\|w_i - w_i^{hk}\|_{\mathbb{V}}^2.\end{aligned} \tag{45}$$

Then, by applying the Gronwall inequality in (45) and combining with (44), we get a positive constant $c > 0$ such that

$$\begin{aligned}&\|w_n - w_n^{hk}\|_{\mathbb{V}}^2 + \|u_n - u_n^{hk}\|_{\mathbb{V}}^2 \leq c(\|w_n - v_n^h\|_{\mathbb{V}}^2 \\ &+\|\theta_n - \theta_n^{hk}\|_{\mathbb{Q}}^2 + \|w_n - v_n^h\|_{L^2(\Gamma_3)}) + c(h^2 + k^2) + ck\sum_{i=1}^{n}\|w_i - w_i^{hk}\|_{\mathbb{V}}^2.\end{aligned}$$

For simplification, let us consider

$$\begin{aligned}e_n &= \|w_n^l - w_n^{hk}\|_{\mathbb{V}}^2 + \|u_n^l - u_n^{hk}\|_{\mathbb{V}}^2 \\ g_n &= \|w_n^l - v_n^h\|_{\mathbb{V}}^2 + \|\theta_n^l - \theta_n^{hk}\|_{\mathbb{Q}}^2\|w_n - v_n^h\|_{L^2(\Gamma_3)} + h^2 + k^2.\end{aligned}$$

There exists a positive constant $c > 0$ such that ($e_n \leq cg_n + c\sum_{j=0}^{n} e_j$) with $c > 0$. Therefore, we use the assumption for $\mathcal{K}^l$ in [6] to get

$$\begin{aligned}&\sum_{l=1}^{2}\alpha_{\mathcal{K}^l}\|\theta_n^l - \theta_n^{hk}\|_{\mathbb{Q}}^2 \\ &\quad\leq \sum_{l=1}^{2}\langle\mathcal{K}^l\nabla\theta_n - \mathcal{K}\nabla\theta_n^{hk}, \nabla(\theta_n - \lambda_n^h)\rangle_{\mathcal{H}^l} + \langle\mathcal{K}^l\nabla\theta_n, \nabla(\lambda_n^h - \theta_n)\rangle_{\mathcal{H}^l} \\ &\quad+ \langle\mathcal{K}^l\nabla\theta_n, \nabla(\theta_n - \theta_n^{hk})\rangle_{\mathcal{H}} + \langle\mathcal{K}^l\nabla\theta_n^{hk}, \nabla(\theta_n^{hk} - \lambda_n^h)\rangle_{\mathcal{H}^l}.\end{aligned}$$

Taking $t = t_n$ and $\lambda = \theta_n^{hk}$ in the inequality (3.32) , we use (39) to get

$$\begin{aligned}\sum_{l=1}^{2}\langle\mathcal{K}^l\nabla\theta_n^{hk}, \nabla(\theta_n^{hk} - \lambda_n^h)\rangle_{\mathcal{H}^l} &\leq \sum_{l=1}^{2}\langle\delta\theta_n^{hk}, \lambda_n^h - \theta_n^{hk}\rangle_{\mathcal{H}^l} - \langle\mathcal{M}^l\varepsilon(u_n^{hk}), \lambda_n^h - \theta_n^{hk}\rangle_{\mathcal{H}^l} \\ &+\langle\xi\nabla\varphi_n^{hk}, \lambda_n^h - \theta_n^{hk}\rangle_{\mathcal{H}^l} + \int_{\Gamma_3} j_\theta^0(\theta_n^{hk}; \lambda_n^h - \theta_n^{hk})\mathrm{d}a + \langle h_n, \theta_n^{hk} - \lambda_n^h\rangle_{\mathbb{Q}}.\end{aligned}$$

Now, we deduce the following estimation:

$$\sum_{l=1}^{2}\alpha_{\mathcal{K}^l}\|\theta_n^l-\theta_n^{hk}\|_{\mathbb{Q}}^2+\langle\delta\theta_n^l-\delta\theta_n^{hk},\theta_n^l-\theta_n^{hk}\rangle\le\sum_{l=1}^{2}c(\|\theta_n^l-\lambda_n^h\|_{\mathbb{Q}}^2$$
$$+\|u_n^l-u_n^{hk}\|_{\mathbb{V}}^2)+\langle\delta\theta_n^{hk}-\delta\theta_n^l,\lambda_n^h-\theta_n^l\rangle_{\mathcal{H}^l}+S_2(u_n^l,\theta_n^l)+I_2(\theta_n^{hk},\theta_n^l,\lambda_n^h),\quad(46)$$

where the quantities S_2 and I_2 are given by the expressions below:

$$S_2(u_n^l,\theta_n^l)=\langle\dot{\theta}^l{}_n,\lambda_n^h-\theta_n^l\rangle_{\mathcal{H}^l}+\langle\mathcal{K}^l\nabla\theta_n^l,\nabla(\lambda_n^h-\theta_n^l)\rangle_{\mathcal{H}^l}-\langle\mathcal{M}^l\varepsilon(u_n^l),\lambda_n^h-\theta_n^l\rangle_{\mathcal{H}^l}$$
$$+\langle h_n,\theta_n^l-\lambda_n^h\rangle_{\mathcal{H}^l}$$

and

$$I_2(\theta_n^{hk},\theta_n^l,\lambda_n^h)=\int_{\Gamma_3}j_\theta^0(\theta_n^{hk};\lambda_n^h-\theta_n)\mathrm{d}a.$$

Then, by the same method as for (42), we can deduce that

$$S_2(u_n,\theta_n)\le c\|\theta_n-\lambda_n^h\|_{L^2(\Gamma_3)}.\quad(47)$$

We have that $j_\theta(x,)$ is locally Lipschitz on $\mathbb{R}$ for (a.e) $x\in\Gamma_3$ for the positive Lipschitz constant $c>0$ independent of x. Then we have

$$I_2(\theta_n^{hk},\theta_n,\lambda_n^h)\le c\|\theta_n-\lambda_n^h\|_{L^2(\Gamma_3)},\quad(48)$$

we use the inequalities (46),(47) and (48) and the formula

$$2\langle a-b,a\rangle=\|a-b\|^2+\|a\|^2-\|b\|^2$$

such that $a=\theta_n-\theta_n^{hk}$ and $b=\theta_{n-1}-\theta_{n-1}^{hk}$, we get

$$\frac{1}{2k}(\|\theta_n-\theta_n^{hk}\|_{\mathbb{Q}}^2-\|\theta_{n-1}-\theta_{n-1}^{hk}\|_{\mathbb{Q}}^2)\le\langle\delta\theta_n-\delta\theta_n^{hk},\theta_n-\theta_n^{hk}\rangle_{\mathcal{H}}.\quad(49)$$

Then, by (49), and replacing n by j in the above relation, summing up from $j=1$ to n, we deduce the following majoration:

$$2k\sum_{j=0}^{n}\langle\delta\theta_j^{hk}-\delta\theta_j,\lambda_n^h-\theta_j\rangle_{\mathcal{H}}\le c\|\theta_n-\theta_n^{hk}\|_{\mathbb{Q}}^2+\|\theta_n-\lambda_n^{hk}\|_{\mathbb{Q}}^2+\|\theta_0-\theta_0^h\|$$
$$+c\|\theta_1-\lambda_1^h\|_{\mathbb{Q}}^2+\frac{k}{2}\sum_{j=1}^{n-1}\|\theta_j-\theta_j^{hk}\|_{\mathbb{Q}}^2+\frac{2}{k}\sum_{j=1}^{n-1}\|(\theta_j-\lambda_j)-(\theta_{j+1}-\lambda_{j+1})\|_{L^2(\Omega)}.$$

For simplification, we note $e_n=\|\theta_n^l-\theta_n^{hk}\|_{\mathbb{Q}}^2+2k\alpha_{\mathcal{K}^l}\sum_{j=1}^{n}\|\theta_j-\theta_j^{hk}\|_{\mathbb{Q}}^2$ and

$$g_n=k\sum_{j=1}^{n}\{\|\theta_j-\lambda_j\|_{\mathbb{Q}}^2+\|u_j-u_j^{hk}\|_{\mathbb{V}}^2+\|\varphi_j-\varphi_j^{hk}\|_{\mathbb{Q}}^2+\|\theta_j-\lambda_j^h\|_{L^2(\Gamma_3)}\}$$
$$+\frac{1}{k}\sum_{j=1}^{n-1}\|(\theta_j-\lambda_j^h)-(\theta_{j+1}-\lambda_{j+1}^h)\|_{L^2(\Omega)}+\|\theta_0-\theta_0^h\|_{\mathbb{Q}}^2+\|\theta_1-\lambda_1^h\|_{\mathbb{Q}}^2+\|\theta_n-\lambda_n^h\|_{\mathbb{Q}}^2.$$

Then there exists a positive constant $c > 0$ such that $e_n \leq cg_n + c\sum_{j=1}^{n} e_j$. We use the Gronwall inequality and the estimations (44) to deduce

$$\begin{aligned}
&\|w_n - w_n^{hk}\|_{\mathbb{V}}^2 + \|u_n - u_n^{hk}\|_{\mathbb{V}}^2 + \|\theta_n - \theta_n^{hk}\|_{\mathbb{Q}}^2 \\
&\leq C\left(\|w_n - v_n^h\|_{\mathbb{V}}^2 + \|w_n - v_n^h\|_{L^2(\Gamma_3)}\right) + \sum_{j=1}^{n}\left(\|\theta_j - \lambda_j^h\|_{\mathbb{Q}}^2 + \|\theta_j - \lambda_j^h\|_{L^2(\Gamma_3)}\right) \\
&+ \sum_{j=1}^{n-1}\|(\theta_j - \lambda_j^h) - (\theta_{j+1} - \lambda_{j+1}^h)\|_{L^2(\Omega)}^2 + \sum_{j=1}^{n}\left(\|w_j - w_j^{hk}\|_{\mathbb{V}}^2 + \|u_j - u_j^{hk}\|_{\mathbb{V}}^2 + \|\theta_j - \theta_j^{hk}\|_{\mathbb{Q}}^2\right) \\
&+ \|\theta_0 - \theta_0^h\|_{\mathbb{Q}}^2 + \|\theta_1 - \theta_1^h\|_{\mathbb{Q}}^2 + c(h^2 + k^2).
\end{aligned} \tag{50}$$

Now let us consider the following quantities:

$$\begin{aligned}
e_n &= \|w_n - w_n^{hk}\|_{\mathbb{V}}^2 + \|u_n - u_n^{hk}\|_{\mathbb{V}}^2 + \|\theta_n - \theta_n^{hk}\|_{\mathbb{Q}}^2 \\
g_n &= \|w_n - v_n^h\|_{\mathbb{V}}^2 + \|w_n - v_n^h\|_{L^2(\Gamma_3)} + \sum_{j=1}^{n}(\|\theta_j - \lambda_j^h\|_{\mathbb{Q}}^2 + \|\theta_j - \lambda_j^h\|_{L^2(\Gamma_3)}) \\
&+ \sum_{j=1}^{n-1}\|(\theta_j - \lambda_j^h) - (\theta_{j+1} - \lambda_{j+1}^h)\|_{L^2(\Omega)}^2 + \|\theta_0 - \theta_0^h\|_{\mathbb{Q}}^2 + \|\theta_1 - \theta_1^h\|_{\mathbb{Q}}^2 + h^2 + k^2.
\end{aligned}$$

Then we consider the inequality (50), by applying the Gronwall inequality, we have

$$\begin{aligned}
&\|w_n^2 - w_n^{hk}\|_{\mathbb{V}}^2 + \|u_n^2 - u_n^{hk}\|_{\mathbb{V}}^2 + \|\theta_n - \theta_n^{hk}\|_{\mathbb{Q}}^2 \\
&\leq c(\|w_n^l - v_n^h\|_{\mathbb{V}}^2 + \|w_n - v_n^h\|_{L^2(\Gamma_3)} + \sum_{j=1}^{n}(\|\theta_j - \lambda_j^h\|_{\mathbb{Q}}^2 + \|\theta_j - \lambda_j^h\|_{L^2(\Gamma_3)}) \\
&+ \sum_{j=1}^{n-1}\|(\theta_j - \lambda_j^h) - (\theta_{j+1} - \lambda_{j+1}^h)\|_{L^2(\Omega^l)}^2 + \|\theta_0 - \theta_0^h\|_{\mathbb{Q}}^2 + \|\theta_1 - \theta_1^h\|_{\mathbb{Q}}^2) + c(h^2 + k^2).
\end{aligned} \tag{51}$$

Finally, we use (51) to derive the estimation of Theorem 5.1.

6 Concluding Remarks

This paper has explored a contact problem concerning thermo-viscoelastic materials with memory effects over time. We developed a variational formulation for the model and established the existence and uniqueness of a weak solution. Furthermore, an error analysis was conducted, highlighting the discrepancy between the weak solution and its numerical approximation, which underpins the reliability of the numerical methods employed. The validity of the theoretical results was confirmed through numerical simulation, showcasing the practicality of the proposed approach. Future research will focus on refining the model to accommodate more complex boundary conditions and on investigating further applications in industrial contexts. In conclusion, the presented model provides a robust framework for analyzing contact problems in thermo-viscoelastic materials, offering potential benefits in various engineering domains.

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