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Electronic Circuit and Complete Synchronization via Active Backstepping Control for a New Chaotic 3-D Jerk System

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Abstract: This paper presents the modeling, electronic circuit implementation, and complete synchronization of a new chaotic 3-D jerk system with two quadratic nonlinearities. The proposed jerk system, characterized by the third derivative of its output being a function of lower-order derivatives, exhibits chaotic behavior under specific parameter conditions. The system's dynamics are analyzed, revealing the presence of chaotic attractors through numerical simulations and Lyapunov exponents. An electronic circuit realizing the jerk system is designed using operational amplifiers, resistors, and capacitors, demonstrating chaos through Multisim and MATLAB simulations. Additionally, a backstepping control technique is employed to achieve complete synchronization between the master and slave jerk systems, with potential applications in secure communications and cryptosystems. Theoretical proofs and simulation results validate the effectiveness of the proposed synchronization method.

Keywords: *chaos theory; chaotic systems; dynamical systems; jerk systems; bifurcation; synchronization; backstepping control.*

Mathematics Subject Classification (2020): 34H10, 65P40, 34D06, 34D08, 94C30, 94C60.

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1 Introduction

Chaos theory has become a fundamental area of study in nonlinear dynamics, with applications spanning across various fields such as physics, engineering, biology, and secure communications [1–3]. A chaotic system is highly sensitive to initial conditions, leading to behavior that appears random and unpredictable despite being deterministic [4, 5]. Among the numerous chaotic systems studied, jerk systems are of particular interest due to their simplicity and ability to exhibit complex dynamical behavior [6]. A jerk system is defined by its output's third derivative (jerk) being a function of lower-order derivatives [8].

The study of chaotic systems has its roots in the pioneering work of Lorenz, who discovered the Lorenz attractor [9], a set of chaotic solutions of the Lorenz system, which has become a classic example of chaos. Since then, numerous chaotic systems have been explored, including the Rossler system, Chua's circuit, and jerk systems [10, 11]. Jerk systems, in particular, are intriguing due to their relatively simple mathematical form and the rich dynamical behavior they exhibit [12].

In a jerk system, the third derivative of a variable with respect to time is expressed as a function of the variable and its first and second derivatives [13]. This form allows for the construction of chaotic systems using basic electronic components such as resistors, capacitors, and operational amplifiers [14]. The electronic realization of chaotic systems not only provides a tangible means of studying chaos but also facilitates practical applications in areas like secure communications, where chaotic signals can be used for encryption.

The introduction of a novel 3-D jerk system with quadratic nonlinearities, as explored in this study, contributes to the ongoing exploration of complex dynamical behaviors such as chaotic attractors and multistability, which are prevalent in both natural and engineered systems. Moreover, the practical implementation of these systems through electronic circuit designs bridges the gap between theoretical models and real-world applications, further enhancing their relevance. The synchronization of such systems, demonstrated here using backstepping control, has significant implications for creating stable, secure systems in various technological domains, thereby underscoring the importance of this research in advancing the understanding and application of chaotic dynamics.

The synchronization of chaotic systems, where two or more chaotic systems are made to exhibit identical behavior over time, has significant implications for practical applications [15]. Techniques such as backstepping control have been developed to achieve synchronization, offering robust methods for controlling chaotic systems.

This paper presents a novel chaotic 3-D jerk system characterized by two quadratic nonlinearities. We explore its dynamic properties through numerical simulations, revealing chaotic behavior under specific parameter settings. The system is then realized electronically, and its chaotic nature is validated through both Multisim and MATLAB simulations. Finally, we employ an active backstepping control technique to achieve complete synchronization between a pair of chaotic jerk systems and demonstrating the method's effectiveness.

The rest of the paper is organized as follows. In Section 2, we present the modeling of the new chaotic 3-D jerk system with two quadratic nonlinearities and analyze its dynamic behavior through numerical simulations. Section 3 describes the electronic circuit implementation of the proposed chaotic system and validates its chaotic behavior using simulations in Multisim and MATLAB. In Section 4, we introduce the backstepping control technique and demonstrate the complete synchronization of master and slave chaotic jerk systems, providing theoretical proofs and simulation results. In Section 5, we conclude the paper by summarizing the contributions of the research and discussing potential applications of the proposed system.

2 Modelling of the New Jerk System

This section presents and examines a new chaotic jerk system with two quadratic nonlinearities. The new jerk system is described as follows:

$$\begin{cases} \dot{x} &= y, \\ \dot{y} &= z, \\ \dot{z} &= -ax + by - z - xy + cy^2. \end{cases} \tag{1}$$

In the jerk system (1), $X = (x, y, z)$ is the 3-D state and a, b, c are positive parameters.

In this paper, we show that the jerk system (1) is chaotic when the parameters are $a = 1$, $b = 0.1$ and $c = 1$.

For numerical simulations in MATLAB, we pick the parameters as $(a, b, c) = (1, 0.1, 1)$ and the initial state as $X(0) = (0.2, 0.2, 0.2)$. Then the Lyapunov exponents (LE) of the jerk system (1) are numerically determined for $T = 1E4$ seconds as

$$l_1 = 0.1252,\ l_2 = 0,\ l_3 = -1.1252. \tag{2}$$

The LE results in Eq.(2) show that the new 3-D jerk system (1) is chaotic and dissipative with the maximal Lyapunov exponent (MLE) found as $l_1 = 0.1252 > 0$. The Kaplan dimension of the new 3-D jerk system can be also determined as

$$D_K = 2 + \frac{1}{|l_3|}(l_1 + l_2) = 2.1113. \tag{3}$$

Figures 1-3 show the phase plots of the jerk system (1) generated in MATLAB using the classical fourth-order Runge-Kutta method for the initial state $(0.2, 0.2, 0.2)$ and the parameter vector $(a, b, c) = (1, 0.1, 1)$.

The equilibrium points of the system described by Equation (1) can be determined by setting in Equation (1) as follows:

$$\begin{cases} 0 &= y, \\ 0 &= z, \\ 0 &= -ax + by - z - xy + cy^2. \end{cases} \tag{4}$$

Thus, the equilibrium points of the system (4) are $E_0 = (0,\ 0,\ 0)$. The Jacobian matrix of the system (1) can be written as

$$J = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 - y & 0.1 - x + 2y & -1 \end{pmatrix}. \tag{5}$$

The Jacobian matrix at the equilibrium point E_0 is expressed as

$$J = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0.1 & -1 \end{pmatrix}. \tag{6}$$

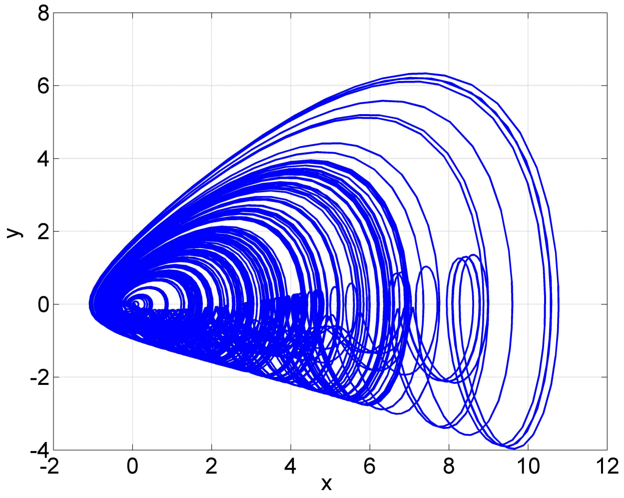


Figure 1: (x, y) plot of the chaotic jerk system (1).

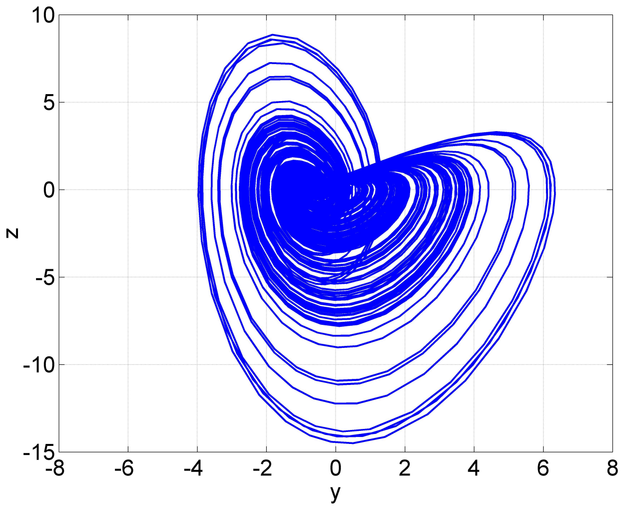


Figure 2: (y, z) plot of the chaotic jerk system (1).

The polynomial characteristic equation of Eq.(6) is given by

$$\lambda^3 + \lambda^2 - 0.1\lambda + 1 = 0. \tag{7}$$

The Jacobian matrix JE_0 has the eigenvalues $-1.5068, 0.2534 \pm 0.7742i$. This shows that the system (1) exhibits the index-2 spiral saddle point, which is unstable.

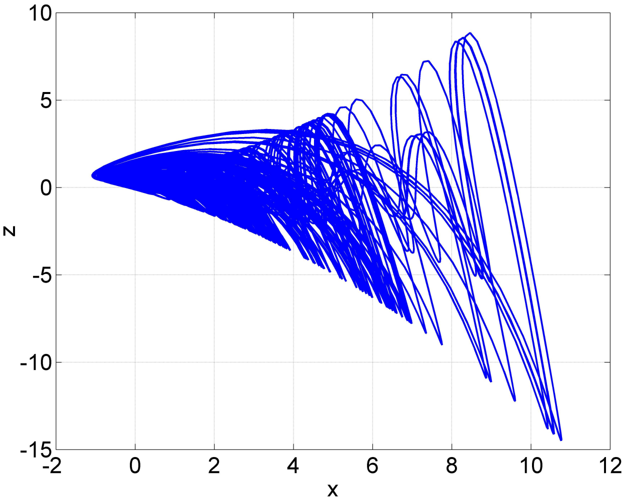


Figure 3: (x, z) plot of the chaotic jerk system (1).

3 Electronic Circuit

The provided circuit diagram represents a jerk system (see Figure 4), which is a type of nonlinear dynamical system known for exhibiting chaotic behavior under certain conditions. A jerk system is characterized by the third derivative of its output (jerk) being a function of lower-order derivatives. The circuit comprises operational amplifiers (op-amps), resistors, and capacitors to realize the jerk function.

In this specific circuit, each section with an op-amp configuration represents different components of the jerk system. The resistors and capacitors determine the time constants and feedback paths, which are crucial for defining the system's dynamic behavior. The input signals (X, Y, and Z) are processed through the network of op-amps to produce a chaotic output. The chaotic nature arises from the nonlinear interactions between the components, causing the system to exhibit sensitive dependence on initial conditions—a hallmark of chaos. The op-amps (TL082CD) are used for their high input impedance and low offset voltage, making them suitable for precise analog computations required in the jerk system.

By using Kirchhoffs circuit laws in Eq.(8), the circuital equations of the designed circuit in Figure 4 are derived as follows:

$$\begin{cases} \dot{x} &= \frac{1}{C_1 R_1} y, \\ \dot{y} &= \frac{1}{C_2 R_2} z, \\ \dot{z} &= -\frac{1}{C_3 R_3} x + \frac{1}{C_3 R_4} y - \frac{1}{C_3 R_5} z - \frac{1}{10 C_3 R_6} xy + \frac{1}{10 C_3 R_7} y^2. \end{cases} \tag{8}$$

Here, x, y, z are the voltages across the capacitors C_1, C_2, and C_3, respectively. We choose the values of the circuital elements as $R_6 = R_7 = 10$ kΩ, $R_4 = 1000$ kΩ, $R_1 = R_2 = R_3 = R_5 = R_8 = R_9 = R_{10} = R_{10} = 100$ kΩ, $C_1 = C_2 = C_3 = 1$ nF. The corresponding phase portraits on the oscilloscope are shown in Fig.5. Multisim simulation has been performed in order to validate the numerical simulation results. A good agreement has been revealed between the results obtained from Multisim software and Matlab software.

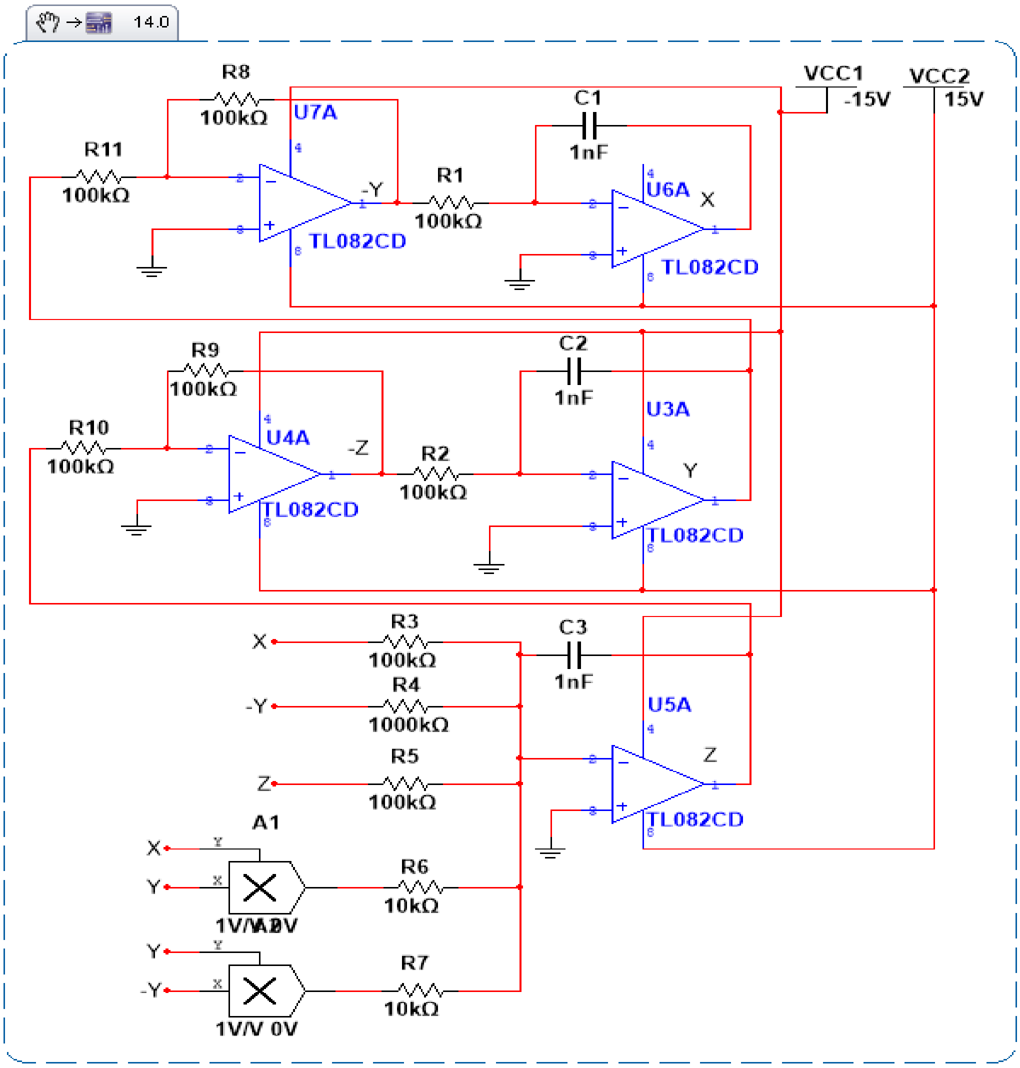
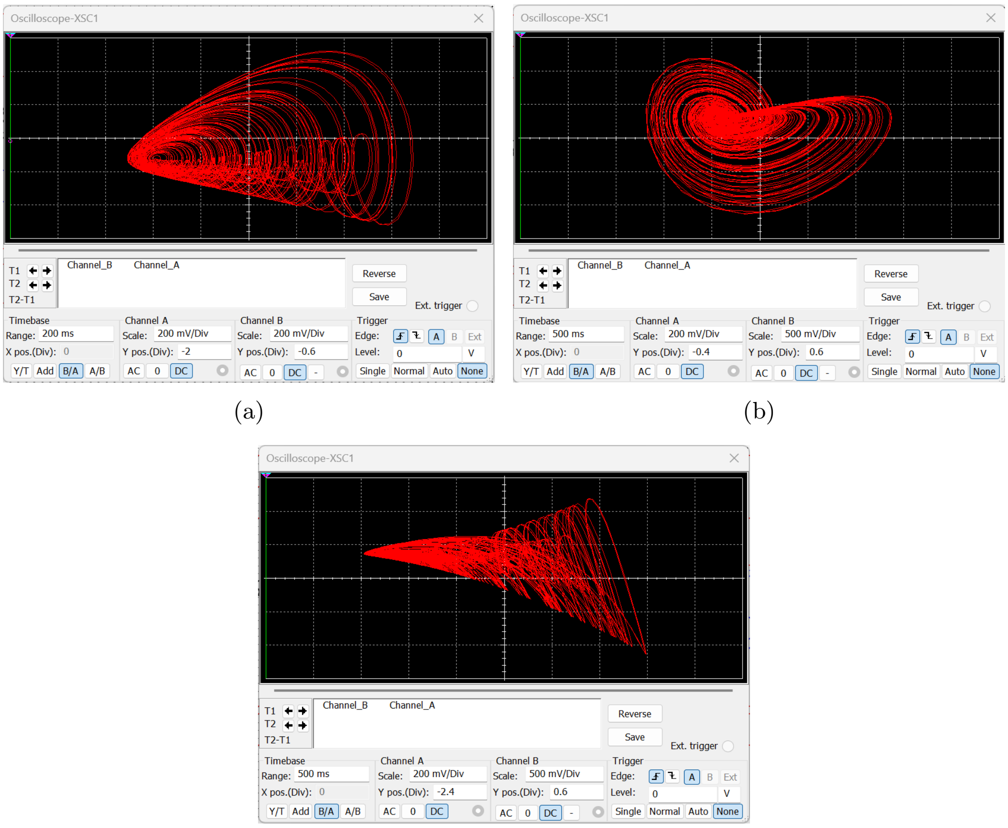


Figure 4: Circuit design of the system (1).

4 Complete Synchronization of the New Chaotic Jerk Systems

In this section, we give a new control application for the chaotic jerk system proposed in Section 2. We consider a pair of the new chaotic jerk systems taken as the *master* and *slave* systems, and we invoke the active backstepping control technique to synchronize the respective states of the master and slave jerk systems. We note that the synchronization of chaotic systems has important applications in engineering, namely in secure communications, cryptosystems, etc.



(c)

Figure 5: 2-D oscilloscope outputs of the new chaotic jerk system: (a) $x - y$ plane, (b) $y - z$ plane, and (c) $x - z$ plane.

As the master system for the synchronization process, we take the new jerk system with the dynamics given by

$$\begin{cases} \dot{x}_1 &= y_1, \\ \dot{y}_1 &= z_1, \\ \dot{z}_1 &= -ax_1 + by_1 - z_1 - x_1 y_1 + c y_1^2. \end{cases} \tag{9}$$

As the slave system for the synchronization process, we take the new jerk system with the dynamics given by

$$\begin{cases} \dot{x}_2 &= y_2, \\ \dot{y}_2 &= z_2, \\ \dot{z}_2 &= -ax_2 + by_2 - z_2 - x_2 y_2 + c y_2^2 + W(t). \end{cases} \tag{10}$$

In this research work, we use backstepping control to devise a feedback control $W(t)$ to asymptotically synchronize the states of the two jerk systems given by the equations (9) and (10).

We define the synchronization error between the master and slave jerk systems as follows:

$$\begin{cases} e_x &= x_2 - x_1, \\ e_y &= y_2 - y_1, \\ e_z &= z_2 - z_1. \end{cases} \tag{11}$$

The error dynamics can be calculated as

$$\begin{cases} \dot{e}_x &= e_y, \\ \dot{e}_y &= e_z, \\ \dot{e}_z &= -ae_x + be_y - e_z - x_2y_2 + x_1y_1 + c\left(y_2^2 - y_1^2\right) + W(t). \end{cases} \tag{12}$$

Next, we state and prove the main control result for the complete synchronization of the new jerk systems given by the Eqs. (9) and (10).

Theorem 4.1 The backstepping feedback control law defined by

$$W = -(3-a)e_x - (5+b)e_y - 2e_z + x_2y_2 - x_1y_1 - c(y_2^2 - y_1^2) - kq_3 \tag{13}$$

with the feedback gain $k > 0$ and $q_3 = 2e_x + 2e_y + e_z$ achieves complete synchronization between the chaotic jerk systems (9) and (10) for all initial states in $\mathbb{R}^3$.

Proof. We set $q_1 = e_x$.
We define the quadratic Lyapunov function as

$$V_1(q_1) = \frac{1}{2}\, e_x^2. \tag{14}$$

Then we get

$$\dot{V}_1 = q_1\dot{q}_1 = e_xe_y = -q_1^2 + q_1(e_x + e_y). \tag{15}$$

Next, we define $q_2 = e_x + e_y$ so that we can simplify Eq.(15) as follows:

$$\dot{V}_1 = -q_1^2 + q_1q_2. \tag{16}$$

Based on the Eq.(16), we define the quadratic Lyapunov function as

$$V_2(q_1, q_2) = V_1(q_1) + \frac{1}{2}q_2^2 = \frac{1}{2}q_1^2 + \frac{1}{2}q_2^2. \tag{17}$$

A simplification results in

$$\dot{V}_2 = -q_1^2 - q_2^2 + q_2(2e_x + 2e_y + e_z). \tag{18}$$

To simplify the notations in Eq.(19), we set

$$q_3 = 2e_x + 2e_y + ez. \tag{19}$$

Then Eq.(18) reduces to

$$\dot{V}_2 = -q_1^2 - q_2^2 + q_2q_3. \tag{20}$$

As a final step, we take the quadratic Lyapunov function given by

$$V(q_1, q_2, q_3) = V_2(q_1, q_2) + \frac{1}{2}q_3^2. \tag{21}$$

It is easy to see that V is a positive definite function on R^3. It is also very clear that

$$V(q_1, q_2, q_3) = \frac{1}{2}q_1^2 + \frac{1}{2}q_2^2 + \frac{1}{2}q_3^2. \tag{22}$$

Based on the Eq.(22), when we calculate the derivative of V, we get

$$\dot{V} = -q_1^2 - q_2^2 - q_3^2 + q_3 Z, \tag{23}$$

where we define Z as

$$Z = q_2 + q_3 + \dot{q}_3. \tag{24}$$

A simple calculation yields

$$Z = (3 - a)e_x + (5 + b)e_y + 2e_z - x_2 y_2 + x_1 y_1 + c(y_2^2 - y_1^2) + W. \tag{25}$$

Substituting the formula given in Eq.(13) for v into Eq.(25), we get

$$Z = -kq_3. \tag{26}$$

From the Eqs.(23) and (26), we get

$$\dot{V} = -q_1^2 - q_2^2 - q_3^2(1 + k). \tag{27}$$

Since $k > 0$, we see that $\dot{V}$ is a quadratic and negative definite function defined on R^3.

By Lyapunov Stability Theory, we deduce that the error dynamics (12) is globally exponentially stable. This completes the proof.

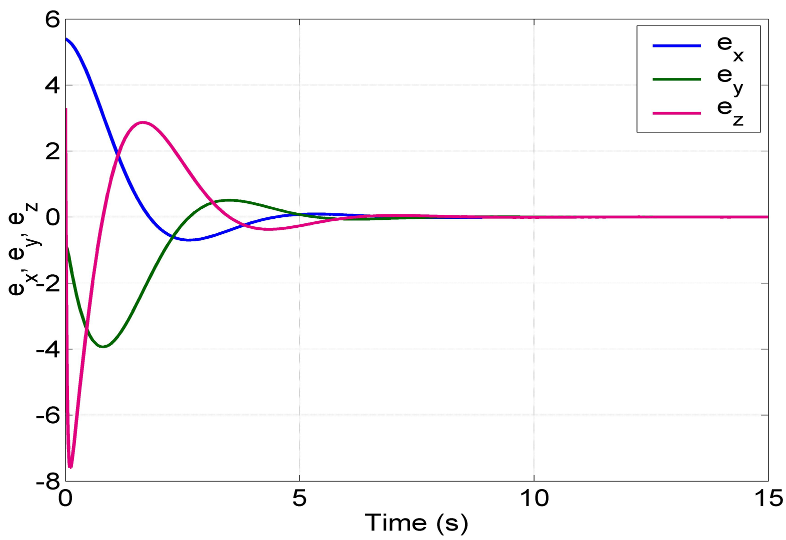


Figure 6: MATLAB plot depicting the exponential convergence of the complete synchronization error between the chaotic jerk systems (9) and (10).

For computer simulations, we consider the chaotic case for the master and slave jerk systems, *viz.* $a = 1$, $b = 0.1$ and $c = 1$. Also, we take $k = 30$.

For simulations, the initial conditions of the master system (9) are taken as $x_1(0) = 1.8,\ y_1(0) = 1.4$ and $z_1(0) = 2.1$.

Also, the initial conditions of the slave system (10) are taken as $x_2(0) = 7.2,\ y_2(0) = 0.5$ and $z_3(0) = 5.4$.

Figure 6 shows the convergence of the synchronization errors ϵ_x, ϵ_y and ϵ_z between the chaotic jerk systems (9) and (10).

5 Conclusion

This study introduces a novel chaotic 3-D jerk system characterized by two quadratic nonlinearities, which sets it apart from previously studied chaotic systems. The specific configuration of the system, including the interplay of its nonlinearities, represents a new contribution to the field of nonlinear dynamics and chaos theory. Additionally, the practical realization of this chaotic system through a custom-designed electronic circuit demonstrates a unique approach to linking theoretical chaos models with physical implementations. The circuit design, validated through Multisim and MATLAB simulations, offers a tangible and reliable method to replicate chaotic behavior in real-world applications.

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